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Abstract

We consider the linearized version of the stationary Navier-Stokes equations on a subdomain Ω of a smooth, compact Riemannian manifold M. The emphasis is on regularity: the boundary of Ω is assumed to be only C^1 and even Lipschitz, and the data are selected from appropriate Sobolev-Besov scales. Our approach relies on the method of boundary integral equations, suitably adapted to the variable coefficient setting we are considering here. Applications to the stationary, non-linear Navier-Stokes equations in this context are also discussed.

Key words. Stationary Navier-Stokes system, Lipschitz domains, boundary problems, Sobolev-Besov spaces

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THE STATIONARY NAVIER-STOKES SYSTEM IN NONSMOOTH MANIFOLDS: THE POISSON PROBLEM IN LIPSCHITZ AND C^1 DOMAINS

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1. Introduction

In this paper we study the linearized version of the stationary Navier-Stokes equations on a fixed subdomain Ω of a smooth, compact Riemannian manifold M. Let Tr denote the trace on $\partial\Omega$. With Def standing for the deformation tensor and with d denoting the exterior derivative operator on M, set $L = 2\,\mathrm{Def}^*\mathrm{Def}$, $\delta = d^*$. We consider the Dirichlet problem for the (modified) Stokes system

$$\begin{aligned} &Lu + \nabla_\omega u + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM),\\ &\delta u = h \in L^p_{s+\frac{1}{p}-1}(\Omega),\\ &\mathrm{Tr}\, u = g \in B^{p,p}_s(\partial\Omega, \Lambda^1 TM). \end{aligned} \tag{1.1}$$

Here, ω, f, h and g are given and ω is assumed to be divergence-free. We are interested in this equation as it is an important first step toward solving the stationary Navier-Stokes system via fixed point techniques. We elaborate more on this in the second part of this introduction.

A related, simpler PDE, the Dirichlet problem for the Stokes system

$$\begin{aligned} &Lu + d\pi = f \text{ in } \Omega,\\ &\delta u = 0 \text{ in } \Omega,\\ &\mathrm{Tr}\, u = f \text{ on } \partial\Omega \end{aligned} \tag{1.2}$$

in arbitrary Lipschitz domains on manifolds, has been studied in [46]. There it has been established that there exists $1 \le q_\Omega < 2$ such that, for each $q_\Omega < p < (1 - 1/q_\Omega)^{-1}$, the solution (u, π) of (1.2) satisfies

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$$\|\mathcal{N}(u)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p(\partial\Omega,\Lambda^1 TM)} \tag{1.3}$$

and

$$\|\mathcal{N}(\nabla u)\|_{L^p(\partial\Omega)} + \|\mathcal{N}(\pi)\|_{L^p(\partial\Omega)} \leq C\|f\|_{L^p_1(\partial\Omega,\Lambda^1 TM)}. \tag{1.4}$$

Here $\mathcal{N}$ is the standard *nontangential maximal operator*; cf. (4.16) and [46] for more details. For further reference, let us also introduce here the critical index

$$p_\Omega := \max\Big\{q_\Omega\,,\,q_{M\setminus\bar{\Omega}}\Big\}. \tag{1.5}$$

See the body of the text for a thorough discussion of notation as well as other relevant definitions; cf. especially the first part of §2.

As an initial, natural step in the direction of understanding (1.1), we study the Poisson problem for the Stokes system in non-smooth domains with data in Sobolev-Besov spaces. This is concerned with finding a velocity field $u \in L^p_{s+\frac{1}{p}}(\Omega,\Lambda^1 TM)$ along with a pressure function $\pi \in L^p_{s+\frac{1}{p}-1}(\Omega)$, solving the equation (1.2). Our main novel technical achievement is proving sharp estimates for this problem both in lower dimensional Lipschitz domains, as well as in C^1 domains of arbitrary dimension.

Let us now momentarily digress and explain why we find it both natural and important to study these (and related) problems on *Riemannian manifolds*. For starters, note that our setting applies to the case of a bounded Euclidean domain in $\mathbb{R}^n$, since such a domain can be embedded in a (sufficiently large) flat torus (equipped with the canonical metric). However, the main advantage of working with a *general* Riemannian metric tensor is that this gives rise to a context in which *variable-coefficient* operators arise naturally. For example, any (scalar) divergence-form operator

$$Lu = \sum_{1\leq i,j\leq n} \partial_i(a_{ij}(x)\partial_j u), \tag{1.6}$$

induced by a positive definite matrix $A(x) = (a_{ij}(x))_{ij}$ can be viewed (modulo a multiple of suitable power of $\det[A(x)]$) as the Laplace-Beltrami operator Δ_g associated with the Riemannian metric tensor

$$g(x) := (\det[A(x)])^{1/(n-2)} \sum_{1\leq i,j\leq n} a^{ij}(x)\,dx_i \otimes dx_j, \tag{1.7}$$

where a^{ij} are the entries of A^{-1}, for $n \geq 3$.

Formulating and studying the Navier-Stokes equations on Riemannian manifolds has a fairly rich history, going back to an influential paper by D. Ebin and J. Marsden, [16], where the correct form of these equations has been first identified. One of the important observations made in [16] is as follows. While in the flat, Euclidean setting, the operator L in (1.1) is the ordinary, constant coefficient Laplacian, the correct replacement −in the context of a Riemannian manifold – is not the Hodge-Laplacian on forms, but rather the deformation-Laplacian $2\,\mathrm{Def}^*\mathrm{Def}$ (the two do not coincide unless the manifold is Ricci-flat). Other papers dealing with fluid dynamics problems on Riemannian manifolds are [7], [39], [48], [51], [53], [60], [59]. In addition, a number of authors have dealt with special geometric settings, such as that of a sphere (cf. [62], [63]). Another distinct advantage of working in a non-zero curvature ambient space is that in that setting we are able to remove certain topological assumptions of artificial nature customarily imposed on the domain Ω. For instance, suitably altering the underlying manifold, matters can be always arranged so that the complement Ω^c is simultaneously *connected* and *compact* (even when $\partial\Omega$ is *not* necessarily connected).

This point of view has been systematically emphasized by Mitrea and Taylor who, starting in the mid 1990's, have treated boundary value problems for elliptic differential operators in Lipschitz subdomains of Riemannian manifolds, via layer potential methods. This program has produced [40], [42], [43], [44], [45], and [46]. In particular, building on the work in [18], they have extended the main results of Jerison and Kenig [34] to include Poisson problems with Neumann and Dirichlet boundary conditions in Lipschitz domains for the Laplace-Beltrami operator associated with a rough metric tensor. In [11], we have extended the scope of this work by allowing scalar, lower-order, nonlinear perturbations.

Here, we are led to considering similar issues for *systems* of PDE's. More specifically, our strategy for dealing with (1.1) is to regard this problem as a linear perturbation of the Poisson problem for the Stokes system

$$\begin{aligned} &Lu + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM),\\ &\delta u = h \in L^p_{s+\frac{1}{p}-1}(\Omega),\\ &\mathrm{Tr}\, u = g \in B^{p,p}_s(\partial\Omega, \Lambda^1 TM). \end{aligned} \tag{1.8}$$

These results are important for theory of partial regularity, as they clarify the (maximal) amount of smoothness exhibited by the solution on Sobolev-Besov scales.

As far as this latter problem is concerned, we are able to prove optimal (in the sense that the range of indices s and p is largest possible) well-posedness results for the following types of subdomains of the manifold M:

(*i*) *arbitrary Lipschitz domains of dimension* ≤ 3;
(*ii*) *Lipschitz domains with a sufficiently small Lipschitz constant of arbitrary dimension.*

Note that the last category above includes the class of all C^1 domains. We also derive certain partial results in the case of Lipschitz domains of arbitrary space dimension; they require $|p-2|$ to be sufficiently small. It is clear that many of our estimates have applications to the dynamic case as well, although this point of view will be pursued elsewhere.

There is a rather extensive literature devoted to the study of the Stokes system (1.8) in domains exhibiting a limited amount of smoothness. Our results unify and extend most of these, at least as far as the range of indices p, s are concerned, in the context when $\partial\Omega$ is rough and when there are no artificial topological assumptions on Ω.

When $\partial\Omega \in C^{1,1}$, the classical approach based on the Agmon-Douglis-Nirenberg theory [1] applies. This point of view has been exploited in [2], which also made essential use of the resolvent estimates from [28]; cf. also [61] and the approach via pseudodifferential operators from [59]. Certain lower dimensional cases (dim ≤ 3) for C^2 domains have been treated earlier in [8] and [25]. For indices related by $s + 1/p = 1$, Euclidean domains with a small Lipschitz constant have been dealt with in [26] based on flattening the boundary and *a priori* estimates. L^p estimates in conical domains have been derived in [12], [13]. See also [5] for regularity issues related to the Stokes system in Lipschitz domains.

Our analysis of the linearized Stokes system relies on the method of layer potentials. This approach has proved very effective in the treatment of elliptic equations for smooth, flat, Euclidean domains. In the case of the Stokes system, this technique goes back to the pioneering work of Odqvist and Lichtenstein and, more recently, to Solonnikov [54] and Ladyzenskaja [38], among others.

In late 1980's, Fabes, Kenig and Verchota [17] have successfully employed the method of layer potentials for the Dirichlet problem for the Stokes system (with nontangential maximal function estimates) in Euclidean Lipschitz domains. Subsequently, their work has been used in [14] (which also has an extensive list of references) to prove short time existence of solutions for the (non-stationary) Navier-Stokes equations in three-dimensional Euclidean Lipschitz domains, based on the Fujita-Kato approach [22]. Related results are in [46] for Lipschitz subdomains of Riemannian manifolds. More recently, and in the same geometrical context, Monniaux [47] has proved the existence of mild solutions $u \in \mathcal{C}([0,T); L^3(\Omega, \mathbb{R}^3))$ for the Navier-Stokes equations (with small initial data in $L^{3/2}_1(\Omega, \mathbb{R}^3)$). This is remarkable since, at the present time, it is not known whether the Stokes semi-group in $L^2(\Omega, \mathbb{R}^3)$ extends to an analytic semi-group in $L^3(\Omega, \mathbb{R}^3)$, for any Lipschitz domain $\Omega \subset \mathbb{R}^3$. For smooth domains, see [28] and [29]. Other papers, dealing with related regularity issues are [24] and [37].

The final step in our analysis –utilizing the results for the linearized problem in the treatment of the stationary Navier-Stokes system– requires a proper understanding of how the solution u of (1.8) depends on lower order perturbations of the Stokes system. *A priori* regularity results and uniform estimates eventually allow us to conveniently implement the Schauder fixed-point theory in the framework of Lebesgue spaces; see §7 for the actual details.

The layout of the first part of the paper (dealing with linear BVP's) is as follows. Section 2 contains a detailed description of the geometric setting, as well as of function spaces and relevant differential operators. Here we also introduce the hydrostatic single layer operator S and summarize some of its mapping properties. The double layer operator is considered in detail in §3. In Section 4 we deal with the issue of inverting S on Besov spaces in arbitrary Lipschitz domains of dimension ≤ 3. Here, the starting point is an adaptation of the Hölder estimate from [52], which we further interpolate with the L^2-theory from [46]. The well-posedness of the Poisson problem for the Stokes system is then established in §5, both for domains with small Lipschitz constant as well as for arbitrary Lipschitz domains of dimension ≤ 3. As a preamble for the nonlinear problem, a perturbation of (1.8) is considered in §6.

Our tools and techniques, including elements of Calderón-Zygmund theory, smoothness spaces, interpolation, pseudodifferential operators, and Hodge theory, are based on the earlier work of many people. In particular, the harmonic analysis approach to problems arising in PDE's as presented in Kenig's book [36] has been particularly influential for our work.

The last section §7 is devoted to the nonlinear problem - the stationary Navier-Stokes equation

$$
\begin{aligned}
&Lu + \nabla_u u + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM), \\
&\delta u = h \in L^p_{s+\frac{1}{p}-1}(\Omega), \\
&\mathrm{Tr}\, u = g \in B^{p,p}_s(\partial\Omega, \Lambda^1 TM),
\end{aligned}
\tag{1.9}
$$

in Sobolev-Besov spaces, via a fixed-point argument.

In the case of smooth domains in $\mathbb{R}^n$ with $n \leq 6$, solutions to the stationary-Navier Stokes equations have been constructed for arbitrary smooth f (and $h = 0$, $g = 0$) by J. Frehse, M. Ružička and M. Struwe in a series of papers [19], [20], [21] and [56], [57]. In the case of periodic boundary conditions the same result is true up to dimension 15.

The solutions these authors produce are smooth in the interior even for large (smooth) data. Their approach avoids using perturbation techniques and, instead, relies on a suitable maximum principle. This is particularly relevant in dimensions ≥ 5, where the standard methods are no longer applicable (for large data). Let us also mention that the issue of establishing regularity *up to the boundary* is open even for smooth large data.

Another important open problem, in the case of the stationary Navier-Stokes equations, is the existence of a solution with arbitrarily large prescribed boundary data on domains with disconnected boundary in dimensions $n \geq 2$ (under the natural condition that the *total* flux through the boundary is zero). The interested reader is referred to [23] for a discussion.

The role of self-similarity in the Euclidean context was studied in [50] in connection with the *interior* regularity of the solution of the evolution problem (where a surprising negative results has been proved in the critical case). Another relevant reference in this regard is [65]. In this connection, a very interesting open problem is that of the existence of a non-trivial self-similar singular solution of the Navier-Stokes equations in a cone, with the singularity located at the boundary (this is not known even in the case of a half-space).

Finally, Stokes and Navier-Stokes equations in nonsmooth domains have also been treated in [17], [5], [52], [58], [46], while Sobolev and Besov data have been emphasized in [39], [60], [30], [31], and [32], to give just a few examples.

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2. Notation, definitions and preliminary results

Let M be a smooth, compact, boundaryless manifold of (real) dimension n. As usual, by TM and T^*M we denote, respectively, the tangent and cotangent bundle on M. Also, $\Lambda^\ell TM$ stands for the corresponding (exterior) power bundle. We assume that M is equipped with a smooth Riemannian metric tensor $g = g_{jk}dx_j \otimes dx_k$, denote by $(g^{jk})_{jk}$ the inverse matrix to (g_{jk}) and set $g := \det (g_{jk})_{jk}$. Thus, in local coordinates, the volume element is given by $d\mathrm{Vol} = \sqrt{g}\, dx_1 ... dx_n$.

The pairing $\langle dx_j, dx_k\rangle := g^{jk}$ defines an inner product in $\Lambda^1 TM$. More generally, if $\{\omega_j\}_j$ is a local orthonormal frame for $\Lambda^1 TM$, then we take $\{\omega_I\}_I$ to be a local orthonormal frame for $\Lambda^\ell TM$. Here, for each increasing multi-index of length ℓ, $I = (i_1, i_2, ..., i_\ell)$, we set $\omega_I := \omega_{i_1} \wedge \omega_{i_2} \wedge ... \wedge \omega_{i_\ell}$, with wedge denoting the ordinary exterior product of forms.

As it is customary, we may identify vector fields with one-forms (i.e., $TM \cong T^*M = \Lambda^1 TM$) via $\partial_j \mapsto g_{jk}dx_k$ (lowering indices). This mapping is an isometry whose inverse (raising indices) is given by $dx_j \mapsto g^{jk}\partial_k$. In the sequel, we shall not make any notational distinction between a vector field and its associated one-form. Under this identification, we have grad $\equiv d$ and div $\equiv -\delta$. Hereafter, we let d and δ stand, respectively, for the exterior derivative and exterior co-derivative operators.

The Hodge Laplacian is then given by

$$\Delta := -d\delta - \delta d. \tag{2.1}$$

Furthermore, if ∇ is the Levi-Civita connection and Ric is the Ricci tensor on M then, under the above identification, the Bochner Laplacian and the Hodge Laplacian are related by

$$-\nabla^*\nabla \equiv \Delta + \mathrm{Ric}, \tag{2.2}$$

a special case of the Weitzenbock identity.

We let OPS^ℓ_{cl} denote the collection of all classical pseudodifferential operators $P(x, D)$ of order ℓ. In particular, their symbols $p(x, \xi) \in S^\ell_{\mathrm{cl}}$ satisfy $p(x, \xi) \sim p_\ell(x, \xi) + p_{\ell-1}(x, \xi) + \cdots$, where $p_j(x, \xi)$ is smooth and homogeneous of degree j in ξ for $|\xi| \geq 1$, $j = \ell, \ell-1, ...$ Also, we denote by $\sigma(P; x, \xi)$ the principal symbol $p_\ell(x,\xi)$. Set $ØPS^\ell_{\rm cl}$ for the class of formal adjoints of operators in $OPS^\ell_{\rm cl}$.

The deformation tensor Def$X$ of a field $X \in TM$ is given by

$$(\mathrm{Def}\, X)(Y,Z) = \tfrac{1}{2}\{\langle \nabla_Y X, Z\rangle + \langle \nabla_Z X, Y\rangle\}, \quad \forall\, X, Y, Z \in T^*M. \tag{2.3}$$

Thus, Def : $C^\infty(M, TM) \to C^\infty(M, S^2T^*M)$, where S^2T^*M stands for symmetric tensor fields of type $(0,2)$. In coordinate notation,

$$(\mathrm{Def}\, X)_{jk} = (\mathrm{Def}\, X)(\partial_j, \partial_k) = \tfrac{1}{2}(X_{j;k} + X_{k;j}), \quad \forall\, j, k. \tag{2.4}$$

Here, for a vector field $X = X^j\partial_j$ it is customary to set $X_{k;j} := \partial_j X_k - \Gamma^l_{kj}X_l$, where Γ^l_{kj} are the Christoffel symbols associated with the metric. In the sequel, we shall find it convenient to denote $TM \ni Z \mapsto (\mathrm{Def}\, X)(Y,Z) \in \mathbb{R}$ by $(\mathrm{Def}\, X)Y \in \Lambda^1 TM$.

At each $x \in M$, the principal symbol of the deformation operator is

$$\sigma(\mathrm{Def}; x, \xi)u = -i\tfrac{1}{2}(\xi \otimes u + u \otimes \xi), \qquad \forall\, \xi \in T^*_x M, \ u \in T_x M. \tag{2.5}$$

The adjoint of the operator Def is $\mathrm{Def}^* v = -\mathrm{div}\, v$, $v \in S^2T^*M$, or in local coordinates, $(\mathrm{Def}^* v)^j = -v^{jk}{}_{;k}$. Thus if we consider the second-order partial differential operator

$$L := 2\,\mathrm{Def}^*\,\mathrm{Def} = \nabla^*\nabla - \mathrm{grad}\,\mathrm{div} - \mathrm{Ric} \equiv -\Delta + d\delta - 2\,\mathrm{Ric} \tag{2.6}$$

then, at each $x \in M$,

$$\sigma(L; x, \xi) = |\xi|^2 I + \xi \wedge (\xi \vee \cdot), \qquad \xi \in T^*_x M \setminus 0, \tag{2.7}$$

where, here and elsewhere, $\xi\vee$ stands for the interior product (with ξ), and I is the identity.

Consider an arbitrary Lipschitz subdomain Ω of M. This means that $\partial\Omega$ can be covered by a finite family of open cylinders $\{Z_i\}_{1\leq i\leq N}$ with the following properties (in local coordinates in $\mathbb{R}^n$). For each i, there exists a Lipschitz function $\varphi_i : \mathbb{R}^{n-1} \to \mathbb{R}$ so that $2\|\varphi_i\|_{L^\infty}$ is less than the height of Z_i and, if $2Z_i$ denotes the concentric double of Z_i, in the rectangular coordinate system defined by Z_i one has

$$\begin{aligned} \Omega \cap 2Z_i &= \{x = (x', x_n); \varphi_i(x') < x_n\} \cap 2Z_i, \\ \partial\Omega \cap 2Z_i &= \{x = (x', x_n); \varphi_i(x') = x_n\} \cap 2Z_i. \end{aligned} \tag{2.8}$$

See, e.g., [49] for more details. The families $\{Z_i\}_{1\leq i\leq N}$, $\{\varphi_i\}_{1\leq i\leq N}$ with the above properties define what we are going to call the Lipschitz character of Ω. In particular,

$$\max\Big\{\|\varphi_i\|_{L^\infty};\, 1\le i\le N\Big\} \tag{2.9}$$

will be referred to as the Lipschitz constant of Ω.

The standard Sobolev scale $L^p_s(M)$, $1<p<\infty$, $s\ge 0$, is obtained by lifting $L^p_s(\mathbb{R}^n):=\{(I-\Delta)^{s/2}f;\ f\in L^p(\mathbb{R}^n)\}$ to M via a smooth partition of unity and pull-back. Let $L^p_s(\Omega)$ denote the restriction of elements in $L^p_s(M)$ to the Lipschitz domain Ω, and $L^p_{s,0}(\Omega)$ stand for the subspace consisting of restrictions to Ω of elements from $L^p_s(M)$ with support contained in $\bar{\Omega}$. Finally, for $s>0$ and $1<p,q<\infty$ with $1/p+1/q=1$, we set $L^p_{-s}(\Omega):=(L^q_{s,0}(\Omega))^*$. Boundary Sobolev spaces $L^p_s(\partial\Omega)$, $1<p<\infty$, $0\le s\le 1$, can be introduced by starting with the Euclidean model, $L^p_s(\mathbb{R}^{n-1})$, via a partition of unity and pull-back. These Sobolev scales are then related to Besov spaces via real interpolation:

$$\begin{aligned}(L^p(\Omega),L^p_k(\Omega))_{s,q}&=B^{p,q}_{sk}(\Omega),\\ (L^p(\partial\Omega),L^p_1(\partial\Omega))_{s,q}&=B^{p,q}_{s}(\partial\Omega),\end{aligned} \tag{2.10}$$

when $0<s<1$, $1<p,q<\infty$ and k is a positive integer. Furthermore, the trace operator Tr is well-defined from either $L^p_s(\Omega)$ or $B^{p,p}_s(\Omega)$ onto $B^{p,p}_{s-1/p}(\partial\Omega)$ for each $1<p<\infty$ and $1/p<s<1+1/p$. For a more detailed exposition, the interested reader is referred to [3], [34], [44], [64]. We set $L^p_s(\Omega,\Lambda^\ell TM):=L^p_s(\Omega)\otimes\Lambda^\ell TM\Big|_\Omega$ and $B^{p,p}_s(\partial\Omega,\Lambda^\ell TM):=B^{p,p}_s(\partial\Omega)\otimes\Lambda^\ell TM\Big|_{\partial\Omega}$ etc., but we shall occasionally drop the dependence of the various spaces of forms on the exterior power bundle.

Next we discuss the single layer potential operator, starting with the construction of a suitable fundamental solution. Let $\Omega\subset M$ be a connected Lipschitz domain (note that we do not insist on $\partial\Omega$ being connected). By eventually altering M away from $\overline{\Omega}$, matters will henceforth be arranged so that:

$$M\ \text{ has no nontrivial Killing fields,} \tag{2.11}$$

and

$$M\setminus\overline{\Omega}\ \text{ is connected.} \tag{2.12}$$

See [46] for more details. Now (2.11) guarantees that $\operatorname{Ker}\operatorname{Def}=\{0\}$. In particular, the elliptic operator

$$L:=2\,\mathrm{Def}^*\mathrm{Def}:L^2_1(M,TM)\to L^2_{-1}(M,TM) \tag{2.13}$$

is invertible. Next, let $\mathcal{H}(M)$ be the (finite dimensional) space of all harmonic one-forms on M, and denote by P_h the orthogonal projection on $\mathcal{H}(M)$. The Green operator at the level of one-forms is then defined by

$G := (-\Delta)^{-1}P_h^{\perp}$ where $(-\Delta)^{-1}$ is the inverse of $-\Delta$ on $\mathcal{H}(M)^{\perp}$. Consequently, $P_d := d\delta G$ and $P_\delta := \delta d G$ are projections onto the ranges of d and δ, respectively. As in [46], consider

$$\widetilde{L} := P_d^{\perp} L P_d^{\perp} + P_d L P_d \in OPS^2(TM, TM), \tag{2.14}$$

which is self-adjoint, elliptic and also invertible from $L^2_1(M, TM)$ onto $L^2_{-1}(M, TM)$. If we now set

$$\begin{aligned} \Phi &:= P_d^{\perp}\widetilde{L}^{-1} \in OPS^{-2}(\Lambda^1 TM, \Lambda^1 TM), \\ \Psi &:= -\delta G(L\Phi - I) \in OPS^{-1}(\Lambda^1 TM, \mathbb{R}), \end{aligned} \tag{2.15}$$

then, in the sense of distributions (and with I denoting the identity operator),

$$L\Phi + d\Psi = I \quad \text{and} \quad \delta\Phi = 0 \quad \text{on } M. \tag{2.16}$$

Denote by $\Gamma(x, y)$ and $\Theta(x, y)$ the Schwartz kernels of Φ and Ψ, respectively. Then the above identity yields

$$L_x\Gamma(x, y) + d_x\Theta(x, y) = \mathrm{Dirac}_y(x), \qquad \delta_x\Gamma(x, y) = 0, \tag{2.17}$$

on M, where Dirac_y is the Dirac distribution with mass at y. At the level of *principal symbols*, a straightforward algebraic calculation gives that

$$\begin{aligned} \sigma(\Phi; x, \xi) &= |\xi|^{-2} - |\xi|^{-4}\xi \wedge (\xi \vee \cdot), \quad x \in M,\ \xi \in T_x^*M \setminus 0, \\ \sigma(\Psi; x, \xi) &= i|\xi|^{-1}\xi \vee \cdot\,, \qquad x \in M,\ \xi \in T_x^*M \setminus 0. \end{aligned} \tag{2.18}$$

Next, denote by $\nu \in T^*M \equiv \Lambda^1 TM$ the outward unit conormal to $\partial\Omega$ and by $d\sigma$ the surface measure on $\partial\Omega$. For a $\Lambda^1 TM$-valued function f on $\partial\Omega$, we introduce the single layer potential

$$\mathcal{S}f(x) := \int_{\partial\Omega} \langle \Gamma(x, y), f(y)\rangle \, d\sigma_y, \quad x \in M \setminus \partial\Omega, \tag{2.19}$$

along with

$$\mathcal{Q}f(x) := \int_{\partial\Omega} \langle \Theta(x, y), f(y)\rangle \, d\sigma_y, \quad x \in M \setminus \partial\Omega. \tag{2.20}$$

Finally, set

$$Sf := \mathrm{Tr}\,(\mathcal{S}f) \quad \text{on} \quad \partial\Omega. \tag{2.21}$$

The theorem below summarizes some of the most important mapping properties of the operators (2.19)-(2.21) on Sobolev-Besov scales in Lipschitz domains.

Theorem 2.1. *For each Ω, Lipschitz domain, the following hold.*

(i) For $0 < s < 1/p < 1$, the operator

$$\mathcal{Q} : B^{p,p}_{-s}(\partial\Omega, \Lambda^1 TM) \to B^{p,p}_{-s+1/p}(\Omega, \Lambda^1 TM) \cap L^p_{-s+1/p}(\Omega, \Lambda^1 TM), \tag{2.22}$$

is well-defined and bounded.

(ii) For $1 < p < \infty$ and $0 < s < 1$,

$$\mathcal{S} : B^{p,p}_{-s}(\partial\Omega, \Lambda^1 TM) \to B^{p,p}_{-s+1+1/p}(\Omega, \Lambda^1 TM) \cap L^p_{-s+1+1/p}(\Omega, \Lambda^1 TM) \tag{2.23}$$

is well-defined and bounded. Also,

$$\mathcal{S}\nu = 0 \quad \textit{and} \quad \mathcal{Q}\nu = \textit{constant} \quad \textit{in} \;\; \Omega. \tag{2.24}$$

(iii) For $0 < s < 1$, the estimates

$$\Big\| \mathrm{dist}(\cdot, \partial\Omega)^{1-s} |\nabla \mathcal{S} f| \Big\|_{L^\infty(\Omega)} \le C \|f\|_{(C^s(\partial\Omega, \Lambda^1 TM))^*}, \tag{2.25}$$

$$\Big\| \mathrm{dist}(\cdot, \partial\Omega)^{1-s} |\mathcal{Q} f| \Big\|_{L^\infty(\Omega)} \le C \|f\|_{(C^s(\partial\Omega, \Lambda^1 TM))^*},$$

hold uniformly in f.

(iv) For $1 < p < \infty$, and $0 < s < 1$,

$$S : L^p_{-s}(\partial\Omega, \Lambda^1 TM) \longrightarrow L^p_{1-s}(\partial\Omega, \Lambda^1 TM), \tag{2.26}$$

$$S : B^{p,p}_{-s}(\partial\Omega, \Lambda^1 TM) \longrightarrow B^{p,p}_{1-s}(\partial\Omega, \Lambda^1 TM)$$

are well-defined and bounded. Furthermore, there exists $\varepsilon = \varepsilon(\Omega) > 0$ so that whenever $2 - \varepsilon < p < 2 + \varepsilon$ and $0 < s < 1$, the above operators are Fredholm with index zero. In each case,

$$\mathrm{Ker}\, S = \mathbb{R}\nu, \tag{2.27}$$

independently of s, p.

The proof is essentially contained in [46].

3. The double layer potential on Sobolev-Besov spaces

In the previous section we have established mapping properties for the single layer potential $\mathcal{S}$ and its corresponding boundary version S. Here we consider the double layer potential $\mathcal{D}$, the pressure operator $\mathcal{P}$, and the (boundary) principal-value double layer K. The main results are the mapping properties of $\mathcal{D}$, $\mathcal{P}$ on Sobolev-Besov scales in Propositions 3.3 and 3.4, and the Fredholmness of K on (essentially) C^1 domains in Proposition 3.5.

Recall that $\mathit{\Gamma}(x,y)$, $\mathit{\Theta}(x,y)$ are the Schwartz kernels of the pseudodifferential operators (2.15). Consider the double layer potential operator

$$\mathcal{D}f(x) := \int_{\partial\Omega} \big\langle -2\big[\mathrm{Def}_y(\mathit{\Gamma}(x,\cdot))\nu\big](y) + \mathit{\Theta}^t(y,x)\nu(y), f(y)\big\rangle \, d\sigma_y, \quad x \in \Omega, \tag{3.1}$$

along with its (principal value) boundary version for $x \in \partial\Omega$

$$Kf(x) := \text{p.v.} \int_{\partial\Omega} \big\langle -2\big[\mathrm{Def}_y(\mathit{\Gamma}(x,\cdot))\nu\big](y) + \mathit{\Theta}^t(y,x)\nu(y), f(y)\big\rangle \, d\sigma_y. \tag{3.2}$$

We shall also need the formal transposed of (3.2), i.e.,

$$K^*f(x) := \text{p.v.} \int_{\partial\Omega} \big\langle -2\big[\mathrm{Def}_x(\mathit{\Gamma}(\cdot,y))\nu\big](x) + \mathit{\Theta}(x,y)\nu(x), f(y)\big\rangle \, d\sigma_y, \tag{3.3}$$

for $x \in \partial\Omega$. The general jump formulas derived in [40] (cf. also the discussion in [46]) give that

$$\mathcal{D}f\big|_{\partial\Omega} = (\tfrac{1}{2}I + K)f, \qquad \text{on } \partial\Omega, \tag{3.4}$$

and

$$u := \mathcal{S}f,\ \pi := \mathcal{Q}f \Longrightarrow \Big[2(\mathrm{Def}\, u)\nu - \pi\nu\Big]\Big|_{\partial\Omega} = (-\tfrac{1}{2}I + K^*)f, \qquad \text{on } \partial\Omega. \tag{3.5}$$

In particular, (2.24) gives

$$K^*\nu \in \mathbb{R}\nu. \tag{3.6}$$

For further reference, let us also note that whenever $Lu + d\pi = 0$ and $\delta u = 0$ in Ω, the following Green formula holds:

$$u = \mathcal{D}(u|_{\partial\Omega}) - \mathcal{S}\Big(\big[2(\mathrm{Def}\, u)\nu - \pi\nu\big]\Big|_{\partial\Omega}\Big), \qquad \text{in } \Omega. \tag{3.7}$$

This is seen by paring u with (2.17) and then successively integrating by parts.

Proposition 3.1. *With the above notation,*

$$\partial\Omega \in C^\infty \Longrightarrow K \in OPS^{-1}_{cl}(\Lambda^1 TM|_{\partial\Omega}, \Lambda^1 TM|_{\partial\Omega}). \tag{3.8}$$

Proof. That K is a zero-order pseudodifferential operator is standard. The subtler fact that this operator is actually smoothing of order one is going to be a consequence of the special structure of the main singularity of the integral kernel in (3.2), which we now proceed to analyze.

Recall the symbol of Φ from (2.18). Working in local coordinates and (occasionally) identifying one-differential forms with vector fields (via the metric tensor), one can regard $\Gamma(x,y)$ as a matrix with entries given (modulo a purely dimensional constant which we suppress) by

$$\begin{aligned}\Gamma_{rs}(x,y) = \; & \frac{1}{n-2}\, e_0(x-y,y)^{-(n-2)/2}\, g_{rs}(x) \\ & + e_0(x-y,y)^{-n/2}(x_\theta - y_\theta)(x_\eta - y_\eta) g_{r\theta}(x) g_{s\eta}(x) \\ & + \{\text{less singular terms}\}.\end{aligned} \tag{3.9}$$

This formula holds for dim $M = n \geq 3$, the case $n = 2$ will be treated separately in the next section. Here, as usual, the summation convention is used, and we set

$$e_0(z,y) := g_{jk}(y) z_j z_k. \tag{3.10}$$

Similarly, given the formula for the symbol of Ψ from (2.18), $\Theta(x,y)$ can be thought of as the vector with components

$$\Theta_s(x,y) = 2e_0(x-y,y)^{-n/2}(x_\tau - y_\tau) g_{s\tau}(x) + \{\text{less singular terms}\}. \tag{3.11}$$

Our aim is to identify the top singularity in

$$\nu_j(x) g^{jk}(x)\Big(\partial_{x_k}\Gamma_{\ell s}(x,y) + \partial_{x_\ell}\Gamma_{ks}(x,y)\Big) - \nu_\ell(x)\Theta_s(x,y). \tag{3.12}$$

Now, from (3.9), we obtain after some algebra

$$\begin{aligned}\nu_j(x) g^{jk}(x) \partial_{x_k}\Gamma_{\ell s}(x,y) = & -e_0(x-y,y)^{-n/2}\nu_j(x)(x_j - y_j) g_{s\ell}(x) \\ & -\tfrac{n}{2}\, e_0(x-y,y)^{-(n+2)/2}\nu_j(x)(x_j-y_j)(x_\theta-y_\theta)(x_\eta-y_\eta) g_{\ell\theta}(x) g_{s\eta}(x) \\ & + e_0(x-y,y)^{-n/2}\nu_\ell(x)(x_\eta - y_\eta) g_{s\eta}(x) \\ & + e_0(x-y,y)^{-n/2}\nu_s(x)(x_\theta - y_\theta) g_{\ell\theta}(x) \\ & + \{\text{less singular terms}\},\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\nu_j(x) g^{jk}(x) \partial_{x_\ell}\Gamma_{ks}(x,y) = & -e_0(x-y,y)^{-n/2}\nu_s(x)(x_\tau - y_\tau) g_{\ell\tau}(x) \\ & -\frac{n}{2}\, e_0(x-y,y)^{-(n+2)/2}\nu_j(x)(x_j-y_j)(x_\theta-y_\theta)(x_\eta-y_\eta) g_{\ell\theta}(x) g_{s\eta}(x) \\ & + e_0(x-y,y)^{-n/2}\nu_\ell(x)(x_\eta - y_\eta) g_{s\eta}(x) \\ & + e_0(x-y,y)^{-n/2}\nu_j(x)(x_j - y_j) g_{s\ell}(x) \\ & + \{\text{less singular terms}\}.\end{aligned} \tag{3.14}$$

Also,

$$-\nu_\ell(x)\Theta_s(x,y) = -2\,e_0(x-y,y)^{-n/2}\nu_\ell(x)(x_\tau-y_\tau)g_{s\tau}(x) + \{\text{less singular terms}\}. \tag{3.15}$$

Note that the first term in the right side of (3.15) cancels the third term in the right side of (3.13) along with the third term in the right side of (3.14). Also, the fourth term in the right side of (3.13) cancels the first term in the right side of (3.14). Finally, the first term in the right side of (3.13) cancels the fourth term in the right side of (3.14). Thus, on account of these cancellations, the main singularity in (3.12) is given by

$$-n\,e_0(x-y,y)^{-(n+2)/2}\nu_j(x)(x_j-y_j)(x_\theta-y_\theta)(x_\eta-y_\eta)g_{\ell\theta}(x)g_{s\eta}(x). \tag{3.16}$$

Let us now denote by $\nu^{\rm e}=(\nu^{\rm e}_j)_j$ the *Euclidean* normal to $\partial\Omega$. In particular, ν, the unit conormal to $\partial\Omega$ with respect to the metric g_{jk} is related to $\nu^{\rm e}$ by

$$\nu_j(x) = \big(g^{\ell k}(x)\nu^{\rm e}_\ell(x)\nu^{\rm e}_k(x)\big)^{-1/2}\nu^{\rm e}_j(x), \qquad 1\leq j\leq n. \tag{3.17}$$

Due to the presence of the factor $\langle\nu^{\rm e}(x),x-y\rangle$ (Euclidean inner product) in (3.16), the desired conclusion, i.e. (3.8), follows much as in the case of the harmonic double layer potential. See also [59, Proposition 11.2, Vol. II, p. 36].

Next, let Δ_0 stand for the Laplace-Beltrami operator on M and denote by G_0 the inverse of $-\Delta_0 : \{u\in L^2_1(M);\,\langle u,1\rangle=0\}\to\{u\in L^2_{-1}(M);\,\langle u,1\rangle=0\}$. If we now introduce

$$\Upsilon := -\delta GL(\Phi L-I)dG_0 \in OPS^0_{\rm cl}(\mathbb{R},\mathbb{R}), \tag{3.18}$$

then, so we claim,

$$\Psi L = \Upsilon\delta. \tag{3.19}$$

To justify the claim, first note that

$$\Psi L = -\delta G(L\Phi-I)L = -\delta GL(\Phi L-I). \tag{3.20}$$

Also, as in (3.10) of [46], $L\Phi-I=P_d(L\Phi-I)$ thus, by adjunction,

$$\Phi L-I=(\Phi L-I)P_d=(\Phi L-I)d\delta G=(\Phi L-I)dG_0\delta. \tag{3.21}$$

Returning with this in (3.20), we finally get (3.19), on account of (3.18).

If now denote by $\Xi(x,y)$ the Schwartz kernel of Υ^t (i.e. the transposed of (3.18)) then, at the level of Schwartz kernels, (3.19) translates into

$$L_x \Theta^t(y,x) = d_x \Xi(x,y). \tag{3.22}$$

Let us also recall from (2.17) that

$$L_x \Gamma(x,y) = -d_x \Theta(x,y) + \mathrm{Dirac}_y(x). \tag{3.23}$$

By adjunction, this further gives

$$L_y \Gamma(x,y) = -d_y \Theta^t(y,x) + \mathrm{Dirac}_y(x). \tag{3.24}$$

We are now ready to compute $L\mathcal{D}f$. Specifically, if we introduce

$$\mathcal{P}f(x) := \int_{\partial\Omega} \big\langle -2\big[\mathrm{Def}_y(\Theta(x,\cdot))\nu\big](y) - \Xi(x,y)\nu(y)\,,\, f(y)\big\rangle \, d\sigma_y, \quad x \in \Omega, \tag{3.25}$$

then from (2.17) and (3.22) we have

$$\begin{aligned} L_x \mathcal{D}f(x) &:= \int_{\partial\Omega} \big\langle -2\big[\mathrm{Def}_y(L_x\Gamma(x,\cdot))\nu\big](y) + L_x\Theta^t(y,x)\nu(y), f(y)\big\rangle \, d\sigma_y \\ &= -d_x \left[\int_{\partial\Omega} \big\langle -2\big[\mathrm{Def}_y(\Theta(x,\cdot))\nu\big](y) - \Xi(x,y)\nu(y)\,,\, f(y)\big\rangle \, d\sigma_y\right] \\ &= -d_x \mathcal{P}f(x). \end{aligned} \tag{3.26}$$

Going further, we claim that

$$\delta_x \Theta^t(y,x) = -\mathrm{Dirac}_y(x). \tag{3.27}$$

To see this, note that $\Phi d = \tilde{L}^{-1} P_d^{\perp} d = 0$ so that, further, $\Psi d = -\delta G(L\Phi - I)d = \delta d G_0 = -I$, modulo constants. Re-writing this last identity at the level of Schwartz kernels yields (3.27). The bottom line is that for every $f : \partial\Omega \to \Lambda^1 TM|_{\partial\Omega}$,

$$u := \mathcal{D}f, \;\; \pi := \mathcal{P}f \Longrightarrow Lu + d\pi = 0, \;\; \delta u = 0 \text{ in } \Omega. \tag{3.28}$$

Proposition 3.2. *Let Ω be a Lipschitz domain and assume that $\{\tau_j\}_j$ is a frame for the tangent bundle $T\partial\Omega \subset TM$. Then there exist singular integral operators $\mathcal{R}_j, \mathcal{R}_0$, $1 \le j \le n-1$, of Calderón-Zygmund type with the property that*

$$\nabla \mathcal{D}f = \mathcal{R}_j(\nabla_{\tau_j} f) + \mathcal{R}_0 f, \tag{3.29}$$

for each $f \in L^p_1(\partial\Omega, \Lambda^1 TM)$.

An explanation is in order here. The class of Calderón-Zygmund kernels referred to above consists of Schwartz kernels of certain pseudodifferential operators. More specifically, if $p(x,\xi)$, $q(\xi,x) \in S^{-1}_{\rm cl}$ have principal symbols that are odd in ξ, then we take the Schwartz kernels of $p(x,D) \in OPS^{-1}_{\rm cl}$ and $q(D,x) \in \emptyset{\rm P}S^{-1}_{\rm cl}$ to be in that class. We may also want to include weakly singular kernels, in order to accommodate residual terms. See also [42] for a discussion.

Proof. In local coordinates, the Radon-Nikodym derivative $\rho := d\sigma/d\sigma^{\rm e}$, of the area element on $\partial\Omega$ inherited from the Riemannian metric with respect to that inherited from the standard Euclidean metric, is given by the formula $\rho(x) = \sqrt{g(x)}(g^{jk}(x)\nu^{\rm e}_j(x)\nu^{\rm e}_k(x))^{1/2}$. If we now recall that $\nu^{\rm e} = (\nu^{\rm e}_j)_j$ denotes the Euclidean normal to $\partial\Omega$, it follows that $\nu_j(x) = \sqrt{g(x)}\rho(x)^{-1}\nu^{\rm e}_j(x)$.

Assume next that, in local coordinates, the first-order differential operator Def is given by

$$({\rm Def}\, u)^\alpha = a^{\alpha\beta}_j \partial_j u^\beta + b^{\alpha\beta} u^\beta, \tag{3.30}$$

where the summation convention is used. In particular, working in local coordinates and with orthonormal frames we have

$$Lu = 2\,{\rm Def}^*{\rm Def}\, u = 2\left(a^{\mu\beta}_j a^{\mu\gamma}_k \partial_{x_j}\partial_{x_k} u^\beta\right)^\gamma + \{\text{lower order terms}\} \tag{3.31}$$

and, via a straightforward calculation,

$$({\rm Def}\, u)\nu = \sigma({\rm Def}^*;\nu){\rm Def}\, u = \left(a^{\mu\beta}_j a^{\mu\gamma}_k \nu_k \partial_{x_j} u^\beta\right)^\gamma + \{\text{lower order terms}\}. \tag{3.32}$$

Identifying one-differential forms with vector fields (via the metric tensor), one can regard $\Gamma(x,y)$ as a matrix with entries $\Gamma_{\alpha\beta}(x,y)$. Similarly, $\Theta(x,y)$ is identified with $(\Theta_\alpha(x,y))_\alpha$. For an arbitrary, fixed $f \in L^p_1(\partial\Omega, \Lambda^1 TM)$ it follows that the α-component of $\mathcal{D}f$ is given by

$$\begin{aligned} &-\int_{\partial\Omega} 2\, a^{\mu\beta}_j(y) a^{\mu\gamma}_k(y)\nu^{\rm e}_k(y)\left[\partial_{y_j}\Gamma_{\alpha\beta}(x,y)\right] f_\gamma(y)\sqrt{g(y)}\, d\sigma^{\rm e}(y) \\ &+\int_{\partial\Omega} \Theta_\alpha(y,x)\nu^{\rm e}_\gamma(y) f_\gamma(y)\sqrt{g(y)}\, d\sigma^{\rm e}(y) + \{\text{lower order terms}\} \\ &=: I + II + \{\text{lower order terms}\}. \end{aligned} \tag{3.33}$$

Before we proceed with the main arguments, we make an important observation to the effect that, for any $1 < p < \infty$ and $j \in \{1,...,n\}$,

$$\begin{aligned} &\nabla_x \left(\partial_{y_j}\Gamma(x,y) + \partial_{x_j}\Gamma(x,y)\right) \quad \text{and} \quad \nabla_y \left(\partial_{y_j}\Gamma(x,y) + \partial_{x_j}\Gamma(x,y)\right) \\ &\quad \text{are Calderón-Zygmund kernels.} \end{aligned} \tag{3.34}$$

Indeed, take for instance the first expression in (3.34). With $[\cdot,\cdot]$ standing for the usual commutator bracket, this is the Schwartz kernel of $\nabla_x\big[\Phi,\partial_{x_j}\big]$. Now, if $\big\{p_1,p_2\big\} := \partial_{\xi_j}p_1\partial_{x_j}p_2 - \partial_{x_j}p_1\partial_{\xi_j}p_2$ denotes the Poisson bracket, then the principal symbol of $\big[\Phi,\partial_{x_j}\big]\in OPS^{-2}_{\rm cl}$ is (cf., e.g., [59], Vol. II, pp. 13)

$$i\,\big\{\xi_j,\sigma(\Phi;x,\xi)\big\} = i\partial_{x_j}\sigma(\Phi;x,\xi). \tag{3.35}$$

Since $\partial_{x_j}\sigma(\Phi;x,\xi)\in S^{-2}_{\rm cl}$ is even (cf. (2.18)), the claim (3.34) follows. Similarly, since $\partial_{x_j}\sigma(\Psi;x,\xi)\in S^{-1}_{\rm cl}$ is odd (cf. (2.18)),

$$\partial_{x_j}\Theta(x,y)+\partial_{y_j}\Theta(x,y)\quad\text{is a Calderón-Zygmund kernel.} \tag{3.36}$$

Turning now to the analysis of $\nabla\mathcal{D}f$, we need to consider the effect of applying ∂_{x_s}, $1\le s\le n$ to I and II in (3.33). First, when ∂_{x_s} hits the lower order terms, the highest singularity comes from terms of the form $\partial_{x_s}\Gamma_{\alpha\beta}(x,y)$. By (3.9), these kernels are of Calderón-Zygmund type.

Second, there is the case when ∂_{x_s} hits I or II in (3.33). The main singularities in $\partial_{x_s}I$ are contained in terms of the form $\partial_{x_s}\partial_{y_j}\Gamma_{\alpha\beta}(x,y)$. In the sequel, we find it convenient to replace these by $-\partial_{y_s}\partial_{y_j}\Gamma_{\alpha\beta}(x,y)$. By (3.34), this can be arranged modulo Calderón-Zygmund kernels which, once again, suits our purposes. Next, for each fixed s, we write

$$\begin{aligned}
&a^{\mu\beta}_j(y)a^{\mu\gamma}_k(y)\nu^{\rm e}_k(y)\partial_{y_s}\partial_{y_j}\Gamma_{\alpha\beta}(x,y)f_\gamma(y)\sqrt{g(y)}\\
&\quad= a^{\mu\beta}_j(y)a^{\mu\gamma}_k(y)\Big\{\big[\nu^{\rm e}_k(y)\partial_{y_s}-\nu^{\rm e}_s(y)\partial_{y_k}\big]\,\partial_{y_j}\Gamma_{\alpha\beta}(x,y)\Big\}f_\gamma(y)\sqrt{g(y)}\\
&\qquad+a^{\mu\beta}_j(y)a^{\mu\gamma}_k(y)\nu^{\rm e}_s(y)\partial_{y_k}\partial_{y_j}\Gamma_{\alpha\beta}(x,y)f_\gamma(y)\sqrt{g(y)}\\
&\quad=: III+IV.
\end{aligned} \tag{3.37}$$

Observe that III is a linear span of tangential derivatives. Hence, when III is integrated against $\int_{\partial\Omega}d\sigma^{\rm e}$, this tangential derivative can be passed on to the other factors. The resulting terms obviously have the right form.

There remains IV. In order to treat this term, we shall use the PDE satisfied by Γ and Θ. The incisive observation is that

$$\begin{aligned}
IV &= \big(L_y\Gamma(x,y)\big)_{\alpha\gamma}\nu^{\rm e}_s(y)f_\gamma(y)\sqrt{g(y)}+\{\text{residual terms}\}\\
&= \big(-d_y\Theta^t(y,x)\big)_{\alpha\gamma}\nu^{\rm e}_s(y)f_\gamma(y)\sqrt{g(y)}+\{\text{residual terms}\}\\
&= \partial_{y_\gamma}\Theta_\alpha(y,x)\nu^{\rm e}_s(y)f_\gamma(y)\sqrt{g(y)}+\{\text{residual terms}\}.
\end{aligned} \tag{3.38}$$

The source of main singularities in the residual terms are of the type $\nabla_y\Gamma(x,y)$, hence acceptable. Also, in the last (non-residual) term in (3.38), we can replace $\nu^{\rm e}_s(y)\partial_{y_\gamma}$ with $\nu^{\rm e}_\gamma(y)\partial_{y_s}$ at the expense of picking up a tangential derivative. When integrating this expression on $\partial\Omega$, this tangential derivative is ultimately passed on to f (via an integration by parts), and the integral operator arising in this fashion is of Calderón-Zygmund type. Finally, we may replace $\partial_{y_s}\Theta_\alpha(x,y)$ by $-\partial_{x_s}\Theta_\alpha(x,y)$ at the expense of picking up further Calderón-Zygmund kernels (cf. (3.36)), and this cancels precisely the main singularity in $\partial_{x_s}II$ (recall that II has been introduced in (3.33)). This takes care of IV (as well as of II) and concludes the proof of the proposition.

The identity (3.29) is instrumental in establishing mapping properties for the double layer potential on Sobolev-Besov spaces.

Proposition 3.3. *Assume that $\Omega \subset M$ is a Lipschitz domain. Then for each $1 < p < \infty$, $0 < s < 1$, the operator*

$$\mathcal{D} : B^{p,p}_s(\partial\Omega, \Lambda^1 TM) \longrightarrow L^p_{s+1/p}(\Omega, \Lambda^1 TM) \cap B^{p,p}_{s+1/p}(\Omega, \Lambda^1 TM) \tag{3.39}$$

is well-defined and bounded.

Proof. For each $0 < s < 1$, we claim that the estimates

$$\|\mathrm{dist}\,(\cdot, \partial\Omega)^{-s} |\nabla \mathcal{D} f|\,\|_{L^1(\Omega)} + \|\mathcal{D} f\|_{L^1(\Omega)} \leq C\|f\|_{B^{1,1}_s(\partial\Omega, TM)} \tag{3.40}$$

and

$$\|\mathrm{dist}\,(\cdot, \partial\Omega)^{1-s} |\nabla \mathcal{D} f|\,\|_{L^\infty(\Omega)} + \|\mathcal{D} f\|_{L^\infty(\Omega)} \leq C\|f\|_{B^{\infty,\infty}_s(\partial\Omega, TM)}, \tag{3.41}$$

hold uniformly for $f \in B^{1,1}_s(\partial\Omega, TM)$ and $f \in B^{\infty,\infty}_s(\partial\Omega, TM)$, respectively. Indeed, with Proposition 3.2 at hand, these are proved much as in the case of Cauchy type operators treated in §3 of [41].

Note that (3.40), (3.41) and Stein's interpolation theorem for analytic families of operators give that for $1 \leq p \leq \infty$, $0 < s < 1$,

$$\|\mathrm{dist}\,(\cdot, \partial\Omega)^{1-s-1/p} |\nabla \mathcal{D} f|\,\|_{L^p(\Omega)} + \|\mathcal{D} f\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega, TM)}, \tag{3.42}$$

uniformly in $f \in B^{p,p}_s(\partial\Omega, TM)$. Now, this already leads to the desired conclusion when $s + 1/p \leq 1$, thanks to real-variable characterizations of the Sobolev-Besov spaces in [34]. In the case when $s + 1/p > 1$, we first invoke standard interior estimates to obtain from (3.42) that

$$\|\mathrm{dist}\,(\cdot, \partial\Omega)^{2-s-1/p} |\nabla^2 \mathcal{D} f|\,\|_{L^p(\Omega)} + \|\nabla \mathcal{D} f\|_{L^p(\Omega)} \leq C\|f\|_{B^{p,p}_s(\partial\Omega, TM)}, \tag{3.43}$$

and then proceed as before.

Next, we consider similar issues for the operator $\mathcal{P}$, introduced in (3.25).

Proposition 3.4. *Let Ω be a Lipschitz domain. Then, for each $1 < p < \infty$, $0 < s < 1$, the operator*

$$\mathcal{P} : B_s^{p,p}(\partial\Omega, \Lambda^1 TM) \longrightarrow L^p_{s+1/p-1}(\Omega, \Lambda^1 TM) \cap B^{p,p}_{s+1/p-1}(\Omega, \Lambda^1 TM) \tag{3.44}$$

is well-defined and bounded.

Proof. For starters, we note that (3.43) and (3.26) give

$$\|\mathrm{dist}\,(\cdot, \partial\Omega)^{2-s-1/p} |\nabla \mathcal{P} f|\,\|_{L^p(\Omega)} \leq C(\Omega, p, s)\|f\|_{B_s^{p,p}(\partial\Omega, TM)}, \tag{3.45}$$

which, much as in the proof of Proposition 3.3, establishes the claim made about (3.44) when $s + 1/p > 1$.

As for the remaining set of indices s, p, we proceed as follows. First we note that the top singularity in the (vector-valued) integral kernel of $\mathcal{P}$ is

$$C_n \left(g^{jk}(y)\nu_j(y)\partial_{y_s}\partial_{y_k}\left(e_0(x-y,y)\right)^{-(n-2)/2}\right)_s. \tag{3.46}$$

This can be justified via a direct calculation based on the explicit definition of $\mathcal{P}$. Alternatively, one may use (3.16), the identity $L\mathcal{D} = -d\mathcal{P}$, along with Weitzenbock's formula (2.2), to the effect that locally, $L = g^{jk}\partial_j\partial_k +$ {lower order terms}.

Next, observe that $\nu_j(y)\partial_{y_s} - \nu_s(y)\partial_{y_j}$ is a tangential derivative, and that

$$g^{jk}(y)\partial_{x_j}\partial_{x_k}\Big(e_0(x-y,y)\Big)^{-(n-2)/2} = 0. \tag{3.47}$$

Based on these remarks and (3.46), we may produce in this context an identity which is similar in spirit to (3.29). Specifically, there exists a family of operators $\mathcal{Q}_j$ whose integral kernels are Schwartz kernels of operators in OPS^{-1}_{cl}, or $\emptyset PS^{-1}_{\mathrm{cl}}$ (and which have odd principal symbols), such that

$$\mathcal{P}f = \mathcal{Q}_j(\nabla_{\tau_j} f) + \{\text{lower order terms}\}, \tag{3.48}$$

for each function $f \in L^p_1(\partial\Omega, \Lambda^1 TM)$ (where, as in (3.29), $\{\tau_j\}_j$ is a fixed frame for $T\partial\Omega$). From this and (3.50) in Proposition 3.3 of [46], it follows that the operator (3.44) is well-defined and bounded whenever $0 < s < 1/p < 1$.

With this at hand, and granted what we have already established in the first part of the proof, the claim about (3.44) for the full range of indices $1 < p < \infty$, $0 < s < 1$, follows via (complex) interpolation.

Next we discuss the following.

Proposition 3.5. *Let Ω be a Lipschitz subdomain of M. For each $1 < p < \infty$ and $0 < s < 1$ the assumption that $\partial\Omega$ has a small Lipschitz constant (depending on s, p) implies that*

$$\begin{aligned}&\pm\tfrac{1}{2}I + K \text{ are Fredholm with index zero on} \\ &L^p_s(\partial\Omega, \Lambda^1 TM) \text{ and on } B^{p,p}_s(\partial\Omega, \Lambda^1 TM).\end{aligned} \tag{3.49}$$

In particular, this is the case for C^1 domains.

Proof. When $\partial\Omega$ is only Lipschitz, then

$$K \text{ is bounded on } L^p(\partial\Omega, \Lambda^1 TM) \text{ and on } L^p_1(\partial\Omega, \Lambda^1 TM). \tag{3.50}$$

with operator norms bounded by constants which depend exclusively on p and the Lipschitz character of Ω. This follows from (3.2), (3.9), (3.11) and the general discussion contained in §§1-2 of [40].

Let us first deal with the case $\partial\Omega \in C^1$, when a stronger conclusion holds, namely that

$$K \text{ is compact on } L^p(\partial\Omega, \Lambda^1 TM) \text{ and on } L^p_1(\partial\Omega, \Lambda^1 TM) \tag{3.51}$$

for each $1 < p < \infty$, $0 < s < 1$. This clearly implies (3.49). For starters, the fact that K is compact when acting on $L^p(\partial\Omega, \Lambda^1 TM)$ follows from the structure of (3.16). See [33], whose main results extend to variable coefficient kernels, much as in §1 of [42].

Next, let $\Omega_j \nearrow \Omega$ be a nested sequence of smooth domains, suitably approximating the C^1 domain Ω. Among other things, we assume that the Lipschitz constants of Ω_j are bounded, and that there exist homeomorphisms $\Lambda_j : \partial\Omega_j \to \partial\Omega$ which are bi-Lipschitz, with constants bounded uniformly in j. Denote by K_j the operator constructed similarly to (3.2) in connection with $\partial\Omega_j$ and which is further identified, via the change of variables mappings Λ_j, with an operator acting on $\partial\Omega$. Then, with $\mathcal{L}(X, Y)$ standing for the (normed) space of all linear bounded operators from X into Y and with $\mathcal{L}(X)$ abbreviating $\mathcal{L}(X, X)$, we have

$$\sup \|K_j\|_{\mathcal{L}(L^p_1(\partial\Omega, TM))} < +\infty, \tag{3.52}$$

and

$$K_j \longrightarrow K \text{ strongly in } L^p(\partial\Omega, \Lambda^1 TM). \tag{3.53}$$

See [33] and [10] for details in similar circumstances. For each $1 < p < \infty$, $0 < s < 1$, standard interpolation inequalities also give

$$\begin{aligned}&\max\left\{\|K - K_j\|_{\mathcal{L}(L^p_s(\partial\Omega, TM))}, \|K - K_j\|_{\mathcal{L}(B^{p,p}_s(\partial\Omega, TM))}\right\} \\ &\quad \leq \|K - K_j\|^{1-s}_{\mathcal{L}(L^p(\partial\Omega, TM))} \|K - K_j\|^s_{\mathcal{L}(L^p_1(\partial\Omega, TM))}.\end{aligned} \tag{3.54}$$

Since, by Proposition 3.1, each K_j is smoothing, hence compact on any Sobolev space, it follows from (3.52)-(3.53) that (3.51) holds.

The case when $\partial\Omega$ only has a small Lipschitz constant is treated as follows. First, for each $1<p<\infty$, $0<s<1$, interpolation inequalities give

$$\max\left\{\|K\|_{\mathcal{L}(L^p_s(\partial\Omega,TM))}\,,\,\|K\|_{\mathcal{L}(B^{p,p}_s(\partial\Omega,TM))}\right\} \\ \leq \|K\|^{1-s}_{\mathcal{L}(L^p(\partial\Omega,TM))}\|K\|^{s}_{\mathcal{L}(L^p_1(\partial\Omega,TM))}. \tag{3.55}$$

Next, work in local coordinates (in which $\partial\Omega$ is a portion of the graph of $\varphi:\mathbb{R}^{n-1}\to\mathbb{R}$) and recall from the proof of Proposition 3.1 that the main singularity in (3.12) is given by (3.16). In concert with Theorems 1.3, 1.10 and (1.11) in [33] (results which extend to variable coefficient kernels, much as in §1 of [42]), this shows that

$$\max\left\{\|K\|_{\mathcal{L}(L^p_s(\Sigma,TM))}\,,\,\|K\|_{\mathcal{L}(B^{p,p}_s(\Sigma,TM))}\right\}\leq C(s,p,\|\nabla\varphi\|_{BMO}), \tag{3.56}$$

for each relatively compact subset Σ of the hypersurface $\{(x',\varphi(x'));\,x'\in\mathbb{R}^{n-1}\}$, and where $C(s,p,\|\nabla\varphi\|_{BMO})\to 0$ as $\|\nabla\varphi\|_{BMO}\to 0$.

In particular, given $\lambda\in\mathbb{R}\setminus 0$, it follows that $\lambda I+K:L^p_s(\Sigma,TM)\to L^p_s(\Sigma,TM)$ and $\lambda I+K:B^{p,p}_s(\Sigma,TM)\to B^{p,p}_s(\Sigma,TM)$ are invertible (via Neumann series) if $\|\nabla\varphi\|_{BMO}$ is small enough. With this at hand, a variant of the the localization technique from §10 in [40] yields that, given $\lambda\in\mathbb{R}\setminus 0$, the operator $\lambda I+K$ is bounded from below, modulo compact operators, both on $L^p_s(\partial\Omega,TM)$ and on $B^{p,p}_s(\partial\Omega,TM)$, granted that $\partial\Omega$ has a small Lipschitz constant (relative to $1<p<\infty$, $0<s<1$, and λ). In particular, by the homotopic invariance of the index, we may conclude that $\pm\frac{1}{2}I+K$ are Fredholm with index zero both on $L^p_s(\partial\Omega,TM)$ and on $B^{p,p}_s(\partial\Omega,TM)$, for $1<p<\infty$, $0<s<1$, provided that $\partial\Omega$ has a small enough Lipschitz constant. This justifies (3.49) and finishes the proof of the proposition.

4. The lower dimensional case

In this section we continue to deal with the same issues as above but in dimensions 2 and 3. In this particular context, the subtle interplay between the dimension of the ambient space and the exponents (integrability and smoothness) used to describe the spaces from which boundary data are selected allows for the sharpest form of our results. Most importantly, all our theorems in this section are proved on *arbitrary* Lipschitz domains (in particular, no smallness assumption on the Lipschitz constant −as in Proposition 3.5− is made). The case $n=2$, considered in detail below, is also special due to the fact that the main singularity displayed by the kernel Γ has logarithmic nature.

Our main results in this section deal with the invertibility of the single layer operator (2.21) on Besov spaces in Lipschitz domains of dimension ≤ 3. More specifically, we have:

Theorem 4.1. *Assume that* $\dim M = 2$ *and let* $\Omega \subset M$ *be a fixed, connected Lipschitz domain. Also, recall the critical Dirichlet exponent* $p_\Omega \in [1,2)$ *from (1.5). Then the operator*

$$S : B^{p,p}_{-s}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \longrightarrow B^{p,p}_{1-s}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \tag{4.1}$$

is an isomorphism for all pairs of indices $(s, 1/p) \in (0,1) \times (0,1)$ *satisfying one of the following three conditions:*

$$\begin{array}{rc} (I): & 1 - \frac{1}{p_\Omega} \leq \frac{1}{p} \leq \frac{1}{p_\Omega}, \\ (II): & \frac{1}{p_\Omega} \leq \frac{1}{p} < s + \frac{1}{p_\Omega}, \\ (III): & s - \frac{1}{p_\Omega} < \frac{1}{p} < 1 - \frac{1}{p_\Omega}. \end{array} \tag{4.2}$$

A key ingredient in the proof of the above result is an adaptation of the Hölder estimate established in [52] to the two dimensional, variable coefficient setting. Specifically, we shall prove the following.

Proposition 4.2. *Retain the same hypotheses as in the above theorem. Then for each* $0 < \alpha < \frac{1}{p_\Omega}$ *there exists a finite constant* $C = C(\partial\Omega, \alpha) > 0$ *such that*

$$\sup_{x\in\Omega} \Big[\mathrm{dist}(x, \partial\Omega)^{1-\alpha} |\nabla u(x)| \Big] \leq C \|u\|_{C^\alpha(\partial\Omega, \Lambda^1 TM)}, \tag{4.3}$$

uniformly for all null-solutions (u, π) *of (1.2).*

In proving Proposition 4.2, our approach closely parallels that in [52], where the case $n = 3$ has been treated. The main differences are: (i) the involvement of the L^2 theory from [46], in the variable coefficient case, (ii) overcoming certain technical difficulties inherent to the case $n = 2$ (e.g., in this case, the fundamental solution exhibits a different behavior at infinity and certain Sobolev embeddings cease to hold, and (iii) keeping a more careful track of the range of α's for which (4.3) is valid (as far as this range is concerned, the two-dimensional case turns out to be different than the three-dimensional one). Here we would like acknowledge once and for all the strong influence that [9] and [52] have played on our main results in this section.

Turning to details, we first note that, in the case $n = 2$, the analogues of (3.9) and (3.11) are

$$\begin{aligned} \Gamma_{rs}(x,y) = \frac{1}{4\pi} \Big\{ & -g_{rs}(x) \log \big[e_0(x-y, y)^{1/2} \big] \\ & + e_0(x-y,y)^{-1/2} (x_\theta - y_\theta)(x_\eta - y_\eta) g_{r\theta}(x) g_{s\eta}(x) \Big\} \\ & + \{\text{less singular terms}\}, \end{aligned} \tag{4.4}$$

and

$$\Theta_s(x,y) = \frac{1}{2\pi} e_0(x-y,y)^{-1}(x_\tau - y_\tau) g_{s\tau}(x) \\ +\{\text{less singular terms}\}, \tag{4.5}$$

respectively.

Fix a Lipschitz domain $\Omega \subset M$. For $x \in \Omega$, denote by x^* the 'reflection' of x across $\partial\Omega$; i.e. a point $x^* \in M \backslash \bar{\Omega}$ such that $\operatorname{dist}(x^*, \partial\Omega) \approx \operatorname{dist}(x, \partial\Omega)$. In particular,

$$\operatorname{dist}(x^*, y) \approx \operatorname{dist}(x, y), \quad \text{uniformly for} \quad y \in \partial\Omega. \tag{4.6}$$

In the sequel, we shall find it convenient to work with a perturbed version of (4.4), i.e.

$$\tilde{\Gamma}(x,y) := \Gamma(x,y) - \Gamma(x^*, y), \qquad x, y \in \Omega. \tag{4.7}$$

A different normalization has been considered in Appendix A of [6]. The idea is that $\tilde{\Gamma}(x,y)$ exhibits better decay properties than $\Gamma(x,y)$; this is made precise in the lemma below. Hereafter, $B_r(x)$ will denote the (geodesic) ball of radius $r > 0$ centered at $x \in M$.

Lemma 4.3. *There exists a constant $C = C(\partial\Omega) > 0$ such that the following hold. For any point $x_0 \in \partial\Omega$ and any $r > 0$,*

$$|\tilde{\Gamma}(x,y)| \leq C, \quad \textit{uniformly for} \\ y \in \Omega \cap \partial B_{100r}(x_0) \ \textit{and} \ x \in \Omega \cap B_{2r}(x_0), \tag{4.8}$$

and

$$|\tilde{\Gamma}(x,y)| \leq C \frac{r}{r + |x_0 - y|}, \quad \textit{uniformly for} \\ x \in \Omega \cap \partial B_r(x_0), \ \operatorname{dist}(x, \partial\Omega) \geq r/2, \ y \in \partial\Omega. \tag{4.9}$$

Furthermore,

$$|\nabla_I \tilde{\Gamma}(x,y)| + |\nabla_{II} \tilde{\Gamma}(x,y)| \leq C \frac{1}{|x-y|}, \quad \textit{uniformly for} \ \ x \in \Omega, \ y \in \partial\Omega, \tag{4.10}$$

where ∇_I, ∇_{II} denote the gradients with respect to the first and second variable, respectively.

Proof. The first inequality follows from

$$|\tilde{\Gamma}(x,y)| \le |x-x^*| \sup_{\xi\in[x,x^*]} \frac{1}{|\xi-y|} \le Cr\,\frac{1}{r} = C. \qquad (4.11)$$

As for the second, let $\hat{x} \in \partial\Omega$ be such that $\mathrm{dist}(x,\partial\Omega) = |x-\hat{x}|$ and assume that $y \in \partial\Omega$ lies to the right of $\hat{x}$. Choose a system of coordinates with $\hat{x}$ at its origin and such that the segment $[x,\hat{x}]$ is vertical. Let $\zeta := \gamma(t) = (r\cos t, r\sin t)$, $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$, a path joining x and x^*. Then

$$\tilde{\Gamma}(x,y) = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{d}{dt}[\Gamma(\gamma(t),y)]\,dt = -\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \langle(\nabla_I\Gamma)(\zeta,y),\dot{\gamma}(t)\rangle\,dt, \quad (4.12)$$

so that

$$|\tilde{\Gamma}(x,y)| \le C\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{r}{|\zeta-y|}\,dt. \qquad (4.13)$$

Now $|\zeta-y| \ge C|x_0-y|$ and $|\zeta-y| \ge Cr$. In particular, $|\zeta-y| \ge C(r+|x_0-y|)$, uniformly in $t \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. Returning with this in (4.13) yields (4.9).

Finally, (4.10) follows from (4.4) and (4.6).

Consider next the Green function

$$G(x,y) := \tilde{\Gamma}(x,y) - w^x(y), \qquad x,y \in \Omega, \qquad (4.14)$$

where

$$\begin{aligned} &Lw^x + dq^x = 0 \text{ in } \Omega,\\ &\mathcal{N}(w^x) \in L^2(\partial\Omega),\\ &\delta w^x = 0 \text{ in } \Omega,\\ &w^x\,|_{\partial\Omega} = \tilde{\Gamma}(x,\cdot)\,|_{\partial\Omega} \text{ on } \partial\Omega. \end{aligned} \qquad (4.15)$$

Here, $\mathcal{N}$ stands for the nontangential maximal operator, i.e.

$$\mathcal{N}w(x) := \sup\{|w(y)|;\, y \in \gamma(x)\}, \quad x \in \partial\Omega, \qquad (4.16)$$

where $\gamma(x) \subseteq \Omega$ is a suitable nontangential approach region; see [40]. Also, $\cdot|_{\partial\Omega}$ denotes the nontangential boundary trace operator. That is,

$$w\big|_{\partial\Omega}(x) := \lim_{y\in\gamma(x)} w(y), \quad x \in \partial\Omega. \qquad (4.17)$$

Finally, set

$$\pi^x(y) := \Theta(x,y) - q^x(y), \qquad x \in \Omega. \qquad (4.18)$$

Generally speaking, for a pair (u,π) satisfying $Lu + dp = 0$ and $\delta u = 0$ (near the boundary of Ω) we define the conormal derivative

$$\frac{\partial u}{\partial \nu} := 2\mathrm{Def}(u)\nu - \pi\nu, \quad \text{on} \quad \partial\Omega. \tag{4.19}$$

In particular, these considerations apply to the pair $(G(x,\cdot),\pi^x)$.

As in [9], [52], the strategy for proving Proposition 4.2 relies on the following key lemma.

Lemma 4.4. *Fix $z \in \Omega$ and $0 \le \alpha < \frac{1}{p_\Omega}$. Then there exists $C > 0$ such that*

$$\int_{\partial\Omega} \left| \frac{\partial G(x,y)}{\partial \nu_y} - \pi^x(z)\nu(y) \right| |y-x_0|^\alpha \, d\sigma_y \le C|x-x_0|^\alpha, \tag{4.20}$$

uniformly for $x_0 \in \partial\Omega$ and $x \in \Omega$ such that $|x-x_0| \le 2\,\mathrm{dist}\,(x,\partial\Omega)$.

Preparatory to the proof of this result, we first establish two preliminary estimates in Lemmas 4.5 and 4.6 below.

Lemma 4.5. *Assume $x_0 \in \partial\Omega$, $x \in \Omega$, $r := |x-x_0|$ and $\mathrm{dist}(x,\partial\Omega) \ge r/2$. Then*

$$\int_{\partial\Omega\setminus B_{100r}(x_0)} |\mathcal{N}(G(x,\cdot))|^p \, d\sigma \le C\, r \tag{4.21}$$

for each $p_\Omega < p < \infty$.

Proof. Set $D := \Omega\setminus\bar{B}_{100r}(x_0)$ so that $\partial\Omega\setminus B_{100r}(x_0) \subseteq \partial D$. Consequently,

$$\int_{\partial\Omega\setminus B_{100r}(x_0)} |\mathcal{N}(G(x,\cdot))|^p \, d\sigma \le \int_{\partial D} |\mathcal{N}(G(x,\cdot))|^p \, d\sigma. \tag{4.22}$$

Since the estimate (1.3) holds for $p > p_\Omega$ in D and $G(x,\cdot)\,|_{\partial\Omega} \equiv 0$, we may write

$$\begin{aligned} \int_{\partial D} |\mathcal{N}(G(x,\cdot))|^p \, d\sigma &\le C \int_{\partial D} |G(x,\cdot)|^p \, d\sigma = C \int_{\Omega\cap\partial B_{100r}(x_0)} |G(x,\cdot)|^p \, d\sigma \\ &\le C \int_{\Omega\cap\partial B_{100r}(x_0)} |\tilde{\Gamma}(x,\cdot)|^p \, d\sigma + C \int_{\Omega\cap\partial B_{100r}(x_0)} |w^x|^p \, d\sigma \\ &=: I + II. \end{aligned} \tag{4.23}$$

Note that the integrand in I is $\le C$ on the domain of integration, by Lemma 4.3. Thus,

$$|I| \le Cr \tag{4.24}$$

which is of the right order. Next,

$$\begin{aligned}\int_{\Omega\cap\partial B_{100r}(x_0)} |w^x|^p\,d\sigma &\leq C\int_{B_{200r}(x_0)\cap\partial\Omega} |\mathcal{N}(w^x)|^p\,d\sigma \\ &\leq C\int_{\partial\Omega} |\mathcal{N}(w^x)|^p\,d\sigma \leq C\int_{\partial\Omega} |w^x|^p d\sigma \\ &= C\int_{\partial\Omega} |\tilde{\Gamma}(x,\cdot)|^p\,d\sigma \leq C\int_{\mathbb{R}} \left(\frac{r}{r+|y|}\right)^p\,dy \\ &= Cr,\end{aligned} \tag{4.25}$$

where the last inequality utilizes Lemma 4.3.

Next, for $\tau, R > 0$ let

$$D_{\tau,R}(x_0) := \left\{y \in \Omega;\ \frac{R}{\tau} < |y - x_0| < \tau R\right\}. \tag{4.26}$$

Lemma 4.6. *Retain the previous notation and definitions. Then, for each $p > p_\Omega$, there holds*

$$\frac{1}{R^2}\int_{D_{2,R}(x_0)} |G(x,\cdot)|^2\,d\,Vol \leq C_p\left(\frac{r}{R}\right)^{\frac{2}{p}}. \tag{4.27}$$

Proof. If $\frac{1}{p} + \frac{1}{p'} = 1$ we have

$$\begin{aligned}&\int_{D_{1,R}(x_0)} |G(x,\cdot)|^2\,d\mathrm{Vol} \\ &\leq \left(\int_{D_{1,R}(x_0)} |G(x,\cdot)|^p\,d\mathrm{Vol}\right)^{\frac{1}{p}} \cdot \left(\int_{D_{1,R}(x_0)} |G(x,\cdot)|^{p'}\,d\mathrm{Vol}\right)^{\frac{1}{p'}} \\ &=: I\cdot II.\end{aligned} \tag{4.28}$$

Also, by Lemma 4.5, if $R \geq 100r$, then

$$|I| \leq C\left(R\int_{\partial\Omega\setminus B_{100r}(x_0)} |\mathcal{N}(G(x,\cdot))|^p\,d\sigma\right)^{\frac{1}{p}} \leq CR^{\frac{1}{p}}\cdot r^{\frac{1}{p}} = CR^{\frac{2}{p}}\left(\frac{r}{R}\right)^{\frac{1}{p}}. \tag{4.29}$$

Now, in general,

$$\|u\|_{L^{p'}(\mathbb{R}^2)} \leq C_p\|u\|_{L^2(\mathbb{R}^2)}^{\frac{2}{p'}}\cdot\|\nabla u\|_{L^2(\mathbb{R}^2)}^{1-\frac{2}{p'}}, \quad \forall p' \in (2,\infty), \tag{4.30}$$

so that

$$II \leq C \left(\int_{D_{1,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \right)^{\frac{1}{p'}} \cdot \left(\int_{D_{1,R}(x_0)} |\nabla G(x,\cdot)|^2 \, d\mathrm{Vol} \right)^{\frac{1}{2}-\frac{1}{p'}} . \tag{4.31}$$

By the boundary Caccioppoli's inequality for the Stokes system (cf. [27, Theorem 2.2, p. 203] and [52]), the last factor above is

$$\leq C \left(\frac{1}{R^2} \int_{D_{1,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \right)^{\frac{1}{2}-\frac{1}{p'}} . \tag{4.32}$$

Consequently,

$$\begin{aligned} \int_{D_{1,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \leq C R^{-1+\frac{2}{p'}} \cdot \left(\frac{r}{R}\right)^{\frac{1}{p}} \cdot R^{\frac{2}{p}} \cdot \\ \left(\int_{D_{1,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \right)^{\frac{1}{p'}} \cdot \left(\int_{D_{2,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \right)^{\frac{1}{2}-\frac{1}{p'}} . \end{aligned} \tag{4.33}$$

Thus, all in all,

$$\begin{aligned} & \int_{D_{1,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \\ & \qquad \leq C R^{p\left(-1+\frac{2}{p'}\right)} \cdot R^2 \cdot \frac{r}{R} \cdot \left(\int_{D_{2,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol} \right)^{p\left(\frac{1}{2}-\frac{1}{p'}\right)} . \end{aligned} \tag{4.34}$$

If we now introduce

$$\Phi(R) := \frac{1}{R^2} \int_{D_{1,R}(x_0)} |G(x,\cdot)|^2 \, d\mathrm{Vol}, \tag{4.35}$$

then the above becomes

$$\Phi(R) \leq C_0 \cdot \frac{r}{R} \cdot \Phi(2r)^{1-\frac{p}{2}}. \tag{4.36}$$

Note that, also by Lemma 4.5,

$$\Phi(R) \leq C_0 \frac{r}{R}. \tag{4.37}$$

Thus, inductively,

$$\Phi(R) \leq C_k \left(\frac{r}{R}\right)^{\alpha_k}, \qquad k = 0, 1, 2, ..., \tag{4.38}$$

where

$$C_{k+1}=C_k\cdot[C_0\cdot 2^{-\alpha_k}]^{2-p} \quad \text{and} \quad \alpha_{k+1}=\alpha_k\Big(1-\frac{p}{2}\Big)+1 \tag{4.39}$$

for $k=0,1,2,\ldots$. It follows that $\alpha_k\to\alpha^*$ as $k\to\infty$, with $\alpha^*=\alpha^*\big(1-\frac{p}{2}\big)+1$, i.e. $\alpha^*=\frac{2}{p}$. As a consequence, (4.27) follows by choosing k sufficiently large.

We are now in a position to complete the

Proof of Lemma 4.4. For starters, fix $z\in\Omega$, $x_0\in\partial\Omega$, $x\in\Omega$ and set $r:=|x-x_0|$. Also, for $R,\tau>0$, recall the domain $D_{\tau,R}(x_0)$ introduced in (4.26) and fix some $x_R\in D_{2,R}(x_0)$ such that $\operatorname{dist}(x_R,\partial\Omega)\approx R$. Then, for each $\tau\in[1,2]$ fixed,

$$\begin{aligned}\int_{R\leq|y-x_0|\leq 2R}&\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right|^2 d\sigma_y\\ &\leq\int_{\partial D_{\tau,R}(x_0)}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right|^2 d\sigma_y\\ &\leq C\int_{\partial D_{\tau,R}(x_0)}|\nabla_{\tan}G(x,\cdot)|^2\,d\sigma+CR\,|\pi^x(z)-\pi^x(x_R)|^2.\end{aligned} \tag{4.40}$$

Here, the last inequality follows from [46] and rescaling.

We continue by estimating the pressure term above. To this end, let $\gamma:[0,1]\to M$ be a C^1 curve so that $\gamma(0)=x_R$, $\gamma(1)=z$ and $|\dot\gamma(t)|\approx 1$, $\operatorname{dist}(\gamma(t),\partial\Omega)\approx|\gamma(t)-x_0|\geq CR$. Matters can also be arranged so that

$$\int_0^1\frac{dt}{\operatorname{dist}(\gamma(t),\partial\Omega)^2}\leq C\,R^{-1}. \tag{4.41}$$

Then, by the Fundamental Theorem of Calculus,

$$|\pi^x(x_R)-\pi^x(z)|\leq\int_0^1|d\pi^x(\gamma(t))|\,dt\leq\int_0^1|(L\Gamma(x,\cdot))(\gamma(t))|\,dt. \tag{4.42}$$

Note that

$$[\Delta\Delta-2\,\delta d\,\mathrm{Ric}\,]\,G(x,\cdot)=0\ \text{ in }\ \Omega\setminus\{x\}, \tag{4.43}$$

and that the fourth-order differential operator in the left side is elliptic. Thus, if $y\in\Omega$ is such that $\rho(y):=\operatorname{dist}(y,\partial\Omega)\approx|y-x_0|$, standard interior estimates give, in concert with Lemma 4.5, that

$$
\begin{aligned}
|(L\Gamma(x,\cdot))(y)| &\leq C\rho(y)^{-4}\int_{B_{\rho(y)/100}(y)}|G(x,\cdot)|\,d\mathrm{Vol}\\
&\leq C\rho(y)^{-3}\int_{\rho(y)\leq|x_0-\cdot|\leq 2\rho(y)}|\mathcal{N}(G(x,\cdot))|\,d\sigma \qquad (4.44)\\
&\leq C\rho(y)^{-2}\Big(\rho(y)^{-1}\int_{\rho(y)\leq|x_0-\cdot|\leq 2\rho(y)}|\mathcal{N}(G(x,\cdot))|^p\,d\sigma\Big)^{1/p}\\
&\leq C\rho(y)^{-2}\rho(y)^{-1/p}\Big(\int_{\partial\Omega\setminus B_{100r}(x_0)}|\mathcal{N}(G(x,\cdot))|^p\,d\sigma\Big)^{1/p}\\
&\leq C\rho(y)^{-2}\Big(\frac{r}{R}\Big)^{1/p},
\end{aligned}
$$

at least for $R \geq 100r$. Invoking (4.41) this ultimately leads to

$$
|\pi^x(x_R)-\pi^x(z)| \leq CR^{-1-\frac{1}{p}}\cdot r^{\frac{1}{p}}, \qquad (4.45)
$$

for each $p > p_\Omega$. Integrating over $\tau \in \left[1, \frac{3}{2}\right]$ in (4.40) then yields

$$
\begin{aligned}
&\int_{R\leq|y-x_0|\leq 2R}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right|\,d\sigma_y\\
&\quad\leq CR^{\frac{1}{2}}\left(\int_{R\leq|y-x_0|\leq 2R}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right|^2\,d\sigma_y\right)^{\frac{1}{2}} \qquad (4.46)\\
&\quad\leq R^{\frac{1}{2}}\left(\frac{1}{R}\int_{D_{\frac{3}{2},R}(x_0)}|\nabla G(x,\cdot)|^2\,d\mathrm{Vol}+R\,|\pi^x(x_p)-\pi^x(x_0)|^2\right)^{\frac{1}{2}}\\
&\quad\leq C\left(\frac{1}{R^2}\int_{D_{2,R}(x_0)}|G(x,\cdot)|^2\,d\mathrm{Vol}+C\left(\frac{r}{R}\right)^{\frac{2}{p}}\right)^{\frac{1}{2}}\\
&\quad\leq C\left(\frac{r}{R}\right)^{\frac{1}{p}},
\end{aligned}
$$

where Caccioppoli's inequality has been used in the next-to-the-last estimate.

Shortly, we shall also need the fact that

$$
\int_{\partial\Omega\cap B_{100r}(x_0)}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right|\,d\sigma_y \leq C. \qquad (4.47)
$$

To justify this, given that $G(x,y)=\tilde{\Gamma}(x,y)$ and $\pi^x=\Theta(x,\cdot)-q^x$, matters are readily reduced -by invoking (4.5), (4.10)- to checking that

$$
\int_{\partial\Omega\cap B_{100r}(x_0)}\left|\frac{\partial w^x}{\partial\nu}-q^x(z)\nu\right|\,d\sigma_y \leq C. \qquad (4.48)
$$

However, by Cauchy-Schwarz's inequality and (4.10),

$$\begin{aligned}
&\int_{\partial\Omega\cap B_{100r}(x_0)}\left|\frac{\partial w^x}{\partial\nu}-q^x(z)\nu\right|\,d\sigma_y \\
&\leq\ Cr\left(\int_{\partial\Omega\cap B_{100r}(x_0)}\left|\frac{\partial w^x}{\partial\nu}-q^x(z)\nu\right|^2 d\sigma\right)^{1/2} \qquad (4.49)\\
&\leq\ Cr\|w^x\|_{L^2_1(\partial\Omega,\Lambda^1TM)}+Cr\|q^x\|_{L^2(\partial\Omega)}\\
&\leq\ Cr\|\tilde{\Gamma}(x,\cdot)\|_{L^2_1(\partial\Omega,\Lambda^1TM)}\leq C,
\end{aligned}$$

as desired.

Having disposed of (4.46) and (4.47), the proof of (4.20) is simple. Fix $x\in\Omega$ and $x_0\in\partial\Omega$ such that $r:=|x-x_0|\approx\mathrm{dist}\,(x,\partial\Omega)$. Next, recall that $0\leq\alpha<\frac{1}{p_\Omega}$ and choose $\alpha<\frac{1}{p}<\frac{1}{p_\Omega}$. Also, set $\epsilon:=-\alpha+\frac{1}{p}>0$, so that $\frac{1}{p}=\alpha+\epsilon$. Then, by (4.46),

$$\begin{aligned}
&\int_{R\leq|y-x_0|\leq 2R}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right||y-x_0|^\alpha\,d\sigma_y \qquad (4.50)\\
&\quad\leq C\left(\frac{r}{R}\right)^{\frac{1}{p}}R^\alpha=Cr^\alpha\cdot\left(\frac{r}{R}\right)^\epsilon,
\end{aligned}$$

uniformly for $R\geq 100r$. If we new pick $R=2^jr$, with $j=10,11,\ldots$, in (4.50) and sum up the resulting estimates, we eventually get

$$\int_{\partial\Omega\setminus B_{100r}(x_0)}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right||y-x_0|^\alpha\,d\sigma_y\leq Cr^\alpha, \qquad (4.51)$$

for each $0\leq\alpha<\frac{1}{p_\Omega}$. Now, (4.20) follows from (4.47) and the above estimate. □

We can now now focus attention on the

Proof of Proposition 4.2. Fix some $z\in\Omega$ and let $x\in\Omega$ be arbitrary. Also, let $x_0\in\partial\Omega$ be such that $|x-x_0|=\mathrm{dist}(x,\partial\Omega)$. Note that there is no loss of generality if we assume that $u(x_0)=0$. Thus, by Lemma 4.4,

$$\begin{aligned}
|u(x)|\ &=\left|\int_{\partial\Omega}\Big(\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\Big)\cdot u(y)\,d\sigma_y\right| \qquad (4.52)\\
&\leq C\|u\|_{C^\alpha(\partial\Omega,\Lambda^1TM)}\int_{\partial\Omega}\left|\frac{\partial G(x,y)}{\partial\nu_y}-\pi^x(z)\nu(y)\right||x_0-y|^\alpha\,d\sigma_y\\
&\leq C|x-x_0|^\alpha\|u\|_{C^\alpha(\partial\Omega,\Lambda^1TM)}.
\end{aligned}$$

Consequently, if $\rho:=\mathrm{dist}(x,\partial\Omega)=|x-x_0|$, then by (4.52) and interior estimates

$$|\nabla u(x)| = C\rho^{-3}\int_{B_{\rho/100}(x))}|u(y)|\,d\mathrm{Vol}_y \leq C\rho^{\alpha-1}\|u\|_{C^\alpha(\partial\Omega,\Lambda^1 TM)}, \quad (4.53)$$

as desired. □

Before turning to the proof of Theorem 4.1, we need one more preliminary result.

Lemma 4.7. *Fix $z \in \Omega$ and assume that u, π solve (1.2) and $\pi(z) = 0$. Then*

$$\Big\|\mathrm{dist}(\cdot,\partial\Omega)^{1-\alpha}\pi\Big\|_{L^\infty(\Omega)} \leq C(\Omega,\alpha)\Big\|\mathrm{dist}(\cdot,\partial\Omega)^{1-\alpha}|\nabla u|\Big\|_{L^\infty(\Omega)}, \quad (4.54)$$

for each $\alpha \in (0,1)$.

Proof. The estimate we seek is local in character so it suffices to assume that Ω is a starlike Lipschitz domain with respect to $z = 0$. Let M denote the right hand-side in (4.54).

Going further, we claim that it is enough to show that

$$\Big\|\mathrm{dist}(\cdot,\partial\Omega)^{2-\alpha}|\nabla\pi|\Big\|_{L^\infty(\Omega)} \leq CM. \quad (4.55)$$

Indeed, granted (4.55), we may then write

$$\begin{aligned} |\pi(x)| = \Big|\textstyle\int_0^1 \frac{d}{dt}[\pi(tx)]\,dt\Big| &\leq C\textstyle\int_0^1 |(\nabla\pi)(tx)||x|\,dt \quad (4.56)\\ &\leq CM\textstyle\int_0^1 \mathrm{dist}(tx,\partial\Omega)^{\alpha-2}|x|\,dt. \end{aligned}$$

Note that $\mathrm{dist}(tx,\partial\Omega) \cong \mathrm{dist}(x,\partial\Omega) + |x|(1-t)$, uniformly for $t \in [0,1]$, and that

$$\begin{aligned} \int_0^1 \frac{|x|\,dt}{[\mathrm{dist}(x,\partial\Omega)+|x|(1-t)]^{2-\alpha}}\,dt &\leq C\int_{\mathrm{dist}(x,\partial\Omega)}^\infty \frac{ds}{s^{2-\alpha}}\\ &\leq C\,\mathrm{dist}(x,\partial\Omega)^{\alpha-1}, \quad (4.57) \end{aligned}$$

after making the change of variables $s := \mathrm{dist}(x,\partial\Omega)+|x|(1-t)$. This proves the claim.

Now, the fact that u solves (4.43) plus standard interior estimates allows us to write

$$\begin{aligned} |\nabla^2 u(x)| &\leq C\,\mathrm{dist}(x,\partial\Omega)^{-1}\sup_{2|x-y|<\mathrm{dist}(x,\partial\Omega)}|\nabla u(y)| \quad (4.58)\\ &\leq C\,\mathrm{dist}(x,\partial\Omega)^{-2+\alpha}M. \end{aligned}$$

Hence, given that $|\nabla\pi| = |Lu|$ in Ω, we may write

$$\mathrm{dist}(x,\partial\Omega)^{2-\alpha}|\nabla\pi(x)| \leq C\,\mathrm{dist}(x,\partial\Omega)^{2-\alpha}|\nabla^2 u(x)| \leq CM, \tag{4.59}$$

uniformly for $x \in \Omega$. The proof of the lemma is therefore completed.

We are finally ready to present the

Proof of the Theorem 4.1. Fix $\alpha \in (0,1)$. For each $\varphi \in B^{1,1}_{1-\alpha}(\partial\Omega, \Lambda^1 TM)$, we let $\Phi \in \mathrm{Lip}_{\mathrm{loc}}(\Omega, \Lambda^1 TM) \cap W^{1,1}(\Omega, \Lambda^1 TM)$ be such that

$$\mathrm{Tr}\,\Phi = \varphi \text{ on } \partial\Omega, \quad \text{and } \mathrm{dist}(\cdot,\partial\Omega)^{\alpha-1}|\nabla\Phi| \in L^1(\Omega). \tag{4.60}$$

Such an extension operator can be constructed following [55], as in [18]. Next, recall the definition of the conormal derivative, i.e. $\frac{\partial u}{\partial\nu} := 2\mathrm{Def}(u)\nu - \pi\nu$ for any null-solution (u,π) of the Stokes system. Mimicking Green's formula, we introduce the pairing

$$\left\langle \frac{\partial u}{\partial\nu}, \varphi \right\rangle := \int_\Omega \langle \mathrm{Def}(u)\,,\, \mathrm{Def}(\Phi)\rangle\, d\mathrm{Vol} + \int_\Omega \langle \pi\,,\, \delta\Phi\rangle\, d\mathrm{Vol}, \tag{4.61}$$

where (u,π) is a null-solution of the Stokes system and φ is a one-form on $\partial\Omega$. Note that, by (4.60) and Lemma 4.7,

$$\begin{aligned}\left|\left\langle \frac{\partial u}{\partial\nu}, \varphi \right\rangle\right| \leq\; & C\|\mathrm{dist}(\cdot,\partial\Omega)^{1-\alpha}|\nabla u|\|_{L^\infty(\Omega)}\|\mathrm{dist}(\cdot,\partial\Omega)^{\alpha-1}|\nabla\Phi|\|_{L^1(\Omega)} \\ & + C\|\mathrm{dist}(\cdot,\partial\Omega)^{1-\alpha}\pi\|_{L^\infty(\Omega)}\|\mathrm{dist}(\cdot,\partial\Omega)^{\alpha-1}|\nabla\Phi|\|_{L^1(\Omega)} \\ \leq\; & C\|\mathrm{dist}(\cdot,\partial\Omega)^{1-\alpha}|\nabla u|\|_{L^\infty(\Omega)}\|\varphi\|_{B^{1,1}_{1-\alpha}(\partial\Omega,\Lambda^1 TM)}, \end{aligned} \tag{4.62}$$

assuming that the pressure function is normalized, i.e.

$$\pi(z) = 0 \quad \text{for some fixed} \quad z \in \Omega. \tag{4.63}$$

Consequently, for each $\alpha \in (0,1)$, we have that

$$\frac{\partial u}{\partial\nu} \in (B^{1,1}_{1-\alpha}(\partial\Omega, \Lambda^1 TM))^* =: B^{\infty,\infty}_{\alpha-1}(\partial\Omega, \Lambda^1 TM) \tag{4.64}$$

and the estimate

$$\left\|\frac{\partial u}{\partial\nu}\right\|_{B^{\infty,\infty}_{\alpha-1}(\partial\Omega,\Lambda^1 TM)} \leq C\|\mathrm{dist}(\cdot,\partial\Omega)^{1-\alpha}|\nabla u|\|_{L^\infty(\Omega)} \tag{4.65}$$

holds, whenever the normalization (4.63) is enforced. In particular, thanks to Proposition 4.2 we may further conclude that, for each $0 < \alpha < 1/p_\Omega$,

$$\left\|\frac{\partial u}{\partial\nu}\right\|_{B^{\infty,\infty}_{\alpha-1}(\partial\Omega,\Lambda^1 TM)} \leq C\|u\|_{C^\alpha(\partial\Omega,\Lambda^1 TM)}, \tag{4.66}$$

granted that (4.63) holds.

Let us point out that similar considerations apply to the complement domain. In fact, when applied to $u := \mathcal{S}f$ with $f \in B^{\infty,\infty}_{\alpha-1}(\partial\Omega, \Lambda^1 TM)$ considered both in $\Omega_+ := \Omega$ and in $\Omega_- := M \setminus \bar{\Omega}$, this yields, via the jump-relation (3.5),

$$\begin{aligned} &\|f\|_{B^{\infty,\infty}_{\alpha-1}(\partial\Omega,\Lambda^1 TM)} \\ \leq\ & \left\| \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega_+} \right\|_{B^{\infty,\infty}_{\alpha-1}(\partial\Omega,\Lambda^1 TM)} + \left\| \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega_-} \right\|_{B^{\infty,\infty}_{\alpha-1}(\partial\Omega,\Lambda^1 TM)} \\ \leq\ & C\|Sf\|_{B^{\infty,\infty}_{\alpha}(\partial\Omega,\Lambda^1 TM)} + \|\mathrm{Comp}(f)\|, \end{aligned} \tag{4.67}$$

for each $0 < \alpha < 1/p_\Omega$. Here Comp stands for a generic compact operator (which arises when trying to accommodate the normalization condition (4.63) for $\pi := \mathcal{Q}f$). In particular, the operator

$$S : B^{\infty,\infty}_{\alpha-1}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \longrightarrow B^{\infty,\infty}_{\alpha}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \tag{4.68}$$

has closed range if $0 < \alpha < 1/p_\Omega$. Since the latter operator is the dual of

$$S : B^{1,1}_{-\alpha}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \longrightarrow B^{1,1}_{1-\alpha}(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \tag{4.69}$$

it follows from Banach's closed range theorem that the above operator also has closed range. In fact, (4.69) is onto as well, since it has been proved in [42] that

$$S : L^p_s(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \longrightarrow L^p_s(\partial\Omega, \Lambda^1 TM)/\mathbb{R}\nu \tag{4.70}$$

is an isomorphism whenever $p_\Omega < p < (1-1/p_\Omega)^{-1}$, $0 \leq s \leq 1$, and $L^2(\partial\Omega)$, $L^2_1(\partial\Omega)$ imbed densely into $B^{1,1}_{-\alpha}(\partial\Omega)$ and $B^{1,1}_{1-\alpha}(\partial\Omega)$, respectively.

At this stage, there remains to show that the operator (4.69) is also one-to-one. To this end, fix $f \in B^{1,1}_{-\alpha}(\partial\Omega, \Lambda^1 TM)$ such that $Sf = c\,\nu$ on $\partial\Omega$ for some $c \in \mathbb{R}$. The goal is to show that $f = c\,\nu$ for a possibly different constant $c \in \mathbb{R}$. We shall see that this indeed the case when f is regarded as an element in $(B^{1,1}_{-\alpha}(\partial\Omega, \Lambda^1 TM))^{**} = (C^\alpha(\partial\Omega, \Lambda^1 TM))^*$.

Let us set $u := \mathcal{S}f$, $\pi := \mathcal{Q}f$ in $\Omega_\pm$ and notice that, by *(iii)* in Theorem 2.1,

$$\Big\| \mathrm{dist}(\cdot, \partial\Omega)^{\alpha-1} |\nabla u| \Big\|_{L^1(\Omega_\pm)} \leq C\|f\|_{(C^\alpha(\partial\Omega,\Lambda^1 TM))^*}. \tag{4.71}$$

For an arbitrary, fixed, $\psi \in C^\alpha(\partial\Omega, \Lambda^1 TM)$ with $\int_{\partial\Omega}\langle\psi,\nu\rangle\, d\sigma = 0$, let (w, q) solve the Dirichlet problem for the Stokes system with boundary datum ψ. This can be done by originally regarding ψ in L^2 (in which case [46] guarantees well-posedness); then Proposition 4.2 and Lemma 4.7 ensure that

$$\Big\|\text{dist}(\cdot,\partial\Omega)^{1-\alpha}|\nabla w|\Big\|_{L^\infty(\Omega)} \leq C\|\psi\|_{C^\alpha(\partial\Omega,\Lambda^1TM)}. \tag{4.72}$$

The point is that we can now define $\partial u/\partial\nu$ as an element in $(C^\alpha(\partial\Omega,\Lambda^1TM))^*$ by setting

$$\Big\langle\frac{\partial u}{\partial\nu},\psi\Big\rangle := \int_\Omega \langle \text{Def}(u)\,,\,\text{Def}(w)\rangle\,d\text{Vol}. \tag{4.73}$$

Note that, once again, this is a natural extension of Green's formula and that

$$\begin{aligned}\Big|\Big\langle\frac{\partial u}{\partial\nu},\psi\Big\rangle\Big| &\leq C\|\text{dist}(\cdot,\partial\Omega)^{\alpha-1}|\nabla u|\|_{L^1(\Omega)}\|\text{dist}(\cdot,\partial\Omega)^{1-\alpha}|\nabla w|\|_{L^\infty(\Omega)}\\ &\leq C\|f\|_{(C^\alpha(\partial\Omega,\Lambda^1TM))^*}\|\psi\|_{C^\alpha(\partial\Omega,\Lambda^1TM)}.\end{aligned} \tag{4.74}$$

Thus, $\partial u/\partial\nu \in (C^\alpha(\partial\Omega,\Lambda^1TM))^*$ and $\|\partial u/\partial\nu\|_{(C^\alpha(\partial\Omega,\Lambda^1TM))^*} \leq C\|f\|_{(C^\alpha(\partial\Omega,\Lambda^1TM))^*}$, as claimed. In turn, this can be used to justify the identity

$$\Big\langle\frac{\partial\mathcal{S}f}{\partial\nu},\psi\Big\rangle = \Big\langle Sf,\frac{\partial w}{\partial\nu}\Big\rangle = 0. \tag{4.75}$$

Consequently, $\big[\frac{\partial\mathcal{S}f}{\partial\nu}\big] = 0$ (as an equivalence class) in $(C^\alpha(\partial\Omega,\Lambda^1TM))^*/\mathbb{R}\nu$, when considered both from Ω_+ and Ω_- and, ultimately,

$$[f] = \Big[\frac{\partial\mathcal{S}f}{\partial\nu}\Big|_{\partial\Omega_-}\Big] - \Big[\frac{\partial\mathcal{S}f}{\partial\nu}\Big|_{\partial\Omega_+}\Big] = 0 \quad\text{in}\quad (C^\alpha(\partial\Omega,\Lambda^1TM))^*/\mathbb{R}\nu, \tag{4.76}$$

as desired. This proves that the operator (4.69) is indeed injective.

At this stage, we may conclude that the operator (4.69) is an isomorphism for any $0<\alpha<1/p_\Omega$. Thus, by duality, we see that (4.68) is also an isomorphism, whenever $0<\alpha<1/p_\Omega$. The full range of indices, as described in $(I)-(III)$ in the statement of Theorem 4.1, is then obtained from the invertibility results proved for the operators (4.68), (4.69) and (4.70), via the complex interpolation method. This finishes the proof of the theorem. □

We conclude this section by recording the three-dimensional version of Theorem 4.1.

Theorem 4.8. *Assume that* $\dim M = 3$ *and let* $\Omega\subset M$ *be a fixed, connected Lipschitz domain. Also, recall the critical Dirichlet exponent* $p_\Omega \in [1,2)$ *from (1.5). Then*

$$S : B^{p,p}_{-s}(\partial\Omega,\Lambda^1TM)/\mathbb{R}\nu \longrightarrow B^{p,p}_{1-s}(\partial\Omega,\Lambda^1TM)/\mathbb{R}\nu \tag{4.77}$$

is an isomorphism for all pairs of indices $(s, 1/p) \in (0,1) \times (0,1)$ *satisfying one of the following conditions:*

$$\begin{array}{rl} (I'): & 1-\frac{1}{p_\Omega} \le \frac{1}{p} \le \frac{1}{p_\Omega}, \\ (II'): & \frac{1}{p_\Omega} \le \frac{1}{p} < \frac{s}{2}+\frac{1}{p_\Omega}, \\ (III'): & \frac{s}{2}-\frac{1}{p_\Omega}+\frac{1}{2} < \frac{1}{p} < 1-\frac{1}{p_\Omega}. \end{array} \tag{4.78}$$

Its proof is akin to that of Theorem 4.1 and, therefore, is omitted.

5. The Poisson problem for the Stokes system

In this section we treat the Poisson problem for the Stokes system in C^1 and Lipschitz domains, for data in Sobolev-Besov spaces. Our first important result in this regard is as follows.

Theorem 5.1. *Let* Ω *be a connected,* C^1 *subdomain of* M*. Then for each* $1 < p < \infty$, $0 < s < 1$, *the boundary value problem*

$$\begin{aligned} &u \in L^p_{s+\frac{1}{p}}(\Omega, \Lambda^1 TM), \\ &\pi \in L^p_{s+\frac{1}{p}-1}(\Omega),\ \langle \pi, 1\rangle = 0, \\ &Lu + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM), \\ &\delta u = h \in L^p_{s+\frac{1}{p}-1}(\Omega), \\ &Tr u = g \in B^{p,p}_s(\partial\Omega, \Lambda^1 TM), \end{aligned} \tag{5.1}$$

has a unique solution, granted that the data satisfy the (necessary) compatibility condition

$$\langle h, 1\rangle = \int_{\partial\Omega} \langle \nu, g\rangle \, d\sigma. \tag{5.2}$$

Furthermore, a natural accompanying estimate holds.

Finally, similar results are valid in the case when $\partial\Omega$ *has a small enough Lipschitz constant (relative to* s, p*).*

Note that $-1 + 1/p < s + 1/p - 1 < 1/p$ so $L^p_{s+\frac{1}{p}-1}(\Omega) = L^p_{s+\frac{1}{p}-1,0}(\Omega)$. In particular, the pairings $\langle \pi, 1\rangle$ and $\langle h, 1\rangle$ are meaningful.

Before proving Theorem 5.1, we state several preparatory lemmas. Our first auxiliary result is inspired by Green's formula and duality; compare with (4.19) and (4.61).

Lemma 5.2. *Let Ω be a Lipschitz domain and assume that $1 < p < \infty$, $0 < s < 1$. Then for any (u, π) solution of the homogeneous version of (5.1), we can define $2(\mathrm{Def}\, u)\nu - \pi\nu$ as an element in $B^{p',p'}_{s-1}(\partial\Omega, \Lambda^1 TM) = (B^{p,p}_{1-s}(\partial\Omega, \Lambda^1 TM))^*$, if $1/p + 1/p' = 1$, via*

$$\langle 2(\mathrm{Def}\, u)\nu - \pi\nu\,,\ Tr\, w\rangle := 2\int_\Omega \langle \mathrm{Def}\, u\,,\, \mathrm{Def}\, w\rangle\, d\,Vol + \int_\Omega \langle \pi, \delta w\rangle\, d\,Vol, \quad (5.3)$$

for any $w \in L^{p'}_{1-s+1/p'}(\Omega, \Lambda^1 TM)$.

Our second lemma is a slight extension of a result in [4]:

Lemma 5.3. *For any Lipschitz domain Ω and each $1 < p < \infty$, $0 < s < 1$, the operator*

$$\delta : L^p_{s+1/p,0}(\Omega, \Lambda^1 TM) \longrightarrow \{f \in L^p_{s+1/p-1}(\Omega);\ \langle f, 1\rangle = 0\} \quad (5.4)$$

is onto. In fact, (5.4) has a (linear, bounded) right inverse, denoted in the sequel by δ^{-1}.

Next, record a version of Theorem 2.9 in [35].

Lemma 5.4. *Let $\{T_z\}_{z\in U}$ be an interpolating family of operators between two interpolation scales of Banach spaces $\{X_z\}_{z\in U}$, $\{Y_z\}_{z\in U}$. Suppose that U, the space of parameters, is connected and that $T_z : X_z \to Y_z$ is a Fredholm operator for each $z \in U$. Then its index is independent of $z \in U$.*

Finally, we record a simple but useful observation.

Lemma 5.5. *Let X, Y be two Banach spaces such that the inclusion $Y \hookrightarrow X$ is continuous with dense range. Also, assume that $T \in \mathcal{L}(X) \cap \mathcal{L}(Y)$ is Fredholm of index zero both when acting on X and when acting on Y. Then* $\mathrm{Ker}\,(T; X) = \mathrm{Ker}\,(T; Y)$.

We are now in a position to present the

Proof (Proof of Theorem 5.1.). First, for each $1 < p < \infty$, $0 < s < 1$, consider the mapping

$$\begin{aligned}\mathcal{A} : L^p_{s+1/p}(\Omega, \Lambda^1 TM) \oplus L^p_{s+1/p-1}(\Omega) \longrightarrow \quad (5.5)\\ L^p_{s+1/p-2}(\Omega, \Lambda^1 TM) \oplus L^p_{s+1/p-1}(\Omega) \oplus B^{p,p}_s(\partial\Omega, \Lambda^1 TM),\end{aligned}$$

given by

$$\mathcal{A}(u, \pi) := (Lu + d\pi, \delta u, \mathrm{Tr}\, u). \quad (5.6)$$

Clearly, this is well-defined, linear and bounded between two complex interpolation scales of Banach spaces. Our intention is to show that this map is Fredholm with index zero.

To this end, let us first prove that $\mathcal{A}$ has finite dimensional kernel. Indeed, if (u,π) solves the homogeneous version of (5.1), Lemma 5.2 gives that

$$\xi := -\Big[2(\mathrm{Def}\,u)\nu - \pi\nu\Big]\Big|_{\partial\Omega} \in B^{p',p'}_{s-1}(\partial\Omega,\Lambda^1TM), \quad \text{where} \quad 1/p+1/p'=1. \tag{5.7}$$

Then, in the current context, Green's formula (3.7) reduces to

$$u=\mathcal{S}\xi \qquad \text{in } \Omega. \tag{5.8}$$

In particular, $d\pi=-Lu=-L\mathcal{S}\xi=d\mathcal{Q}\xi$ which forces $\pi=\mathcal{Q}\xi+c$, for some constant c. Using this and (5.7) in (3.5) then gives $(\frac{1}{2}I+K^*)\xi\in\mathbb{R}\nu$. Hence the mapping

$$\text{Null-space of } \mathcal{A} \ni (u,\pi)\mapsto[\xi]\in \mathrm{Ker}\left(\tfrac{1}{2}I+K^*;\, B^{p',p'}_{s-1}(\partial\Omega,TM)/\mathbb{R}\nu\right) \tag{5.9}$$

is well-defined and, by (5.8), one-to-one. In particular,

$$\dim\left[\text{Null-space of } \mathcal{A}\right] \le \dim\mathrm{Ker}\left(\tfrac{1}{2}I+K^*;\, B^{p',p'}_{s-1}(\partial\Omega,TM)/\mathbb{R}\nu\right). \tag{5.10}$$

Now, since the operator

$$\tfrac{1}{2}I+K^*: B^{p',p'}_{s-1}(\partial\Omega,TM)/\mathbb{R}\nu \longrightarrow B^{p',p'}_{s-1}(\partial\Omega,TM)/\mathbb{R}\nu \tag{5.11}$$

is well-defined and Fredholm with index zero (by Proposition 3.5), it follows that the dimension of its kernel is finite. In light of (5.11), this proves the claim that (5.5)-(5.6) has finite dimensional kernel.

Next, for a generic function $k(x,y)$, we introduce the Newtonian type potential operator

$$\Pi_k f(x) := \int_\Omega \langle k(x,y), f(y)\rangle\, d\mathrm{Vol}_y, \qquad x\in\Omega. \tag{5.12}$$

In particular, $L\Pi_\Gamma + d\Pi_\Theta = I$ and $\delta\Pi_\Gamma=0$. Taking

$$\begin{aligned} u &:= \Pi_\Gamma f_1 + \delta^{-1} f_2 + \mathcal{D} f_3 \in L^p_{s+1/p}(\Omega,\Lambda^1TM), \\ \pi &:= \Pi_\Theta f_1 + \mathcal{P} f_3 \in L^p_{s+1/p-1}(\Omega), \end{aligned} \tag{5.13}$$

for $f_1\in L^p_{s+1/p-2}(\Omega,\Lambda^1TM)$, $f_2\in L^p_{s+1/p-1}(\Omega)$ with $\langle f_2,1\rangle=0$ and $f_3\in B^{p,p}_s(\partial\Omega,\Lambda^1TM)$, arbitrary, gives that the range of $\mathcal{A}$ contains that of

$$(f_1, f_2, f_3) \mapsto \Big(f_1 + L\delta^{-1} f_2\,,\, f_2\,,\, \mathrm{Tr}\,[\Pi_\Gamma f_1] + (\tfrac{1}{2}I + K) f_3\Big). \tag{5.14}$$

However, this latter operator is Fredholm since, in matrix form, it reads

$$\begin{pmatrix} I & L\delta^{-1} & 0 \\ 0 & I & 0 \\ \mathrm{Tr}\,\Pi_\Gamma & 0 & \frac{1}{2}I + K \end{pmatrix} \tag{5.15}$$

(here Proposition 3.5 is invoked). This observation allows us to further conclude that $\mathcal{A}$ has closed range, of finite codimension. Thus, the mapping (5.5)-(5.6) is Fredholm. From [46] we know that its index is zero for $p = 2$, $s = 1/2$, hence its index is zero for any $1 < p < \infty$, $0 < s < 1$, by Lemma 5.4. Also, by Lemma 5.5, the null-space of $\mathcal{A}$ for $p \in (1, \infty)$, $s \in (0, 1)$, is identical to that corresponding to $p = 2$, $s = 1/2$. That is, $(u, \pi) \in \mathrm{Ker}\,\mathcal{A}$ if and only if $u = 0$ and π is a constant. As this space is one-dimensional, we infer that $\mathcal{A}$ has a range of codimension one. On the other hand, by the Divergence Theorem, the range of $\mathcal{A}$ is always contained in the subspace

$$\begin{aligned} X := \{(f, h, g) \in L^p_{s+1/p-2}(\Omega, TM) \oplus L^p_{s+1/p-1}(\Omega, TM) \oplus B^{p,p}_s(\partial\Omega, TM); \\ \langle h, 1\rangle = \textstyle\int_{\partial\Omega} \langle \nu, g\rangle\, d\sigma\}, \end{aligned} \tag{5.16}$$

codimension one in $L^p_{s+1/p-2}(\Omega, TM) \oplus L^p_{s+1/p-1}(\Omega) \oplus B^{p,p}_s(\partial\Omega, TM)$. Consequently, this forces $\mathrm{Range}\,\mathcal{A} = X$. This finishes the proof of the theorem.

Theorem 5.6. *The conclusion in Theorem 5.1 remains valid when $\partial\Omega$ is only Lipschitz, provided we make the extra assumption that $p \in (2-\varepsilon, 2+\varepsilon)$, for some $\varepsilon = \varepsilon(\partial\Omega) > 0$. If dim $M = 2$ then the conclusion holds for all pairs $(s, 1/p) \in (0,1) \times (0,1)$ satisfying one of the conditions (4.2). A similar conclusion holds if dim $M = 3$ and $(s, 1/p) \in (0,1) \times (0,1)$ satisfies one of the three conditions listed in (4.78).*

Proof. We shall use the same approach as in the proof of Theorem 5.1, so we will only stress the main novel points. Much as before, the crucial step is proving that $\mathcal{A}$ (defined in (5.5)-(5.6)) is Fredholm with index zero. To see that $\mathcal{A}$ has a finite dimensional kernel we rely again on (5.8). Taking the trace of both sides yields $0 = \mathrm{Tr}\,u = S\xi$ so that $\xi \in \mathbb{R}\nu$, by (2.27) in Theorem 2.1. Returning with this back in (5.8) gives that $u = 0$, thanks to (2.24). Hence π is a constant which proves that $\dim(\mathrm{Ker}\,\mathcal{A}) \leq 1$. The desired conclusion in dimensions 2 and 3 follows from Theorems 4.1 and 4.8, respectively.

There remains to prove that $\mathcal{A}$ has closed range. This is done as in the second half of the proof of Theorem 5.1, the most notable exception being the replacement of (5.13) by

$$u := \Pi_\Gamma f_1 + \delta^{-1} f_2 + \mathcal{S} f_3 \in L^p_{s+1/p}(\Omega, \Lambda^1 TM), \tag{5.17}$$
$$\pi := \Pi_\Theta f_1 + \mathcal{Q} f_3 \in L^p_{s+1/p-1}(\Omega),$$

where f_1, f_2 are as before and, this time, $f_3 \in B^{p,p}_{s-1}(\partial\Omega, \Lambda^1 TM)$. In this scenario, the role of (5.15) is played by the operator

$$\begin{pmatrix} I & L\delta^{-1} & 0 \\ 0 & I & 0 \\ \mathrm{Tr}\,\Pi_\Gamma & 0 & S \end{pmatrix} \tag{5.18}$$

which, by (iv) in Theorem 2.1, is Fredholm. Again, in dimensions 2 and 3 the corresponding result follows from the theorems proved in Section 4. This concludes the proof.

6. A perturbation of the Stokes system

In this section we discuss the Poisson problem for the linearization (1.1) of the stationary Navier-Stokes system. This problem can be viewed as a perturbation of the Stokes system (1.2). Our main result is the following.

Theorem 6.1. *Let Ω be a connected, C^1 subdomain of M, and assume that $\omega \in L^n(\Omega, TM)$ is divergence-free. Then for each $1 < p < \infty$, $0 < s < 1$, the boundary value problem*

$$\begin{aligned} & u \in L^p_{s+\frac{1}{p}}(\Omega, \Lambda^1 TM), \\ & \pi \in L^p_{s+\frac{1}{p}-1}(\Omega),\ \langle \pi, 1\rangle = 0, \\ & Lu + \nabla_\omega u + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM), \\ & \delta u = h \in L^p_{s+\frac{1}{p}-1}(\Omega), \\ & Tr\, u = g \in B^{p,p}_s(\partial\Omega, \Lambda^1 TM), \end{aligned} \tag{6.1}$$

has a unique solution, provided that the data satisfy the (necessary) compatibility condition (5.2). In fact, the same results are valid in the case when $\partial\Omega$ has a small enough Lipschitz constant (relative to s, p).

Furthermore, similar conclusions hold when Ω is a Lipschitz domain and $\dim M = 2$, *for all pairs $(s, 1/p) \in (0,1) \times (0,1)$ satisfying one of the conditions (4.2). The same is true if dim $M = 3$ and $(s, 1/p) \in (0,1) \times (0,1)$ satisfies one of the three conditions in (4.78). Finally, if dim $M \geq 4$ then the result holds provided that $p \in (2-\varepsilon, 2+\varepsilon)$, where $\varepsilon = \varepsilon(\partial\Omega) > 0$ depends only on the Lipschitz character of the domain.*

Before presenting the proof of the main theorem we discuss a couple of preliminary results. Recall that $n := \dim M \geq 2$.

Lemma 6.2. *Let Ω be a Lipschitz domain and assume that $\omega \in L^r(\Omega, TM)$ is a divergence-free field. Then the operator*

$$\nabla_\omega : L^p_\theta(\Omega, TM) \longrightarrow L^p_{\theta-1-n/r}(\Omega, TM) \tag{6.2}$$

is well-defined and bounded, in fact compact, for each r, p, θ such that

$$1 < p < \infty, \quad 0 < \theta < 1 + \frac{n}{r}, \quad -\frac{1}{n} < \frac{1}{p} - \frac{\theta}{n} < 1 - \frac{1}{r}. \tag{6.3}$$

Proof. First, if $\omega = \omega^j \partial_j$ and $u = u^k \partial_k$, then

$$(\nabla_\omega u)^k = (\omega^j \partial_j u^k + \Gamma^k_{\ell j} u^\ell \omega^j)^k, \tag{6.4}$$

where $\Gamma^k_{\ell j}$ are the Christoffel symbols associated with the metric. If the divergence-free condition on ω is taken into account then the above becomes

$$(\nabla_\omega u)^k = \partial_j(\omega^j u^k) + \Gamma^k_{\ell j} u^\ell \omega^j - (\partial_j \log g^{1/2}) \omega^j u^k. \tag{6.5}$$

We first consider two distinguished cases, i.e.

$$1 < p < \infty, \quad 0 < \theta \leq \min\{n/r, 1\}, \quad 0 < \frac{1}{p} - \frac{\theta}{n} < 1 - \frac{1}{r}, \tag{6.6}$$

and

$$1 < p < \infty, \quad 1 < \theta < 1 + \frac{n}{r}, \quad -\frac{1}{n} < \frac{1}{p} - \frac{\theta}{n} < 1 - \frac{1}{r} - \frac{1}{n}, \tag{6.7}$$

respectively. When (6.6) holds, we use (6.5) in concert with the sequence of embeddings

$$\begin{aligned} &L^p_\theta \hookrightarrow L^q, \quad \frac{1}{q} = \frac{1}{p} - \frac{\theta}{n}, \\ &L^r \cdot L^q \hookrightarrow L^t, \quad \frac{1}{t} = \frac{1}{r} + \frac{1}{q}, \\ &L^t_{-1} \hookrightarrow L^p_{\theta - 1 - \frac{n}{r}}, \end{aligned} \tag{6.8}$$

in order to justify the estimate

$$\|\nabla_\omega u\|_{L^p_{\theta-1-n/r}(\Omega, TM)} \leq C \|\omega\|_{L^r(\Omega, TM)} \cdot \|u\|_{L^p_\theta(\Omega, TM)}. \tag{6.9}$$

On the other hand, if (6.7) holds, then we utilize (6.4) together with

$$\begin{aligned} &L^p_{\theta-1} \hookrightarrow L^q, \quad \frac{1}{q} = \frac{1}{p} - \frac{\theta - 1}{n}, \\ &L^r \cdot L^q \hookrightarrow L^t, \quad \frac{1}{t} = \frac{1}{r} + \frac{1}{q}, \\ &L^t \hookrightarrow L^p_{\theta - 1 - \frac{n}{r}}, \end{aligned} \tag{6.10}$$

in order to conclude that (6.9) holds in this case as well. By interpolating between the regions (6.6) and (6.7) it follows that (6.9) holds for any θ, p, r as in (6.3).

This proves the fact that the operator in (6.2) is well-defined and bounded. Finally, that (6.2) is in fact compact, follows readily from the estimate (6.9), Rellich's selection lemma and a standard approximation argument.

Next we prove some Korn type estimates.

Proposition 6.3. *Let Ω be a Lipschitz domain and assume that $1 < p < \infty$. Then there exists a constant $C = C(\Omega, p) > 0$ such that*

$$\|u\|_{L^p_1(\Omega,TM)} \leq C\Big\{\|\mathrm{Def}\, u\|_{L^p(\Omega,S^2TM)} + \|u\|_{L^p(\Omega,TM)}\Big\}, \tag{6.11}$$

uniformly for $u \in L^p_1(\Omega, TM)$. Furthermore,

$$\|u\|_{L^p_1(\Omega,TM)} \approx \|\mathrm{Def}\, u\|_{L^p(\Omega,S^2TM)}, \tag{6.12}$$

uniformly for $u \in L^p_{1,0}(\Omega, TM)$.

It is worth pointing out that our proof of the estimate (6.11) works under the rather weak assumption that the metric tensor has Lipschitz coefficients. The case when $p = 2$ and both $\partial\Omega$ and the metric are C^∞, has been treated in [59, Vol. I, Corollary 12.3, p. 400], via pseudodifferential techniques.

Proof. Working in local coordinates and assuming that $u = u^j\partial_j$, the components of the deformation tensor become $(\mathrm{Def}\, u)_{jk} = \varepsilon_{jk}(u) + \mathcal{O}(|u|)$, where we set

$$\varepsilon_{jk}(u) := \tfrac{1}{2}(\partial_j u_k + \partial_k u_j), \qquad \text{and} \quad u_\ell := g_{\ell j}u^j. \tag{6.13}$$

A direct calculation shows that

$$\partial_i\partial_j u_k = \partial_i\varepsilon_{jk}(u) + \partial_j\varepsilon_{ik}(u) - \partial_k\varepsilon_{ij}(u), \qquad \forall\, i, j, k. \tag{6.14}$$

In particular,

$$\begin{aligned}
\sum_{j,k}\|\partial_j u_k\|_{L^p(\Omega)} &\leq C\sum_{j,k}\sum_i\|\partial_i\partial_j u_k\|_{L^p_{-1}(\Omega)} + C\sum_{j,k}\|\partial_j u_k\|_{L^p_{-1}(\Omega)}\\
&\leq C\sum_{i,j,k}\|\partial_i\varepsilon_{jk}(u)\|_{L^p_{-1}(\Omega)} + C\sum_k\|u_k\|_{L^p(\Omega)}\\
&\leq C\sum_{j,k}\|\varepsilon_{jk}(u)\|_{L^p(\Omega)} + C\sum_{j,k}\|\partial_i\varepsilon_{jk}(u)\|_{L^p_{-1}(\Omega)}\\
&\quad + C\|u\|_{L^p(\Omega,TM)} \qquad\qquad (6.15)\\
&\leq C\|\mathrm{Def}\, u\|_{L^p(\Omega,S^2TM)} + C\|u\|_{L^p(\Omega,TM)}.
\end{aligned}$$

Now (6.11) readily follows from this.

Next, (6.12) amounts to proving that the operator

$$\mathrm{Def} : L^p_{1,0}(\Omega, TM) \to L^p(\Omega, S^2TM) \tag{6.16}$$

is continuous and bounded from below. While continuity is clear from definitions, the latter claim can be further reduced (via the Open Mapping Theorem) to showing that

(i) *the operator (6.16) has closed range*, and (6.17)

(ii) *the operator (6.16) is one-to-one.*

That (6.16) has closed range follows from (6.11), as this estimate entails the boundedness of this operator from below, modulo compact operators. As for the injectivity of (6.16), we note that if $u \in L^p_{1,0}(\Omega, TM)$ is such that $\mathrm{Def}\, u = 0$ in Ω, then $\mathrm{Def}\, \tilde{u} = 0$ in M, where tilde denotes extension by zero outside Ω. Consequently, due to the absence of global Killing fields on M (cf. (2.11)), this forces $\tilde{u} = 0$ and, ultimately, $u = 0$ on Ω. This takes care of (ii) in (6.17) and finishes the proof of the proposition.

We now establish some L^2-*a priori* estimates which are needed for the nonlinear problem.

Proposition 6.4. *Let Ω be an arbitrary Lipschitz domain and assume that $\omega \in L^n(\Omega, TM)$ is divergence-free. Then*

$$\|u\|^2_{L^2_1(\Omega,\Lambda^1TM)} \leq C\Big(\|f\|^2_{L^2_{-1}(\Omega,\Lambda^1TM)} + \|\pi\|_{L^2(\Omega)}\|h\|_{L^2(\Omega)}\Big), \tag{6.18}$$

uniformly for $u \in L^2_{1,0}(\Omega, \Lambda^1TM)$, $\pi \in L^2(\Omega)$, satisfying

$$Lu + \nabla_\omega u + d\pi = f, \qquad \delta u = h \qquad \text{in } \Omega. \tag{6.19}$$

The constant C appearing in (6.18) depends exclusively on Ω; in particular, it is independent of the vector field ω.

Proof. Recall that, in general, $(\nabla_X)^t = -\nabla_X - (\mathrm{div}\, X)$ for any $X \in TM$. Also, with $p = 2$, $\theta = 1$ and $r = n$, Lemma 6.2 gives that $\nabla_\omega u \in L^2_{-1}(\Omega, TM) = \Big(L^2_{1,0}(\Omega, TM)\Big)^*$. Thus, the pairings below are meaningful, and integrating by parts yields

$$\langle f, u\rangle = \langle Lu, u\rangle + \langle \nabla_\omega u, u\rangle + \langle d\pi, u\rangle = 2\,\|\mathrm{Def}\, u\|^2_{L^2(\Omega,S^2TM)} + \langle \pi, h\rangle. \tag{6.20}$$

With this at hand, the desired conclusion follows from (6.12) in Proposition 6.3.

Finally, we are now ready to tackle the

Proof of Theorem 6.1. Let us start by recalling the map (5.5)-(5.6) and then introduce the perturbation

$$\mathcal{A}_\omega := \mathcal{A} + (\nabla_\omega, 0, 0). \tag{6.21}$$

From the proof of Theorem 5.1 we know that $\mathcal{A}$ is Fredholm with index zero. Thus, on account of Lemma 6.2 (used here with $\theta := s + 1/p$, and $r = n$), the map (6.21) is also Fredholm with index zero (in the context of (5.5)). Furthermore, from Lemma 5.5, its kernel is independent of p, s. Consequently, by Proposition 6.4, the null-space of $\mathcal{A}_\omega$ is simply the one-dimensional space $\{0\} \times \mathbb{R}$. With these at hand, we can now finish the proof of the well-posedness of (6.1) much as we did in the non-perturbed case (treated in Theorem 5.1).

When Ω is only Lipschitz and $(s, 1/p)$ satisfy appropriate conditions, we argue analogously, by relying on Theorem 5.6. □

7. The stationary Navier-Stokes equations

Having dealt with the linearization of the stationary Navier-Stokes equations, we are now ready to discuss the Poisson problem in the nonlinear case.

Theorem 7.1. *Let Ω be a connected, C^1 subdomain of M and assume that $\dim M = n \in \{2,3,4\}$. Then for each pair of indices s, p satisfying*

$$1 < p < \infty, \quad 0 < s < 1, \quad 1 \leq s + \frac{1}{p}, \quad \frac{1}{p} - \frac{s}{n-1} \leq \frac{n-2}{2(n-1)}, \tag{7.1}$$

the boundary value problem

$$\begin{aligned} & u \in L^p_{s+\frac{1}{p},0}(\Omega, \Lambda^1 TM), \\ & \pi \in L^p_{s+\frac{1}{p}-1}(\Omega), \\ & Lu + \nabla_u u + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM), \\ & \delta u = 0 \ \text{in } \Omega, \end{aligned} \tag{7.2}$$

has at least one solution. Furthermore, there exits $\kappa > 0$ so that the solution is unique whenever

$$\|f\|_{L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1 TM)} \leq \kappa. \tag{7.3}$$

In fact, the same results are valid in the case when $\partial\Omega$ has a small enough Lipschitz constant (relative to s, p).

Furthermore, similar conclusions hold when Ω is a Lipschitz domain and dim $M = 2$ for all pairs $(s, 1/p) \in (0,1) \times (0,1)$ satisfying (7.1) and *one of the conditions (4.2). The same is true if dim* $M = 3$ *and* $(s, 1/p) \in (0,1) \times (0,1)$ *satisfies (7.1) and one of the conditions in (4.78). Finally, if dim* $M = 4$ *then the same result holds granted that, in addition to (7.1), we also have* $p \in (2-\varepsilon, 2+\varepsilon)$ *for some* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *depending only on the Lipschitz character of the domain.*

Proof. For some fixed, sufficiently large R (to be specified momentarily), consider the closed, convex set

$$\mathcal{O} := \{u \in L^2_{1,0}(\Omega, \Lambda^1 TM);\ \delta u = 0 \text{ in } \Omega \text{ and } \|u\|_{L^2_{1,0}(\Omega,\Lambda^1 TM)} \leq R\}. \tag{7.4}$$

Also, for a fixed $f \in L^2_{-1}(\Omega, \Lambda^1 TM)$, introduce the (nonlinear) operator $T_f : \mathcal{O} \to \mathcal{O}$ as follows. For each $u \in \mathcal{O}$, we let (v, π) be the unique solution of the boundary value problem

$$\begin{aligned} & v \in L^2_{1,0}(\Omega, \Lambda^1 TM), \\ & \pi \in L^2(\Omega),\ \langle \pi, 1\rangle = 0, \\ & Lv + \nabla_u v + d\pi = f \in L^2_{-1}(\Omega, \Lambda^1 TM), \\ & \delta v = 0 \text{ in } \Omega, \end{aligned} \tag{7.5}$$

and then set $T_f(u) := v$. Since

$$L^2_1(\Omega) \hookrightarrow L^n(\Omega) \quad \text{for} \ \ n \leq 4, \tag{7.6}$$

Proposition 6.4 and Theorem 6.1 show that

$$T_f : \mathcal{O} \longrightarrow \mathcal{O} \tag{7.7}$$

is well-defined if $R = R(\Omega, f) > 0$ is sufficiently large. In fact, in terms of the mapping $\mathcal{A}$ introduced in (5.5)-(5.6) with $p = 2$, $s = 1/2$, we have

$$T_f(u) = \mathrm{pr}_1\Big[\Big(\mathcal{A} + (\nabla_u, 0, 0)\Big)^{-1}(f, 0, 0)\Big], \qquad \forall\, u \in \mathcal{O}, \tag{7.8}$$

where $\mathrm{pr}_1(a, b) := a$ is the canonical projection onto the first factor of a Cartesian product. It follows that the mapping (7.7) is continuous for $n \leq 4$, given that (7.6) holds and

$$L^n(\Omega, TM) \ni u \mapsto \nabla_u \in \mathcal{L}\Big(L^2_1(\Omega, TM), L^2_{-1}(\Omega, TM)\Big) \tag{7.9}$$

is well-defined, linear and bounded for $n \leq 4$. Furthermore, since the embedding (7.6) is compact for $n = 2, 3$ we see, on account of (7.9), that T_f is also compact when $n = 2, 3$. Consequently, the standard version of Schauder's Fixed Point Theorem applies and yields some $u \in \mathcal{O}$ such that $T_f(u) = u$. This proves the existence part in the theorem in the special case $p = 2$, $s = 1/2$, $n \leq 3$.

Consider now the case when $n \leq 3$ and s, p are as in (7.1), and observe that $L^p_{s+1/p-2}(\Omega) \hookrightarrow L^2_{-1}(\Omega)$. Thus, for any $f \in L^p_{s+1/p-2}(\Omega, \Lambda^1 TM)$ we can produce, thanks to what we have proved so far, a one-form u and a scalar function π such that

$$\begin{aligned} & u \in L^2_{1,0}(\Omega, \Lambda^1 TM), \\ & \pi \in L^2(\Omega),\ \langle \pi, 1\rangle = 0, \\ & Lu + \nabla_u u + d\pi = f \in L^p_{s+1/p-2}(\Omega, \Lambda^1 TM), \\ & \delta u = 0 \text{ in } \Omega. \end{aligned} \tag{7.10}$$

By virtue of (7.6) and Theorem 6.1, it follows *a posteriori* that the solution $u \in L^p_{s+1/p}(\Omega, \Lambda^1 TM)$ and $\pi \in L^p_{s+1/p-1}(\Omega)$. Hence, (u, π) solves (7.2).

As for the case when $n = 4$, the problem with the approach above is the lack of compactness of the map T_f in (7.7). Consequently, we shall to treat this case separately. Our approach is based on the following version of the Schauder-Tychonoff Theorem (see, e.g., [15, Thm. 3.6.1, p. 161]):

Theorem 7.2. *Let $\mathcal{E}$ be a separated, locally convex topological vector space, and let $\mathcal{O}$ be a nonempty, closed, convex subset of $\mathcal{E}$. Also, assume that $\mathcal{T} : \mathcal{O} \to \mathcal{O}$ is a continuous map such that $\mathcal{T}(\mathcal{O})$ is relatively compact in $\mathcal{E}$. Then $\mathcal{T}$ has a fixed point.*

For the applications we have in mind, $\mathcal{E}$ will be the separable Hilbert space $L^2_{1,0}(\Omega, \Lambda^1 TM)$ equipped with the weak topology, and keep $\mathcal{O}$ as in (7.4). The subspace of all divergence free fields in $L^2_{1,0}(\Omega, \Lambda^1 TM)$ is closed in the strong norm, hence also weakly closed. Also, since any closed ball in a reflexive Banach space is weakly compact, it follows that

$$\mathcal{O} \text{ is a compact subset of } \mathcal{E}. \tag{7.11}$$

Thus, by employing the *weak* topology on the space $L^2_{1,0}(\Omega, \Lambda^1 TM)$, the demand that $T_f(\mathcal{O})$ is relatively compact is automatically taken care of. Instead another issue arises in this setting. Specifically, we need to establish the continuity of the map (7.7) in the *weak* topology.

For starters, we note that while the topology of $\mathcal{E}$ is not 'globally' metrizable, it is so at a 'local' level –that is, the topology induced by $\mathcal{E}$ on any bounded subset of $L^2_{1,0}(\Omega, \Lambda^1 TM)$ is metrizable (c.f. [59] Vol. I, p. 486). Consequently, it suffices to check the continuity of T_f by working with sequences. To this end, fix $f \in L^2_{-1}(\Omega, \Lambda^1 TM)$, let $u_j \in \mathcal{O}$ converge to u_0 in $\mathcal{E}$, and assume that (v_j, π_j) solve (7.5) for $u = u_j$. In particular, $\|v_j\|_{L^2_1(\Omega, \Lambda^1 TM)}$, $\|v_0\|_{L^2_1(\Omega, \Lambda^1 TM)} \leq R$. Then, if we set $\tilde{v}_j := v_j - v_0$, $\tilde{\pi}_j := \pi_j - \pi_0$, we have

$$\begin{aligned} & L\tilde{v}_j + \nabla_{u_0} \tilde{v}_j + d\tilde{\pi}_j = f_j := \nabla_{u_0 - u_j} v_j \text{ in } \Omega, \\ & \delta \tilde{v}_j = 0 \text{ in } \Omega, \\ & \tilde{v}_j \in L^2_{1,0}(\Omega, \Lambda^1 TM), \end{aligned} \tag{7.12}$$

for each j. Our goal is to show that $\tilde{v}_j \to 0$ in $\mathcal{E}$. As $\tilde{v}_j = \text{pr}_1\left[\mathcal{A}^{-1}(f_j,0,0)\right]$ and $\mathcal{A}^{-1}$ is weakly continuous since it is linear, it is enough to show that $f_j \to 0$ weakly in $L^2_{-1}(\Omega,\Lambda^1TM)$. Continuing our series of reductions and observing that $L^{4/3}(\Omega,\Lambda^1TM) \hookrightarrow L^2_{-1}(\Omega,\Lambda^1TM)$, it suffices to prove that

$$f_j \to 0 \text{ weakly in } L^{4/3}(\Omega,\Lambda^1TM). \tag{7.13}$$

With this objective in mind, observe first that the inclusion

$$\iota : L^2_{1,0}(\Omega,\Lambda^1TM) \hookrightarrow L^p(\Omega,\Lambda^1TM) \tag{7.14}$$

is well-defined and bounded for $1 < p \leq 4$, as well as compact for $1 < p < 4$. In particular,

$$u_j \to u_0 \text{ in any } L^p(\Omega,\Lambda^1TM) \text{ with } p < 4. \tag{7.15}$$

Going further,

$$\begin{aligned} \|f_j\|_{L^q(\Omega,\Lambda^1TM)} &\leq C\|u_j - u_0\|_{L^p(\Omega,\Lambda^1TM)}\|v_j\|_{L^2_{1,0}(\Omega,\Lambda^1TM)} \\ &\leq C\|u_j - u_0\|_{L^p(\Omega,\Lambda^1TM)}, \end{aligned} \tag{7.16}$$

if $1/q = 1/p+1/2$, and $2 < p \leq 4$, (which forces $q \leq 4/3$). Hence, on account of (7.15)-(7.16),

$$f_j \to 0 \text{ in any } L^q(\Omega,\Lambda^1TM) \text{ with } q < 4/3. \tag{7.17}$$

However, $\|f_j\|_{L^{4/3}(\Omega,\Lambda^1TM)} \leq C$ as seen from (7.16) with $p = 4$, and since $L^{4/3}(\Omega,\Lambda^1TM)$ is a reflexive space, (7.13) follows with the help of (7.17). This finishes the proof of the continuity of T_f in (7.7) when $n = 4$. The remaining steps in the existence part when $n = 4$ are then carried out much as before.

Turning to uniqueness in the case when (7.3) holds, note that it suffices to treat the case $p = 2$, $s = 1/2$, regardless of the smoothness of the domain (C^1 or just Lipschitz). To this end, let us assume that (u_j,π_j), $j = 1,2$, solve (7.2) for $p = 2$, $s = 1/2$, and set $w := u_1 - u_2$, $\pi := \pi_1 - \pi_2$. Then $Lw + \nabla_{u_1}w + d\pi = -\nabla_w u_2$, and $w \in L^2_{1,0}(\Omega,TM)$, $\delta w = 0$. It follows then from Proposition 6.4 and the estimates (6.9), (7.3) that

$$\begin{aligned} \|w\|_{L^2_1(\Omega,TM)} &\leq C\|\nabla_w u_2\|_{L^2_{-1}(\Omega,TM)} \leq C\|w\|_{L^2_1(\Omega,TM)}\|u_1\|_{L^2_1(\Omega,TM)} \\ &\leq C\|w\|_{L^2_1(\Omega,TM)}\|f\|_{L^2_{-1}(\Omega,TM)} \leq C\kappa\|w\|_{L^2_1(\Omega,TM)}. \end{aligned} \tag{7.18}$$

Clearly, (7.18) forces $w = 0$ if $\kappa > 0$ is sufficiently small (so that $C\kappa < 1$). This concludes the proof of the theorem.

Next we discuss well-posedness for small data.

Theorem 7.3. *Let Ω be a connected, C^1 subdomain of M, with $\dim M = n \geq 2$. Let $1 < p < \infty$, $0 < s < 1$ such that $s + \frac{1}{p} \geq \frac{1}{2}$ and*

$$\frac{1}{p} - \frac{s}{n-1} \leq min\left\{\frac{1}{2}, \frac{1}{n-1}\right\}, \tag{7.19}$$

(with strict inequality if $n = 3$ and $s + \frac{1}{p} \geq 1$).

Then there exist two constants κ, $\kappa^* > 0$, depending only on Ω, s, p, and which have the following significance. For each f, g, h satisfying the compatibility condition (5.2) and with

$$\|f\|_{L^p_{s+\frac{1}{p}-2}(\Omega,\Lambda^1TM)} + \|h\|_{L^p_{s+\frac{1}{p}-1}(\Omega)} + \|g\|_{B^{p,p}_s(\partial\Omega,\Lambda^1TM)} \leq \kappa, \tag{7.20}$$

the boundary value problem

$$\begin{aligned}
&u \in L^p_{s+\frac{1}{p}}(\Omega, \Lambda^1TM),\\
&\pi \in L^p_{s+\frac{1}{p}-1}(\Omega),\ \langle \pi, 1\rangle = 0,\\
&Lu + \nabla_u u + d\pi = f \in L^p_{s+\frac{1}{p}-2}(\Omega, \Lambda^1TM),\\
&\delta u = h \in L^p_{s+\frac{1}{p}-1}(\Omega),\\
&Tr\, u = g \in B^{p,p}_s(\partial\Omega, \Lambda^1TM),
\end{aligned} \tag{7.21}$$

has a unique solution for which

$$\|u\|_{L^p_{s+\frac{1}{p}}(\Omega,\Lambda^1TM)} + \|\pi\|_{L^p_{s+\frac{1}{p}-1}(\Omega,\Lambda^1TM)} \leq \kappa^*. \tag{7.22}$$

Also, this solution depends on the data in a C^1 fashion.

In fact, the same results are valid in the case when $\partial\Omega$ has a small enough Lipschitz constant (relative to s, p). Moreover, similar conclusions hold when Ω is an arbitrary Lipschitz domain provided that, in addition to the above assumptions, one of the conditions in (4.2) holds if $n = 2$, one of conditions in (4.78) holds if $n = 3$, and p is sufficiently close to 2 if $n \geq 4$.

The starting point is the following abstract perturbation result.

Lemma 7.4. *Let $\mathcal{A} : X \to Y$ be a linear isomorphism between two Banach spaces and let $\mathcal{B} : X \oplus X \to Y$ be a bounded, bilinear map. Then there exist two small neighborhoods of the origin, $U \subseteq X$ and $V \subseteq Y$, respectively, so that the application*

$$X \ni x \mapsto \mathcal{A}x + \mathcal{B}(x, x) \in Y \tag{7.23}$$

is a C^1-diffeomorphism of U onto V.

Proof. By composing with $\mathcal{A}^{-1}$ we see that there is no loss of generality in assuming that $X = Y$ and $\mathcal{A} = I$, the identity operator. In this setting, consider the application

$$\mathcal{F} : X \longrightarrow X, \qquad \mathcal{F}(x) := x + \mathcal{B}(x, x), \quad x \in X, \tag{7.24}$$

and observe that $\mathcal{F}(0) = 0$, and $\mathcal{F} \in C^1$ with derivative

$$d\mathcal{F} : X \longrightarrow \mathcal{L}(X), \quad d\mathcal{F}(x)y = y + \mathcal{B}(x, y) + \mathcal{B}(y, x). \tag{7.25}$$

In particular, $d\mathcal{F}(0) = I$, so the desired conclusion now follows from the Inverse Mapping Theorem.

We shall also need a bilinear estimate in Sobolev spaces.

Lemma 7.5. *Assume that Ω is a Lipschitz domain and that s, p are as in the statement of Theorem 7.3. Then*

$$\|\nabla_u v\|_{L^p_{s+1/p-2}(\Omega,\Lambda^1 TM)} \leq C\|u\|_{L^p_{s+1/p}(\Omega,\Lambda^1 TM)} \cdot \|v\|_{L^p_{s+1/p}(\Omega,\Lambda^1 TM)}, \tag{7.26}$$

uniformly in u and v.

Proof. First we record a pointwise multiplication result for Sobolev spaces in Lipschitz domains, to the effect that for $1 < p_0, p_1, p^* < \infty$ and $s_0, s_1, s^* \geq 0$,

$$L^{p_0}_{s_0}(\Omega) \cdot L^{p_1}_{s_1}(\Omega) \hookrightarrow L^{p^*}_{s^*}(\Omega), \tag{7.27}$$

continuously, whenever

$$\begin{aligned} & s^* \leq \min\{s_0, s_1\}, \qquad s_0 + s_1 > n(1/p_0 + 1/p_1 - 1), \\ & s^* - n/p^* < \min\big\{s_0 - n/p_0,\ s_1 - n/p_1,\ s_0 + s_1 - n/p_0 - n/p_1\big\}. \end{aligned} \tag{7.28}$$

In fact, equality can be allowed in the second line of (7.28) provided that $s_i \neq n/p_i$, $i = 0, 1$. Indeed, this is well known when Ω is replaced by $\mathbb{R}^n$, and this latter case readily implies the desired result.

We now observe that for $0 < s + 1/p < 1$, the estimate (7.26) reduces to verifying that

$$L^p_{s+\frac{1}{p}}(\Omega) \cdot L^{p'}_{2-s-\frac{1}{p}}(\Omega) \hookrightarrow L^{p'}_{1-s-\frac{1}{p}}(\Omega), \tag{7.29}$$

where $1/p + 1/p' = 1$. This, in turn, can be checked with the help of (7.27)-(7.28) which work in this context, given our assumptions on s, p. Going further, if $1 \leq s + 1/p$, then it suffices to ensure that

$$L^p_{s+\frac{1}{p}}(\Omega) \cdot L^p_{s+\frac{1}{p}-1}(\Omega) \cdot L^{p'}_{2-s-\frac{1}{p}}(\Omega) \hookrightarrow L^1(\Omega). \tag{7.30}$$

Once again, we employ (7.27)-(7.28), which leads to the kind of restrictions on the indices s, p described in the lemma.

We are finally ready for the

Proof (Proof of Theorem 7.3.). The idea is to use Lemma 7.5 in which we take $\mathcal{A}$ as in (5.6) and

$$\mathcal{B}\Big((u_1,\pi_1),(u_2,\pi_2)\Big) := (\nabla_{u_1} u_2, 0, 0). \tag{7.31}$$

Then the desired conclusion follows from Theorem 5.1, Lemma 7.5 and Lemma 7.3.

Remark. When $h = 0$ in (7.21), the assumptions on the indices s, p made in Theorem 7.3 can be relaxed to

$$1 < p < \infty, \quad 0 < s < 1, \quad \frac{1}{p} - \frac{s}{n-1} \leq \frac{1}{n-1}, \tag{7.32}$$

if $\partial\Omega \in C^1$; when $\partial\Omega$ is merely Lipschitz, one further assumes the extra hypotheses listed in the last part of the statement of Theorem 7.3.

The proof is virtually the same as before with the notable difference that, this time, one invokes Lemma 6.2 instead of Lemma 7.5.

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