A FAMILY OF SINGULAR OSCILLATORY INTEGRAL OPERATORS AND FAILURE OF WEAK AMENABILITY

Michael Cowling, Brian Dorofaeff, Andreas Seeger, and James Wright

1. Weak amenability and $\Lambda(G)$

Let G be a locally compact Hausdorff topological group, equipped with a left-invariant Haar measure, written dx or dy in integrals. We write $L^p(G)$ for the usual Lebesgue space of (equivalence classes of) functions on G. In this section, the symbol λ will denote the left regular representation of G on $L^2(G)$, and $f * g$ will denote the convolution of functions f and g on G.

1.1. The Fourier algebra and pointwise multipliers.

A matrix coefficient of the left regular representation is a function of the form

$$x \mapsto \langle \lambda(x)h, k\rangle = \int h(x^{-1}y)\overline{k(y)}\, dy$$

where h and k lie in $L^2(G)$. The *Fourier algebra* of G, denoted by $A(G)$, is defined to be the Banach space of all these, that is,

$$A(G) = \{\langle \lambda(\cdot)h, k\rangle : h, k \in L^2(G)\}$$

(which is actually a linear space), equipped with the norm

$$\|\varphi\|_A = \inf\{\|h\|_2\|k\|_2 : \varphi = \langle \lambda(\cdot)h, k\rangle\}.$$

The infimum is in fact attained, see [12]. If G is abelian, a function in $A(G)$ is the Fourier transform of a function in $L^1(\widehat{G})$, where $\widehat{G}$ is the dual group of G.

All functions in $A(G)$ are continuous and vanish at infinity. The Fourier algebra forms a commutative Banach algebra under pointwise operations, with Gel'fand spectrum G. It has a unit (the function 1) if and only if G is compact. For proofs of these results and for much more information about the Fourier algebra, see the original article by Eymard [12] or the book by Pier [29].

The *group von Neumann algebra* $VN(G)$ is defined to be the set of all bounded linear operators on $L^2(G)$ commuting with right translations. Suppose that $f \in L^1(G)$. We associate to f the left convolution operator $\lambda[f]$ on $L^2(G)$, defined by

$$\lambda[f]h(x) = \int_G \lambda(y)h(x)\, f(y)\, dy = f * h(x).$$

This operator lies in $VN(G)$. The function f also gives rise to an element of the topological dual space $A(G)^*$ of $A(G)$, by integration: one defines L_f in $A(G)^*$ to be the linear functional $\varphi \mapsto$

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$\int_G \varphi(x)\, f(x)\, dx$. The association between the operator $\lambda[f]$ and the linear functional L_f extends to identify the group von Neumann algebra $VN(G)$ with the dual space $A(G)^*$. More precisely, for any F in $A(G)^*$, there exists a unique F' in $VN(G)$ such that

$$\langle F'(h), k\rangle = F(\langle \lambda(\cdot)h, k\rangle) \qquad \forall h, k \in L^2(G).$$

The mapping $F \mapsto F'$ is an isometric isomorphism; it also carries the weak-star topology of $A(G)^*$ to the ultraweak topology of $VN(G)$. The set $\{L_f : f \in L^1(G)\}$ is weak-star dense in $A(G)^*$ and the set $\{\lambda[f] : f \in L^1(G)\}$ is ultraweakly dense in $VN(G)$. The correspondence between F and F' is the unique continuous extension of the map $\lambda[f] \mapsto L_f$. For proofs of these facts, see [12] or [29].

On a Lie group G, $\mathcal{D}(G) \subset A(G)$, where $\mathcal{D}$ denotes the space of compactly supported smooth functions. We may think of elements of $A(G)^*$ as distributions on G, and of elements of $VN(G)$ as convolutions by these distributions.

We shall need the notion of a completely bounded operator on a von Neumann algebra. Suppose that $\mathcal{M}$ is a von Neumann algebra and $T : \mathcal{M} \to \mathcal{M}$ is a continuous linear operator. Let $\mathcal{M}^{(n)}$ be the algebra of $n \times n$ matrices with entries in $\mathcal{M}$ and let I_n be the $n \times n$ identity matrix. Define the extension $T \otimes I_n$ to $\mathcal{M}^{(n)}$ by $(T \otimes I_n F)_{ij} = T(F_{ij})$. Then T is said to be *completely bounded* if

$$c_T = \sup_n \|T \otimes I_n\| < \infty.$$

We write $\|T\|_{cb}$ for the completely bounded operator norm c_T. Much more about completely bounded operators may be found in [28].

We define $MA(G)$, the *space of (pointwise) multipliers* of $A(G)$, to be the set of all continuous functions φ on G such that the pointwise product $\varphi\psi$ lies in $A(G)$ for all ψ in $A(G)$. A multiplier $\varphi \in MA(G)$ may be identified with the multiplication operator m_φ on $A(G)$ given by $m_\varphi : \psi \mapsto \varphi\psi$, and we equip $MA(G)$ with the corresponding operator norm.

We also define $M_0A(G)$, the *space of completely bounded multipliers of* $A(G)$, also called Herz–Schur multipliers (see, e.g., [4]), to be the set of all continuous functions φ on G such that the adjoint operator m_φ^* is completely bounded as an operator on $VN(G)$. We define $\|\varphi\|_{M_0A(G)}$ to be the completely bounded operator norm $\|m_\varphi^*\|_{cb}$. This space is smaller than $MA(G)$, and the norm is larger than the $MA(G)$-norm. For further information about these spaces, see the articles by Cowling [4] and De Cannière and Haagerup [8]; in particular it is shown in [8] that $\|m\|_{M_0A(G)} = \sup_H \|m \otimes 1_H\|_{MA(G\times H)}$ where the supremum is taken over all locally compact groups H.

Both $MA(G)$ and $M_0A(G)$ form Banach algebras under pointwise multiplication. We have the inclusions $A(G) \subseteq B(G) \subseteq M_0A(G) \subseteq MA(G)$ where $B(G)$ is the Fourier–Stieltjes algebra consisting of matrix coefficients of unitary representations. If the group G is *amenable*, i.e., there exists a left invariant mean on $L^\infty(G)$, then both of these algebras coincide with the Fourier–Stieltjes algebra $B(G)$; in fact the equality $B(G) = MA(G)$ is a characterization of amenability, see [26]. In general, these inclusions are proper; in fact, specific examples of functions in $M_0A(G)$ arise as matrix coefficients of uniformly bounded representations which need not be equivalent to unitary ones (see [25], [30]).

1.2. Approximate units.

Let L be a positive real number. Then $A(G)$ is said to have an *approximate unit bounded by* L if there exists a directed set I and a net $\{\varphi_i : i \in I\}$ of functions in $A(G)$ such that

$$\lim_{i\in I} \|\psi - \varphi_i\psi\|_A = 0 \qquad \forall \psi \in A(G) \tag{1.2.1}$$

and

$$\|\varphi_i\|_A \le L \qquad \forall i \in I. \tag{1.2.2}$$

It is known that $A(G)$ has an approximate unit bounded by a positive real number L if and only if $A(G)$ has an approximate unit bounded by 1; this is one of the many equivalent conditions for G to be *amenable* (Leptin [24], see also Herz [17]). When G is amenable, the existence of the approximate unit implies that

$$\|\varphi\|_A = \|\varphi\|_{M_0A} = \|\varphi\|_{MA} \qquad \forall \varphi \in A(G).$$

For more information about amenability, see [29].

One may weaken the existence criterion on the approximate unit as follows. Given a positive real number L, we say that $A(G)$ has an *L-completely bounded approximate unit*, if there exists a net $\{\varphi_i : i \in I\}$ of functions in $A(G)$ such that (1.2.1) holds and

$$\|\varphi_i\|_{M_0A} \le L \qquad \forall i \in I \tag{1.2.3}$$

We define the number $\Lambda(G)$ to be the infimum of all the numbers L for which there exists an L-completely bounded approximate unit on $A(G)$, with the convention that $\Lambda(G) = \infty$ if no such approximate unit exists. The group G is said to be *weakly amenable* if $\Lambda(G) < \infty$.

Finally we say that $A(G)$ has an *L-multiplier bounded approximate unit*, if there is a net $\{\varphi_i : i \in I\}$ of functions in $A(G)$ such that (1.2.1) holds and

$$\|\varphi_i\|_{MA} \le L \qquad \forall i \in I.$$

A multiplier bounded approximate unit is simply an L-multiplier bounded approximate unit, for some $L < \infty$.

Clearly $\Lambda(G) \in [1, \infty]$, because $\|\cdot\|_\infty \le \|\cdot\|_{M_0A(G)}$, but in every known case, $\Lambda(G)$ is an extended integer. Much of what is known about $\Lambda(G)$ for locally compact groups is summarized in the following list. For details see the articles by Haagerup [13], [14], Cowling [4], [5], De Cannière and Haagerup [8], Cowling and Haagerup [6], Lemvig Hansen [23], Bożejko and Picardello [1], Dorofaeff [9], [10].

1.2.1. *Suppose that G, G_1, and G_2 are locally compact groups.*

(i) If G_1 is isomorphic to G_2, then $\Lambda(G_1) = \Lambda(G_2)$.

(ii) If K is a compact normal subgroup of G, then $\Lambda(G) = \Lambda(G/K)$.

(iii) If G_1 is a closed subgroup of G_2, then $\Lambda(G_1) \le \Lambda(G_2)$, with equality if G_2/G_1 admits a finite G_2 invariant measure.

(iv) If G is the direct product group $G_1 \times G_2$, then $\Lambda(G) = \Lambda(G_1)\,\Lambda(G_2)$.

(v) If G is discrete and Z is a central subgroup of G, then $\Lambda(G) \le \Lambda(G/Z)$.

(vi) If G is amenable, then $\Lambda(G) = 1$.

(vii) If G is a free group, then $\Lambda(G) = 1$.

(viii) If G is an amalgamated product $G = *_A G_i$, where each G_i is an amenable locally compact group, and A is a compact open subgroup of all G_i, then $\Lambda(G) = 1$.

(ix) If G is locally isomorphic to $\mathrm{SO}(1,n)$ or to $\mathrm{SU}(1,n)$, then $\Lambda(G) = 1$.

(x) If G is locally isomorphic to $\mathrm{Sp}(1,n)$, then $\Lambda(G) = 2n - 1$.

(xi) If G is locally isomorphic to $F_{4(-20)}$, then $\Lambda(G) = 21$.

(xii) If G is a simple Lie group of real rank at least two, then $\Lambda(G) = \infty$.

For generalizations of these ideas to von Neumann algebras, see Haagerup [13], [14], Cowling and Haagerup [6] and for generalizations to ergodic systems and dynamical systems, see Cowling and Zimmer [7] and Jolissaint [19]. These ideas are loosely related to Property (T) and the Haagerup Property, which are investigated in detail in the books by Zimmer [36], by de la Harpe and Valette [16] and by Chérix, Cowling, Jolissaint, Julg and Valette [3].

We shall make use of the following results, without further reference.

1.2.2. *Suppose that H is a closed subgroup of the locally compact group G, that T is a distribution on H, and that φ is a function on G. Then:*

(i) if $\varphi \in A(G)$, then $\varphi\big|_H \in A(H)$ and $\|\varphi\big|_H\|_{A(H)} \leq \|\varphi\|_{A(G)}$

(ii) if $T \in A(H)^*$, then, considered as a distribution on G, $T \in A(G)^*$ and $\|T\|_{A(G)^*} = \|T\|_{A(H)^*}$

(iii) if $\varphi \in M_0A(G)$, then $\varphi\big|_H \in M_0A(H)$ and $\|\varphi\big|_H\|_{M_0A(H)} \leq \|\varphi\|_{M_0A(G)}$

(iv) if $\varphi \in MA(G)$, then $\varphi\big|_H \in MA(H)$ and $\|\varphi\big|_H\|_{MA(H)} \leq \|\varphi\|_{MA(G)}$.

See [17, Thm. 1] and [8, Prop. 1.12] for the proofs.

1.2.3. *If $\{\varphi_i : i \in I\}$ is an L-completely bounded approximate unit on $A(G)$, then $\varphi_i \to 1$ uniformly on compact subsets of G. Conversely, if there exists a net $\{\varphi_i : i \in I\}$ of $A(G)$-functions such that $\|\varphi_i\|_{M_0A(G)} \leq L$ and $\varphi_i \to 1$ uniformly on compact sets, then there exists an L-completely bounded approximate unit of compactly supported $A(G)$-functions, $\{\tilde{\varphi}_j : j \in J\}$ say. If G is a Lie group, then we may also assume that $\tilde{\varphi}_j \in \mathcal{D}(G)$ for all j in J.*

This result also holds when "L-completely bounded" is replaced by "L-multiplier bounded".

For the proof, see [6, Prop. 1.1].

1.2.4. *Let K be a compact normal subgroup of the locally compact group G.*

(i) Let $m \in MA(G)$ and define for $\widetilde{m}(gK) = \int_K m(gk)dk$ (where dk is normalized Haar measure). Then $\widetilde{m} \in MA(G/K)$ with $\|\widetilde{m}\|_{MA(G/K)} \leq \|m\|_{MA(G)}$.

(ii) The statement (i) remains true for $MA(G)$ replaced with $MA_0(G)$; moreover the space $MA_0(G/K)$ may be isometrically identified with the subspace of functions in $MA_0(G)$ which are constants on the cosets of K in G. Furthermore $\Lambda(G/K) = \Lambda(G)$.

(i) is immediate. For (ii) see [6, Prop. 1.3] (one uses the definition [6, (0.3)] to verify the nontrivial part of (ii)).

1.2.5. *Suppose that $G = SK$ is a (set) decomposition of G as a product of an amenable closed subgroup S and a compact subgroup K, and that ν is normalized Haar measure on K. Suppose further that $\tilde{A}(G)$ is one of $A(G)$ or $M_0A(G)$ or $MA(G)$. Then for any $\varphi \in \tilde{A}(G)$ the average $\dot{\varphi}$, defined by*

$$\dot{\varphi}(x) = \int_{K\times K} \varphi(kxk')\, d\nu(k)\, d\nu(k'),$$

belongs to $\tilde{A}(G)$. Further, $\|\dot{\varphi}\|_{\tilde{A}(G)} \leq \|\varphi\|_{\tilde{A}(G)}$.

For the proof, see [6, Prop. 1.6]. The point of the lemma is that, by averaging, we may assume that any given approximate unit of $A(G)$-functions bounded in the $\tilde{A}(G)$-norm is K-biinvariant, with the same bound. The above lemma also holds if we choose compactly supported smooth functions, and these properties are preserved by averaging.

What lies ahead.

For a connected noncompact simple Lie group G with finite center and real rank at least two, the invariant $\Lambda(G)$ takes the value infinity. This result was proved by Haagerup [14]. His proof involves investigating certain semidirect products, namely $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ and $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^1$, where H^1 is the Heisenberg group of dimension three. He shows that these semidirect products do not admit multiplier bounded approximate units, and hence deduces that Λ is infinite for both the semidirect products and then, by structure theory, for any noncompact simple Lie group G with finite center and real rank at least two. These semidirect products are the smallest members of two families of semidirect products, for which it turns out to be interesting to calculate Λ (see Section 8). The first family is formed with the action of the unique irreducible representation of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^n$. It was shown by Dorofaeff [9] that all these groups have infinite Λ; this was used to show that that the original hypothesis of finite center in Haagerup's proof of 1.2.1 (xii) is redundant ([10]). The second family is where $\mathrm{SL}(2,\mathbb{R})$ acts on the Heisenberg group H^n of dimension $2n+1$ by fixing the center and operating on the vector space $\mathbb{R}^{2n}$ by the unique irreducible representation of dimension $2n$.

We consider this family of semidirect products and show they do not admit multiplier bounded approximate units; in particular $\Lambda(\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^n) = \infty$. Given this and earlier results, and some structure theory, it is now possible to compute $\Lambda(G)$ for any real algebraic Lie group G, or indeed for any Lie group G whose Levi factor has finite center.

Main Theorem. *Let G be a real Lie group with Lie algebra $\mathfrak{g}$, and let $\mathfrak{s}\oplus\mathfrak{r}$ be the Levi decomposition of $\mathfrak{g}$, where $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$ and $\mathfrak{s}$ is a semisimple summand, and let $\mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$ be the decomposition of $\mathfrak{s}$ as a sum of simple ideals. Let S be a maximal analytic semisimple subgroup of G corresponding to $\mathfrak{s}$, and let S_i be the subgroup associated to $\mathfrak{s}_i$, where $i = 1, \dots, m$. Suppose that S has finite center.*

Then G is weakly amenable if and only if one of the following two conditions is satisfied for each $i = 1, \dots, m$:
Either
(*) S_i is compact
or
(**) S_i is noncompact, of real rank 1, and the action of $\mathfrak{s}_i$ on $\mathfrak{r}$ is trivial, i.e., $[\mathfrak{s}_i, \mathfrak{r}] = 0$.

If for every $i \in \{1, \dots, m\}$, either (*) or (**) is satisfied then $\Lambda(G) = \prod_{i=1}^m \Lambda(S_i)$ and $\Lambda(G)$ can be computed by consulting the list (1.2.1).

If for at least one $i \in \{1, \dots, m\}$ neither (*) nor (**) holds, then $A(G)$ does not admit any multiplier bounded approximate unit.

Structure of the paper. The main part of this paper (Sections 2–7) is devoted to the proof that the Fourier algebra of $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^n$ does not admit multiplier bounded approximate units, and consequently we have $\Lambda(\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^n) = \infty$. Using a modification of Haagerup's approach for the case $n = 1$ [14], one can reduce matters to the estimation of a singular oscillatory integral operator; this reduction is described in Section 2. The estimation of the integral operator, which is rather nontrivial, is carried out in Sections 3–7. In Section 8 we consider general Lie groups under the assumption that the Levi part has finite center. Here we use facts from the structure theory of Lie groups to show that if for at least one $i \in \{1, \dots, m\}$ neither condition (*) nor condition (**) in the Theorem holds, then G does not admit multiplier bounded approximate units. This will be combined with previously known results to complete the proof of the main theorem.

2. A family of semidirect products

Fix a positive integer n. Throughout this chapter we shall consider the group

$$G_n = \mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^n,$$

where $\mathrm{SL}(2,\mathbb{R})$ acts on the Heisenberg group H^n by the unique irreducible representation of dimension $2n$, fixing the center. We shall reduce the proof that $\Lambda(G_n) = \infty$ to the estimation of a family of singular oscillatory integral operators. The four subsequent sections will then be dedicated to estimating these operators.

2.1. The action of $\mathrm{SL}(2,\mathbb{R})$ on the Heisenberg group.

Recall that H^n is a Lie group whose underlying manifold is $\mathbb{R}^{2n} \times \mathbb{R}$. The group multiplication may be given by the formula

$$(u,t)(u',t') = (u+u', t+t'+u^T B u'),$$

where the symplectic matrix B is defined by

$$B_{ij} = \begin{cases} (-1)^j & \text{if } i+j = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

We shall write $\{e_1, \ldots, e_{2n}\}$ for the standard basis of $\mathbb{R}^{2n}$.

We shall now describe the action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^{2n}$ by the irreducible representation π_{2n} of dimension $2n$ which is unique up to isomorphism (see, e.g., [22, p. 107]). For $j = 1, \ldots, 2n$, let

$$\alpha_j = \binom{2n-1}{j-1}^{1/2}.$$

We identify $\mathbb{R}^{2n}$ with the space $\mathbb{P}_{2n}$ of homogeneous polynomials in two variables of degree $2n-1$ by associating $(u_1, \ldots, u_{2n})$ with the polynomial

$$P : (x,y) \mapsto \sum_{j=1}^{2n} \alpha_j u_j x^{2n-j} y^{j-1}, \tag{2.1.1}$$

and define the action of A in $\mathrm{SL}(2,\mathbb{R})$ by

$$\pi_{2n}(A)P(x,y) = P((x,y)A) = P(ax+cy, bx+dy) \qquad \forall (x,y) \in \mathbb{R}^2,$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ (see [20]). If P is as in (2.1.1), then a computation shows that

$$\pi_{2n}(A)P(x,y) = \sum_{i=1}^{2n} \left[Z(A)u\right]_i \alpha_i x^{2n-i} y^{i-1},$$

where the $2n \times 2n$ matrix $Z(A)$ is given by

$$\left(Z(A)\right)_{ij} = \sum_{l=0}^{2n} \binom{j-1}{l} \binom{2n-j}{2n-i-l} \alpha_i^{-1} \alpha_j \, a^{2n-i-l} \, b^l \, c^{i+l-j} \, d^{j-l-1} \tag{2.1.2}$$

(see [9]). Here we use the standard convention that $\binom{k}{l} = 0$ if l is negative or $l > k$.

In order to extend the action on $\mathbb{R}^{2n}$ to an action on H^n we need to show that the action on $\mathbb{R}^{2n}$ is symplectic.

Lemma 2.1.1. *The map Z is a symplectic action on $\mathbb{R}^{2n}$, i.e.,*

$$Z(A)^T \, B \, Z(A) = B \tag{2.1.3}$$

for each $A \in \mathrm{SL}(2,\mathbb{R})$. Define $\bar{Z}(A) : \mathrm{H}^n \to \mathrm{H}^n$ by

$$\bar{Z}(A)(u,t) = (Z(A)u, t);$$

then $\bar{Z}(A)$ is an automorphism of H^n and $\bar{Z}$ is an action of $\mathrm{SL}(2,\mathbb{R})$ on H^n.

Proof. Recall that $\alpha_j = \binom{2n-1}{j-1}^{1/2}$. From (2.1.2) and our choice of α one checks that

$$Z(A^T) = Z(A)^T.$$

Observe also that

$$B = Z(J) \qquad \text{where} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For any A in $\mathrm{SL}(2,\mathbb{R})$, a direct matrix calculation shows $A^T J A = J$ and so

$$Z(A)^T BZ(A) = Z(A)^T Z(J) Z(A) = Z(A^T J A) = Z(J)$$

and therefore (2.1.3) holds. The fact that $\bar{Z}(A)$ is an automorphism of H^n follows immediately from (2.1.3); hence $\bar{Z}$ is an action on H^n. $\square$

We may now describe the semidirect product group G_n. As a manifold, this is $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^{2n} \times \mathbb{R}$. The product in G_n is defined by

$$(A,u,t)(A',u',t') = (AA', u + Z(A)u', t + t' + u^T BZ(A)u')$$

and the inverse is given by

$$(A,u,t)^{-1} = (A^{-1}, -Z(A^{-1})u, -t), \tag{2.1.4}$$

for all (A,u,t) and (A',u',t') in G_n. The closed subgroups $\{(I,u,t) : u \in \mathbb{R}^{2n},\ t \in \mathbb{R}\}$ (where I is the identity of $\mathrm{SL}(2,\mathbb{R})$) and $\{(A,0,0) : A \in \mathrm{SL}(2,\mathbb{R})\}$ may be identified with H^n and $\mathrm{SL}(2,\mathbb{R})$. Given (A',u,t) and $(A,0,0)$ in G_n, it follows that

$$(A,0,0)(A',u,t)(A,0,0)^{-1} = (AA'A^{-1}, Z(A)u, t),$$

which shows that H^n is normalized by $\mathrm{SL}(2,\mathbb{R})$.

There are several important subgroups and elements of G_n which we now identify. We denote by K the compact subgroup $\mathrm{SO}(2,\mathbb{R})$ of $\mathrm{SL}(2,\mathbb{R})$, considered as a subgroup of G_n. For b in $\mathbb{R}$, we define

$$k_b^{\pm} = \pm\beta(b)^{-1} \begin{pmatrix} b/2 & 1 \\ -1 & b/2 \end{pmatrix}, \qquad n_b = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad h_b = \begin{pmatrix} 0 & \beta(b) \\ -\beta(b)^{-1} & 0 \end{pmatrix},$$

where $\beta(b) = (1 + b^2/4)^{1/2}$. Then $k_b^{\pm} \in K$. We write N for the nilpotent subgroup $\{n_b : b \in \mathbb{R}\}$. For future purposes, we observe the following lemma.

Lemma 2.1.2. *For all b in $\mathbb{R}$, we have*

$$\begin{aligned} k_b^+ \, n_b \, k_b^- &= n_{-b}, \\ n_{b/2} \, k_b^+ \, n_{b/2} &= h_b. \end{aligned} \tag{2.1.5}$$

Further,

$$Z(h_b)(u_n e_n + u_{n+1} e_{n+1}) = (-1)^n (\beta(b)^{-1} u_n e_{n+1} - \beta(b)\, u_{n+1} e_n).$$

Finally,

$$\big(Z(n_b)\big)_{ij} = \begin{cases} \alpha_i^{-1} \alpha_j \binom{j-1}{j-i} b^{j-i} & \text{if } j > i \\ 1 & \text{if } j = i \\ 0 & \text{if } j < i \end{cases} \tag{2.1.6}$$

and, in particular, $Z(n_b)_{n,n+1} = nb$.

Proof. These are all straightforward computations which will be omitted. $\square$

2.2. Two nilpotent subgroups.

We write G for G_n, and H for the subgroup of G of all elements of the form (n_b, u, t), where $b \in \mathbb{R}$ and $u \in \mathbb{R}^{2n}$. Let V_k denote the subspace $\operatorname{span}\{e_1, \dots, e_k\}$ of $\mathbb{R}^{2n}$ (when $k = 1, \dots, 2n$). Since N is a subgroup of $\mathrm{SL}(2,\mathbb{R})$ and the matrix $Z(n_b)$ is upper triangular for all b in $\mathbb{R}$, this subspace is invariant under all the maps $Z(n_b)$, and the subset of G of all elements of the form (n_b, v, t), where $b \in \mathbb{R}$ and $v \in V_k$, is a subgroup of H. We write H_0 for the subgroup of G obtained in this way when $k = n+1$.

We need to understand the behavior of the restrictions of K-bi-invariant functions on G to H. It follows from formula (2.1.4) that

$$(k_b^+, 0, 0)(n_b, u, t)(k_b^-, 0, 0) = (n_{-b}, Z(k_b^+)u, t). \tag{2.2.1}$$

We define the diffeomorphism $\Omega : H \to H$ by the formula

$$\Omega(n_b, u, t) = (n_{-b}, Z(k_b^+)u, t). \tag{2.2.2}$$

Lemma 2.2.1. *If $\varphi \in \mathcal{D}(G)$ and φ is K-bi-invariant, then $\varphi|_H \circ \Omega = \varphi|_H$.*

Proof. Since φ is K-bi-invariant, we have

$$\varphi(n_b, u, t) = \varphi\big((k_b^+, 0, 0)(n_b, u, t)(k_b^-, 0, 0)\big)$$

for all (n_b, u, t) in H and the assertion follows from formulae (2.2.1) and (2.2.2). □

2.3. Some distributions on H_0.

We will define a family of distributions on H_0, using two iterated principal value integrals. To clarify the sense in which these are to be interpreted, and because it will be useful later, we first discuss certain principal value integrals on $\mathbb{R}^2$. For Schwartz functions $\psi \in \mathcal{S}(\mathbb{R}^2)$, let

$$\begin{aligned} D(\psi) &= \text{p.v.} \iint \frac{\psi(s_1, s_2)}{s_2^2 - s_1^2}\, ds_1\, ds_2 \\ &= \lim_{\epsilon \to 0+} \lim_{\delta \to 0+} \int_{|s_2| > \epsilon} \frac{1}{2s_2} \Big(\int_{|s_2 + s_1| > \delta} \frac{\psi(s_1, s_2)}{s_2 + s_1} ds_1 \; + \int_{|s_2 - s_1| > \delta} \frac{\psi(s_1, s_2)}{s_2 - s_1} ds_1 \Big) ds_2. \end{aligned} \tag{2.3.1}$$

It is routine to show that D is a tempered distribution. We shall also need a modification $\widetilde{D}$ defined by

$$\widetilde{D}(\psi) = D(\widetilde{\psi}) \text{ where } \widetilde{\psi}(y_1, y_2) = \psi(y_2, y_1). \tag{2.3.2}$$

The distributions D and $\widetilde{D}$ satisfy

$$D(\psi) + \widetilde{D}(\psi) = \pi^2 \psi(0,0) \tag{2.3.3}$$

for all Schwartz functions; this fact was used by Haagerup [14] and called the failure of Fubini's theorem, since it can be rewritten in the form

$$\text{p.v.} \iint \frac{\psi(s_1, s_2)}{s_2^2 - s_1^2} \big(ds_1 ds_2 - ds_2 ds_1 \big) = \pi^2 \psi(0,0).$$

The verification of formula (2.3.3) can be found in [9]; it relies on a Fourier transform calculation and the fact that $D(e^{-i\langle \cdot, \tau \rangle})$ is equal to π^2 if $\tau_1^2 > \tau_2^2$ and to 0 if $\tau_1^2 < \tau_2^2$.

For fixed $b \in \mathbb{R}$, define $Q_b : \mathbb{R}^2 \to H_0 \subset G$ by

$$Q_b(s_1, s_2) = (n_b, Z(n_{b/2})(s_1 e_n + s_2 \beta(b)^{-1} e_{n+1}), 0) \tag{2.3.4}$$

where, as before, $\beta(b) = (1 + b^2/4)^{1/2}$. For a test function $\phi \in \mathcal{D}(G)$, let $Q_b^* \phi$ be the pullback of ϕ to $\mathbb{R}^2$ defined (as usual) by $Q_b^*\phi(s_1, s_2) = \phi(Q_b(s_1, s_2))$. We now define the distribution D_R on G, for all R in $\mathbb{R}^+$, by the formula

$$D_R(\phi) = \int_{-R}^{R} D(Q_b^* \phi) \frac{db}{\beta(b)} \tag{2.3.5}$$

We may view D_R as a distribution on H or on G, with support in H_0, if we wish.

Lemma 2.3.1. *Suppose that $\varphi \in \mathcal{D}(G)$ and φ is K-bi-invariant. Then*

$$D_R(\varphi) = \frac{\pi^2}{2} \int_{-R}^{R} \frac{\varphi(n_b, 0, 0)}{(1 + b^2/4)^{1/2}} \, db. \tag{2.3.6}$$

In particular, if $\{\varphi_n\}_{n \in \mathbb{N}}$ is a sequence of K-bi-invariant $\mathcal{D}(G)$-functions, and $\varphi_n \to 1$ uniformly on compact subsets of G as $n \to \infty$, then

$$\lim_{n \to \infty} D_R(\varphi_n) = 2\pi^2 \sinh^{-1}(R/2). \tag{2.3.7}$$

Both formulae remain valid if D_R is considered as a distribution on H or H_0 and applied to restrictions of K-bi-invariant functions to H or H_0.

Proof. Recall from Lemma 2.2.1 that if $\varphi \in \mathcal{D}(G)$ and φ is K-bi-invariant, then $\varphi|_H = \varphi|_H \circ \Omega$. Now we compute for arbitrary $\phi \in \mathcal{D}(G)$

$$\begin{aligned} \phi \circ \Omega(Q_b(s_1, s_2)) &= \phi(n_{-b}, Z(k_b^+) Z(n_{b/2})(s_1 e_n + s_2 \beta(b)^{-1} e_{n+1}), 0) \\ &= \phi(n_{-b}, Z(n_{-b/2}) Z(h_b)(s_1 e_n + s_2 \beta(b)^{-1} e_{n+1}), 0) \\ &= \phi(n_{-b}, Z(n_{-b/2})(-1)^n (\beta(b)^{-1} s_1 e_{n+1} - s_2 e_n), 0). \end{aligned}$$

Here we have used the definition of Ω and Q_b, and the relation $k_b^+ n_{b/2} = n_{-b/2} h_b$ (which follows from formula (2.1.5)). Since D is even on $\mathbb{R}^2$, it follows that

$$D(Q_b^*(\phi \circ \Omega)) = \widetilde{D}(Q_{-b}^* \phi),$$

and therefore, since β is even,

$$D_R(\phi \circ \Omega) = \int_{-R}^{R} \widetilde{D}(Q_b^* \phi) \beta(b)^{-1} db. \tag{2.3.8}$$

Now we assume that φ is K-bi-invariant and use (2.3.3). Then

$$\begin{aligned} D_R(\varphi) &= \frac{1}{2} \int_{-R}^{R} \big(D(Q_b^* \varphi) + \widetilde{D}(Q_b^* \varphi)\big) \beta(b)^{-1} db \\ &= \frac{\pi^2}{2} \int_{-R}^{R} Q_b^* \varphi(0, 0) \beta(b)^{-1} db \\ &= \frac{\pi^2}{2} \int_{-R}^{R} \varphi(n_b, 0, 0) \beta(b)^{-1} db, \end{aligned}$$

as required.

The formula (2.3.7) follows by passing to the limit and evaluating the integral.

The last assertion follows from our computation, since Ω maps the subset of G (or of H or H_0) consisting of all $(n_b, s_1 e_n + s_2 e_{n+1}, 0)$ into itself. $\square$

2.4. Failure of weak amenability.

We are now in a position to reduce the question of the weak amenability of G to a question of boundedness of the operators $\lambda[D_R]$ of convolution with D_R.

Proposition 2.4.1. *Suppose that $\lambda[D_R]$ lies in $VN(H_0)$, and that $\|\lambda[D_R]\| = o(\log R)$ as $R \to \infty$. Then G is not weakly amenable, i.e., $\Lambda(G) = \infty$, and further, there does not exist a multiplier bounded approximate unit on G.*

Proof. If G were weakly amenable, then there would exist L in $[1, \infty)$ and a sequence $\{\varphi_n : n \in \mathbb{N}\}$ of $\mathcal{D}(G)$-functions such that $\|\varphi_n\|_{M_0A} \leq L$ for all n in $\mathbb{N}$ and $\varphi_n \to 1$, uniformly on compact subsets of G, as $n \to \infty$. By averaging if necessary, we could suppose that all the functions φ_n were K-bi-invariant; see (1.2.5). *A fortiori*, for some L in $\mathbb{R}^+$, there would be a sequence $\{\varphi_n : n \in \mathbb{N}\}$ of K-bi-invariant $\mathcal{D}(G)$-functions satisfying the conditions $\|\varphi_n\|_{MA} \leq L$ and $\varphi_n \to 1$ as $n \to \infty$. The same would be true if there existed a multiplier bounded approximate unit on G.

Consider the sequence $\{D_R(\varphi_n|_{H_0}) : n \in \mathbb{N}\}$. Since H_0 is amenable, $A(H_0)$ has an approximate unit, whence

$$\|\varphi_n|_{H_0}\|_A = \|\varphi_n|_{H_0}\|_{MA} \leq \|\varphi_n\|_{MA} \leq L. \tag{2.4.1}$$

Thus

$$\left|D_R(\varphi_n|_{H_0})\right| \leq \|\lambda[D_R]\|_{VN}\|\varphi_n|_{H_0}\|_A \leq L\|\lambda[D_R]\|_{VN} = o(\log R).$$

However, by (2.3.7)

$$\lim_{n\to\infty} \left|D_R(\varphi_n|_{H_0})\right| = 2\pi^2 \log\left(\tfrac{R}{2} + \sqrt{\tfrac{R^2}{4} + 1}\right).$$

The last two formulae are contradictory, so the original hypothesis of the weak amenability of G must be incorrect. □

Most of this paper is dedicated to verifying the hypothesis of Proposition 2.4.1; more precisely, we shall obtain the estimate

$$\|\lambda[D_R]\|_{VN(H_0)} = O(\log\log R) \qquad \text{as } R \to \infty. \tag{2.4.2}$$

To do this, we will use Fourier analysis on H_0 to study the distributions D_R when acting on $A(H_0)$. The first stage in this process is to find a family of unitary representations $\{\pi_{\eta,\zeta} : \eta \in \mathbb{R}, \zeta \in V_n\}$ of H_0; we then describe the Plancherel formula for this group. It is a consequence of the Plancherel formula that $\|\lambda[D_R]\|_{VN}$ is equal to the supremum of the operator norms $\|\pi_{\eta,\zeta}[D_R]\|$ as η and ζ vary. We shall then identify the operators $\pi_{\eta,\zeta}[D_R]$ as singular oscillatory integral operators, which will be estimated in Sections 3–7.

2.5. Representations of the group H_0.

To simplify notation, from now on we write (b, u, t) instead of (n_b, u, t), and $P(b)$ instead of $Z(n_b)$, see (2.1.6). Then the group law may be rewritten in the form

$$(b, u, t)(b', u', t') = (b + b', u + P(b)u', t + t' + u^T BP(b)u')$$

and

$$(b, u, t)^{-1} = (-b, -P(-b)u, -t),$$

for all (b, u, t) and (b', u', t') in H_0. From formula (2.1.3), it follows that $P(-b)^T BP(-b) = B$, so

$$\begin{aligned}(b, u, t)^{-1}(b', u', 0) &= (b' - b, P(-b)(u' - u), -t - u^T Bu') \\ &= (b' - b, P(-b)(u' - u), -t + (-1)^n(u_n u'_{n+1} - u_{n+1}u'_n)).\end{aligned} \tag{2.5.1}$$

It is easy to see that the subgroup H_1 of H_0, given by

$$H_1 = \{(0, w, s) : w \in V_n,\ s \in \mathbb{R}\},$$

is normal in H_0 and abelian. Let $\mathfrak{S}$ be the subset $\{(c, ve_{n+1}, 0) \in H_0 : c, v \in \mathbb{R}\}$ of H_0. As a set, we may identify $\mathfrak{S}$ with $\mathbb{R}^2$. Any element h of H_0 may be expressed uniquely in the form σh_1, where $\sigma \in \mathfrak{S}$ and $h_1 \in H_1$. Indeed, if c, s, t, and v are in $\mathbb{R}$, while $w \in V_n$ and $u \in V_{n+1}$, then

$$(c, ve_{n+1}, 0)(0, w, s) = (c, ve_{n+1} + P(c)w, s + ve_{n+1}^T BP(c)w),$$

so

$$\begin{gathered} (c, ve_{n+1}, 0)(0, w, s) = (b, u, t) \qquad \text{if and only if} \\ c = b, \quad v = u_{n+1}, \quad w = P(-b)\operatorname{Proj}_{V_n} u, \quad \text{and} \quad s = t - (-1)^n u_{n+1} u_n, \end{gathered} \tag{2.5.2}$$

where Proj_V denotes the standard orthogonal projection onto the subspace V of $\mathbb{R}^{2n}$. As a consequence, we also note the integration formula

$$\int_{H_0} F(y)dy = \int_{H_1} \int_{\mathfrak{S}} F(\sigma z) d\sigma dz. \tag{2.5.3}$$

We define the characters $\chi_{\eta,\zeta}$ of H_1 by the formula

$$\chi_{\eta,\zeta}(0, w, s) = \exp(i(-1)^n \eta s + i\langle \zeta, w\rangle), \tag{2.5.4}$$

where $\eta \in \mathbb{R}$ and $\zeta \in V_n^*$, and induce the character $\chi_{-\eta,-\zeta}$ from H_1 to H_0. The induced representation $\pi_{\eta,\zeta}$ acts on the Hilbert space $\mathcal{H}_{\eta,\zeta}$ of all complex-valued functions ξ on H_0 such that

$$\xi((b, u, t)(0, w, s)) = \chi_{\eta,\zeta}(0, w, s)\xi(b, u, t) \qquad \forall (0, w, s) \in H_1 \quad \forall (b, u, t) \in H_0,$$

and

$$\left(\int |\xi(c, ve_{n+1}, 0)|^2 \, dc \, dv \right)^{1/2} < \infty.$$

We equip this space with the norm equal to the left hand side of this inequality. As $H_0 = \mathfrak{S} H_1$, each function in $\mathcal{H}_{\eta,\zeta}$ is determined by its restriction to $\mathfrak{S}$, and so this really is a norm on $\mathcal{H}_{\eta,\zeta}$, modulo the usual issues of identification of functions which differ on null sets. Clearly $\mathcal{H}_{\eta,\zeta}$ can be identified with $L^2(\mathfrak{S})$.

The action of the unitary representation $\pi_{\eta,\zeta}$ on a function ξ in $\mathcal{H}_{\eta,\zeta}$ is defined by the formula

$$\pi_{\eta,\zeta}(b, u, t)\xi(b', u', t') = \xi\big((b, u, t)^{-1}(b', u', t')\big).$$

In particular, using formulae (2.5.1) and (2.5.2) and we see that

$$\begin{aligned} \pi_{\eta,\zeta}(b, u, t)\xi(c, ve_{n+1}, 0) &= \xi\big((b, u, t)^{-1}(c, ve_{n+1}, 0)\big) \\ &= \xi\big(c - b, P(-b)(ve_{n+1} - u), -t + (-1)^n u_n v\big) \\ &= \xi\big((c - b, (v - u_{n+1})e_{n+1}, 0)(0, w, s)\big), \end{aligned}$$

where $(0, w, s)$ in H_1 is defined by

$$\begin{aligned} w &= P(b - c)\operatorname{Proj}_{V_n} P(-b)(ve_{n+1} - u) \\ &= P(b - c)\big[P(-b)(ve_{n+1} - u) - (v - u_{n+1})e_{n+1}\big] \\ &= P(-c)(ve_{n+1} - u) + P(b - c)(u_{n+1} - v)e_{n+1}, \end{aligned}$$

and since $P(b)_{n,n+1} = nb$ by Lemma 2.1.2,

$$\begin{aligned} s &= -t + (-1)^n u_n v - (-1)^n (v - u_{n+1})(P(-b)_{n,n+1}(v - u_{n+1}) - u_n) \\ &= -t + (-1)^n \big(nb(v - u_{n+1})^2 + u_n(2v - u_{n+1})\big). \end{aligned}$$

In conclusion,

$$\begin{aligned} &\pi_{\eta,\zeta}(b,u,t)\xi(c, ve_{n+1}, 0) \\ &= \xi(c-b, (v-u_{n+1})e_{n+1}, 0) \\ &\qquad \times \exp\big(i\eta\big[(-1)^{n+1}t + nb(v-u_{n+1})^2 + u_n(2v - u_{n+1})\big]\big) \\ &\qquad \times \exp\big(i\langle \zeta, P(-c)(ve_{n+1} - u) + P(b-c)(u_{n+1} - v)e_{n+1}\rangle\big). \end{aligned} \tag{2.5.5}$$

The elements of $\mathfrak{S}$ act by translations (here we think of $\mathfrak{S}$ as $\mathbb{R}^2$), combined with multiplications, while the action of the elements of H_1 is as follows:

$$\begin{aligned} &\pi_{\eta,\zeta}(0,w,t)\xi(c, ve_{n+1}, 0) \\ &= \xi(c, ve_{n+1}, 0)\exp(i\eta[(-1)^{n+1}t + 2w_n v] - i\langle\zeta, P(-c)w\rangle). \end{aligned}$$

Finally we extend the representation $\pi_{\eta,\zeta}$ to functions f in $L^1(H_0)$. For each η in $\mathbb{R}$ and ζ in V_n^*, we associate an operator $\pi_{\eta,\zeta}[f]$ on $L^2(\mathfrak{S})$ in the usual way by the formula

$$\pi_{\eta,\zeta}[f]\xi(\sigma) = \int_{H_0} f(x)\,\pi_{\eta,\zeta}(x)\xi(\sigma)\,dx. \tag{2.5.6}$$

This formula extends by continuity to define a Fourier transform of certain distributions on H_0.

2.6 A Plancherel formula.

In what follows we shall write χ for $\chi_{\eta,\zeta}$ and $\mathcal{H}_\chi$ for $\mathcal{H}_{\eta,\zeta}$; we also denote by $d\chi$ the measure $(2\pi)^{-n-1}d\eta\,d\zeta$ on the dual space $\widehat{H}_1$.

For Ξ in $\mathcal{D}(H_0)$ and χ in $\widehat{H}_1$, define the function Ξ_χ on H_0 by

$$\Xi_\chi(x) = \int_{H_1} \Xi(xz)\overline{\chi}(z)dz. \tag{2.6.1}$$

We note that

Lemma 2.6.1. *For all Ξ in $\mathcal{D}(H_0)$, the function Ξ_χ belongs to the Hilbert space $\mathcal{H}_\chi$. Further*

$$\|\Xi\|_{L^2(H_0)} = \Big(\int_{\widehat{H}_1} \|\Xi_\chi\|^2_{\mathcal{H}_\chi} d\chi\Big)^{1/2}$$

and the map $\Xi \mapsto (\chi \mapsto \Xi_\chi)$ extends to an isometric bijection of $L^2(H_0)$ to $L^2(\widehat{H}_1, \mathcal{H})$.

Proof. For $\Xi \in \mathcal{D}(H_0)$ and $\chi \in \widehat{H}_1$, we compute:

$$\begin{aligned} \Xi_\chi(xz') &= \int_{H_1} \Xi(xz'z)\overline{\chi}(z)dz = \int_{H_1} \Xi(xz)\overline{\chi}({z'}^{-1}z)dz \\ &= \chi(z')\int_{H_1} \Xi(xz)\overline{\chi}(z)dz = \chi(z')\Xi_\chi(x), \end{aligned}$$

so that Ξ_χ has the required covariance property. Further as σ varies over $\mathfrak{S}$, the function $\Xi_\chi(\sigma)$ varies smoothly, and as a function on $\mathfrak{S}$ it has compact support, contained in $\operatorname{supp}(\Xi)H_1 \cap \mathfrak{S}$. Moreover by the Plancherel theorem for H_1, and Fubini's theorem,

$$\begin{aligned}\int_{\widehat{H}_1} \|\Xi_\chi\|^2_{\mathcal{H}_\chi} d\chi &= \int_{\widehat{H}_1}\int_{\mathfrak{S}} |\Xi_\chi(\sigma)|^2 d\sigma d\chi = \int_{\mathfrak{S}}\int_{\widehat{H}_1} |\Xi_\chi(\sigma)|^2 d\chi d\sigma \\ &= \int_{\mathfrak{S}}\int_{H_1} |\Xi(\sigma z)|^2 dz d\sigma = \int_{H_0} |\Xi(y)|^2 dy.\end{aligned}$$

The extension to $L^2(H_0)$ is straightforward. □

Lemma 2.6.2. *Suppose that D is a distribution in H_0 and suppose that the operator norm on $L^2(\mathcal{H}_\chi)$ satisfies $\|\pi_\chi[D]\| \le A$ for all $\chi \in \widehat{H}_1$. Then $\lambda[D]$ is in $VN(H_0)$ and $\|\lambda[D]\|_{VN} \le A$.*

Proof. We shall assume that D is given by integration against a $\mathcal{D}(H_0)$ function k; the general case follows by a regularization argument. Now let Ξ and Γ be in $L^2(H_0)$. Then

$$\begin{aligned}\langle \lambda[k]\Xi, \Gamma\rangle &= \int_{H_0}\int_{H_0} k(x)\Xi(x^{-1}y)\overline{\Gamma}(y) dy dx \\ &= \int_{H_0}\int_{\mathfrak{S}}\int_{H_1} k(x)\Xi(x^{-1}\sigma z)\overline{\Gamma}(\sigma z) dz d\sigma dx \\ &= \int_{H_0}\int_{\mathfrak{S}}\int_{\widehat{H}_1} k(x)\Xi_\chi(x^{-1}\sigma)\overline{\Gamma_\chi(\sigma)} d\chi d\sigma dx \\ &= \int_{\widehat{H}_1}\int_{\mathfrak{S}}\int_{H_0} k(x)\Xi_\chi(x^{-1}\sigma)\overline{\Gamma_\chi(\sigma)} dx d\sigma d\chi \\ &= \int_{\widehat{H}_1}\int_{\mathfrak{S}} \pi_\chi[k]\Xi_\chi(\sigma)\overline{\Gamma_\chi(\sigma)} d\sigma d\chi \\ &= \int_{\widehat{H}_1} \langle \pi_\chi[k]\Xi_\chi, \Gamma_\chi\rangle_{\mathcal{H}_\chi} d\chi,\end{aligned}$$

where we used (2.5.3), the Plancherel theorem on the abelian group H_1, Fubini's theorem and the definitions of $\pi_\chi(f)$ and $\mathcal{H}_\chi$. From the hypothesis and the Cauchy–Schwarz inequality it follows that

$$\begin{aligned}\big|\langle \lambda[k]\Xi, \Gamma\rangle\big| &\le \int_{\widehat{H}_1} |\langle \pi_\chi[k]\Xi_\chi, \Gamma_\chi\rangle_{\mathcal{H}_\chi}| d\chi \\ &\le \int_{\widehat{H}_1} A\|\Xi_\chi\|_{\mathcal{H}_\chi}\|\Gamma_\chi\|_{\mathcal{H}_\chi} d\chi \\ &\le A\Big(\int_{\widehat{H}_1} \|\Xi_\chi\|^2_{\mathcal{H}_\chi} d\chi\Big)^{1/2}\Big(\int_{\widehat{H}_1} \|\Gamma_\chi\|^2_{\mathcal{H}_\chi} d\chi\Big)^{1/2} \\ &= A\|\Xi\|_{L^2(H_0)}\|\Gamma\|_{L^2(H_0)},\end{aligned}$$

by Lemma 2.6.1. Taking the supremum over all Ξ and Γ with norm ≤ 1 shows that $\|\lambda[k]\|_{VN} \le A$. □

2.7. The oscillatory singular integral operators $\pi_{\eta,\zeta}[D_R]$.

We now compute the operator-valued Fourier transform of the distributions D_R. We change notation slightly, and for ξ in $\mathcal{H}_{\eta,\zeta}$, we write $\xi(c, v)$ instead of $\xi(c, ve_{n+1}, 0)$. We also set

(2.7.1) $$q(b) = n^{-1}\langle \zeta, P(b)e_{n+1}\rangle,$$

and write $\mathcal{M}_q$ for the operator on $L^2(\mathfrak{S})$ of pointwise multiplication by the function $(c,v)\mapsto \exp(invq(-c))$. Observe that

$$\begin{aligned}
q'(b) &= \frac{\partial}{\partial b}\, n^{-1}\,\langle \zeta, \sum_{i=1}^{n+1} P(b)_{i,n+1}\, e_i\rangle \\
&= n^{-1}\,\frac{\partial}{\partial b}\,\langle \zeta, \sum_{i=1}^{n+1} \binom{n}{n+1-i}\binom{2n-1}{n}^{1/2}\binom{2n-1}{i-1}^{-1/2} b^{n+1-i}\, e_i\rangle \\
&= \langle \zeta, \sum_{i=1}^{n} \binom{n-1}{n-i}\binom{2n-1}{n-1}^{1/2}\binom{2n-1}{i-1}^{-1/2} b^{n-i}\, e_i\rangle \\
&= \langle \zeta, \sum_{i=1}^{n} P(b)_{in}\, e_i\rangle = \langle \zeta, P(b)e_n\rangle .
\end{aligned}$$

Then, rewriting formula (2.5.5) we have shown that

$$\begin{aligned}
&\pi_{\eta,\zeta}(b,u,t)\xi(c,v) \\
&= \xi(c-b, v-u_{n+1}) \exp\bigl(i\eta\bigl[(-1)^{n+1}t + nb(v-u_{n+1})^2 + u_n(2v-u_{n+1})\bigr]\bigr) \\
&\qquad \times \exp\bigl(i\langle \zeta, P(-c)(ve_{n+1}-u) + P(b-c)(u_{n+1}-v)e_{n+1}\rangle\bigr).
\end{aligned}$$

Thus

$$\begin{aligned}
&\pi_{\eta,\zeta}\bigl(b, P(b/2)(u_ne_n + u_{n+1}e_{n+1}), 0\bigr)\xi(c,v) \\
&= \xi(c-b, v-u_{n+1}) \exp\bigl(i\eta\bigl[nb(v-u_{n+1})^2 + (u_n + u_{n+1}nb/2)(2v-u_{n+1})\bigr]\bigr) \\
&\qquad \times \exp\bigl(i\langle \zeta, P(-c)(ve_{n+1} - P(b/2)(u_ne_n + u_{n+1}e_{n+1})) + P(b-c)(u_{n+1}-v)e_{n+1}\rangle\bigr) \\
&= \xi(c-b, v-u_{n+1}) \exp\bigl(i\eta\bigl[nb(v^2 - vu_{n+1} + u_{n+1}^2/2) + u_n(2v-u_{n+1})\bigr]\bigr) \\
&\qquad \times \exp\bigl(i\bigl[nvq(-c) - u_nq'(b/2-c) - nu_{n+1}q(b/2-c) + n(u_{n+1}-v)q(b-c)\bigr]\bigr),
\end{aligned}$$

and so

$$\begin{aligned}
&\pi_{\eta,\zeta}[D_R]\xi(c,v) \\
&= \int_{-R}^{R} \text{p.v.} \iint \pi_{\eta,\zeta}\bigl(b, P(b/2)(u_ne_n + \tfrac{u_{n+1}}{\beta(b)}e_{n+1}), 0\bigr)\xi(c,v)\, \frac{1}{u_{n+1}^2 - u_n^2}\, du_n\, du_{n+1}\, \frac{db}{\beta(b)} \\
&= \int_{-R}^{R} \text{p.v.} \iint \pi_{\eta,\zeta}\bigl(b, P(b/2)(u_ne_n + u_{n+1}e_{n+1}), 0\bigr)\xi(c,v)\, \frac{1}{\beta(b)^2\, u_{n+1}^2 - u_n^2}\, du_n\, du_{n+1}\, db \\
&= \int_{-R}^{R} \text{p.v.} \iint \xi(c-b, v-u_{n+1}) \exp\bigl(i\eta\bigl[nb(v^2 - vu_{n+1} + u_{n+1}^2/2) + u_n(2v-u_{n+1})\bigr]\bigr) \\
&\qquad \times \exp\bigl(i\bigl[nvq(-c) + n(u_{n+1}-v)q(b-c) - u_nq'(b/2-c) - nu_{n+1}q(b/2-c)\bigr]\bigr) \\
&\qquad \times \frac{1}{\beta(b)^2\, u_{n+1}^2 - u_n^2}\, du_n\, du_{n+1}\, db,
\end{aligned}$$

and consequently

$$\begin{aligned}
&\mathcal{M}_q^{-1}\pi_{\eta,\zeta}[D_R]\mathcal{M}_q\xi(c,v) \\
&= \int_{-R}^{R} \text{p.v.} \iint \xi(c-b, v-u_{n+1}) \exp\bigl(-iu_nq'(b/2-c) + i\eta u_n(2v-u_{n+1})\bigr) \\
&\qquad \times \exp\bigl(i\bigl[n\eta b(v^2 - vu_{n+1} + u_{n+1}^2/2) - nu_{n+1}q(b/2-c)\bigr]\bigr)\, \frac{1}{\beta(b)^2\, u_{n+1}^2 - u_n^2}\, du_n\, du_{n+1}\, db.
\end{aligned}$$

We can calculate the innermost integral exactly: indeed

$$\begin{aligned}\text{p.v.}\int \exp(i\lambda z)\,\frac{dz}{w^2-z^2} &= \frac{1}{2w}\text{p.v.}\int \exp(i\lambda z)\Big[\frac{1}{z+w}-\frac{1}{z-w}\Big]\,dz\\ &= -\frac{\exp(i\lambda w)-\exp(-i\lambda w)}{2w}\text{p.v.}\int \exp(i\lambda z)\,\frac{dz}{z}\\ &= \frac{\pi\,\mathrm{sign}(\lambda)\sin(\lambda w)}{w} = \frac{\pi\sin(|\lambda|w)}{w}.\end{aligned}$$

We deduce that

$$\begin{aligned}&\mathcal{M}_q^{-1}\pi_{\eta,\zeta}[D_R]\mathcal{M}_q\xi(c,v)\\ &= \pi\int_{-R}^{R}\text{p.v.}\int \xi(c-b,v-u_{n+1})\sin\big(\beta(b)u_{n+1}|(\eta(2v-u_{n+1})-q'(b/2-c))|\big)\\ &\quad\times\exp\big(in\big[\eta b(v^2-vu_{n+1}+u_{n+1}^2/2)-u_{n+1}q(b/2-c)\big]\big)\frac{1}{\beta(b)\,u_{n+1}}\,du_{n+1}\,db\end{aligned}\tag{2.7.2}$$

Since the sine term vanishes when u_{n+1} vanishes, the principal value of the inner integral is the usual integral.

2.8 Equivalent formulation of the oscillatory integrals.
In (2.7.2), we make the change of variables $y_1 = c-b$, $y_2 = v-u_{n+1}$, $x_1 = c$, and $x_2 = v$, and set $p(t) = 2q(-t/2)$ (so that $p'(t) = -q'(t/2)$). Then

$$\begin{aligned}&\mathcal{M}_q\pi_{\eta,\zeta}[D_R]\mathcal{M}_q^{-1}f(x_1,x_2)\\ &= \pi\iint_{\mathbb{R}^2} f(y_1,y_2)\sin\big(\beta(x_1-y_1)(x_2-y_2)|\eta(x_2+y_2)+p'(x_1+y_1)|\big)\\ &\quad\times\exp\big(\frac{in}{2}\big[\eta(x_1-y_1)(x_2^2+y_2^2)-(x_2-y_2)p(x_1+y_1)\big]\big)\frac{\chi_{[-R,R]}(x_1-y_1)}{\beta(x_1-y_1)\,(x_2-y_2)}\,dy_1\,dy_2.\end{aligned}\tag{2.8.1}$$

Now if $\eta \neq 0$, one can conjugate with a dilation in the second variable by a factor of $|\eta|^{1/2}$, to reduce to the case where $\eta = \pm 1$. Further, changing the sign of η and of the polynomial p has the effect of changing the kernel to its complex conjugate, and a kernel operator is bounded on L^2 if and only if the operator with conjugate kernel is. In short, to establish the uniform boundedness of the operators $\pi_{\eta,\zeta}[D_R]$, as (η,ζ) varies over $\mathbb{R}\times V_n$, we may suppose that η is equal to 1.

3. The oscillatory integral

Notation. From now on, fix a positive integer n and Γ in $(1/2,\infty)$. Let p be a real polynomial of degree at most n. An *admissible* constant means a constant which depends only on n and Γ. We write $A \lesssim B$ if $A \leq CB$ and C is an admissible constant in this sense. All "constants" C below will be admissible, and may vary from place to place.

Define the functions $\Psi : \mathbb{R}^2 \to \mathbb{R}$ and $\theta : \mathbb{R}^2 \to \mathbb{R}$ by the formulae

$$\Psi(x,y) = (x_1-y_1)(x_2^2+y_2^2)-(x_2-y_2)p(x_1+y_1)\tag{3.1}$$

$$\theta(x,y) = \beta(x_1-y_1)|x_2+y_2+p'(x_1+y_1)|(x_2-y_2);\tag{3.2}$$

further, recall that

$$\beta(t) = (1+t^2/4)^{1/2}.$$

Suppose $\min\{1, n/2\} \le |\gamma| \le \Gamma$ (the relevant value of γ will be $n/2$). For $R > 0$, we define the family of singular oscillatory integral operators $\mathcal{O}^R$ by

$$\mathcal{O}^R f(x) = \iint \frac{e^{i\gamma\Psi(x,y)} \sin\theta(x,y)}{\beta(x_1 - y_1)(x_2 - y_2)} \chi_{[-R,R]}(x_1 - y_1) f(y_1, y_2)\, dy_1\, dy_2 \tag{3.3}$$

for all f in $C_0^\infty(\mathbb{R}^2)$. We shall see easily that $\|\mathcal{O}^R\|_{L^2 \to L^2} = O(\log R)$ as $R \to \infty$ (this follows from Lemma 3.3 below). However, we have the following result.

Theorem 3.0. *Suppose that* $\min\{1, n/2\} \le |\gamma| \le \Gamma$ *and* $R \ge 100$. *Then the operator* $\mathcal{O}^R$ *extends to a bounded operator on* $L^2(\mathbb{R}^2)$, *and*

$$\|\mathcal{O}^R\|_{L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)} \le C_{n,\Gamma} \log\log(10 + R),$$

where $C_{n,\Gamma}$ *is admissible. If* $n = 1$, *then this estimate may be improved to* $\|\mathcal{O}^R\| = O(1)$.

It is conceivable that the bound $\|\mathcal{O}^R\| = O(1)$ holds in the general case, but this has not been proved so far. For our application, the assertion of the Theorem is (more than) enough.

Corollary 3.1. *(i)* $\sup_{\eta,\zeta} \|\pi_{\eta,\zeta}[D_R]\|_{\mathcal{H}_{\eta,\zeta} \to \mathcal{H}_{\eta,\zeta}} \lesssim \log(\log(10 + R))$.
(ii) $\|\lambda[D_R]\|_{VN(H_0)} \lesssim \log\log(10 + R)$.
(iii) The Fourier algebra of $\mathrm{SL}(2, \mathbb{R}) \ltimes \mathrm{H}^n$ *does not admit multiplier bounded approximate units.*

Proof. By the results of Subsection 2.8, it suffices to prove (i) with $\eta = 1$, but then formula (2.8.1) shows that the statement is implied by Theorem 3.0. The calculations in Section 2.7 together with Lemma 2.6.2 show that (i) implies (ii), and (ii) implies (iii), by Proposition 2.4.1.

Remarks.
(i) The assumption $\min\{1, n/2\} \le |\gamma| \le \Gamma$ in Theorem 3.0 can be replaced by $1/2 \le |\gamma| \le \Gamma$. However the proof for the case where $|\gamma| = 1/2$ and $n \ge 2$ turns out to be substantially more complicated. Fortunately this case is irrelevant for our application.

(ii) There are many results concerning singular oscillatory integral operators with kernels of the form $k(x - y)e^{iP(x,y)}$, where P is a polynomial. If k is a standard Calderón–Zygmund kernel, the oscillatory variants are L^p bounded ($1 < p < \infty$), see Ricci and Stein [31]. If k is a *multiparameter* Calderón–Zygmund kernel the technique in [31], which uses induction on the degree of the polynomial, no longer applies. In fact the L^2 boundedness may then hold or fail depending on the properties of the polynomial P; see, e.g., [2], where a complete characterization of boundedness is obtained for the special case where $P(x, y) = q(x - y)$ and q is a polynomial of two variables.

No theory for general polynomials is currently available. Moreover, our operator is not included in the general class of operators just discussed, because of the positivity of β. Our proof of Theorem 3.0 relies on a subtle global cancellation property of the distribution D defined in (2.3.1) and the noncommutativity of the convolution structure.

(iii) It is instructive to examine the analogue of $\lambda[D_R]$ in the commutative setting, where we identify H_0 as a set with $\mathbb{R}^{n+3}$, writing (b, u, t) for (n_b, u, t), and replace the matrix $Z(b)$ by the identity throughout. Thus define $q_b : \mathbb{R}^2 \to \mathbb{R}^{n+3}$ by $q_b(s_1, s_2) = (b, s_1 e_n + s_2 \beta(b)^{-1} e_{n+1}, 0)$ and define a distribution by $\mathfrak{D}_R(\phi) = \int_{-R}^{R} D(q_b^* \phi)\beta(b)^{-1} db$. Denote by $\mathcal{C}_R f$ the convolution $\mathfrak{D}_R *_E f$ on $\mathbb{R}^{n+3}$; here $*_E$ refers to the standard commutative convolution in Euclidean space. The operator $\mathcal{C}_R$ is bounded on $L^2(\mathbb{R}^{n+3})$; however there is a lower bound for the operator norm of the form $\|\mathcal{C}_R\| \ge c \log R$ as $R \to \infty$. This can be quickly seen by applying the partial Fourier transform $\mathcal{F}_{n+2}$ in the (u, t)-variables. Indeed for fixed $(\xi, \tau) \in \mathbb{R}^{n+1} \times \mathbb{R}$ let $\mathcal{C}_R^{\xi,\tau}$ be the operator on $L^2(\mathbb{R})$ of convolution with $\mathcal{F}_{n+2}[\mathfrak{D}_R](\cdot, \xi, \tau)$; then the operator norm of $\mathcal{C}_R$ is equal to $\sup_{\xi,\tau} \|\mathcal{C}_R^{\xi,\tau}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$.

A quick calculation using the formula for the Fourier transform of D mentioned in §2.3 shows that $\|\mathcal{C}_R^{\xi,\tau}\| \lesssim \log R$ and in particular

$$\mathcal{C}_R^{\xi,\tau} g(b) = \pi^2 \int_{-R}^{R} g(b-c)\frac{dc}{\beta(c)} \quad \text{if } \xi_{n+1}^2 \le \xi_n^2.$$

Testing $\mathcal{C}_R^{\xi,\tau}$ on $g = \chi_{[-R,R]}$ implies that $\|\mathcal{C}_R^{\xi,\tau}\| \ge c\log R$ if $|\xi_{n+1}| \le |\xi_n|$ and the asserted lower bound on $\mathcal{C}_R$ is proved. Thus the better bound of Corollary 3.1 indicates a strictly noncommutative phenomenon.

A first decomposition. In view of the product type singularity of the kernel it is natural to introduce a dyadic decomposition in the variables $x_2 - y_2$ and $x_1 - y_1$ (if the latter is large). For this let η_0 be a smooth nonnegative even function on the real line so that $\eta_0(s) = 1$ if $|s| \le 1/2$ and $\eta_0(s) = 0$ if $|s| \ge 3/4$. We also assume that η_0' has only a finite number of sign changes. Let

$$\eta(s) = \eta_0(s/2) - \eta_0(s)$$

so that η is supported in $[1/2, 3/2] \cup [-3/2, -1/2]$. For pairs of integers $j = (j_1, j_2) \in \mathbb{Z}^2$, with $j_1 > 0$ let

$$\chi_j(x) = \chi_{1,j_1}(x_1)\chi_{2,j_2}(x_2) = \frac{2^{j_1}}{\beta(x_1)}\eta(2^{-j_1}x_1)\frac{2^{j_2}}{x_2}\eta(2^{-j_2}x_2). \tag{3.4}$$

In particular χ_j has the cancellation property

$$\int \chi_j(x_1, x_2)dx_2 = 0 \quad \text{for all } x_1.$$

It will sometimes be useful (see Section 4 below) to use the cut-off function

$$\widetilde{\chi}_j(x) = \chi_j(x)\,\text{sign}(x_2), \tag{3.5}$$

together with the relation

$$\chi_j(x-y)\sin(\theta(x,y)) = \widetilde{\chi}_j(x-y)\sin(|\theta(x,y)|), \tag{3.6}$$

which follows from the evenness of the function $t \mapsto t^{-1}\sin(At)$ and the positivity of β.

Let

$$T_j(x,y) = 2^{-j_1-j_2}\chi_j(x-y)e^{i\gamma\Psi(x,y)}\sin\theta(x,y); \tag{3.7}$$

then we wish to estimate the L^2 operator norm of

$$\mathcal{T}^R f(x) = \sum_{\substack{10<j_1\le\log R\\ j_2\in\mathbb{Z}}} \int T_j(x,y)f(y)dy. \tag{3.8}$$

Preliminary estimates. We shall now verify that the operator norm of $\mathcal{O}^R - \mathcal{T}^R$ is uniformly bounded. To this end, we consider, for fixed (x_1, y_1), the operator $\mathcal{B}^{x_1,y_1}$ acting on functions in $C_0^\infty(\mathbb{R})$, which has the distribution kernel

$$B^{x_1,y_1}(x_2,y_2) = e^{i\gamma\Psi(x,y)}\sin\theta(x,y)(x_2-y_2)^{-1}. \tag{3.9}$$

Lemma 3.2. *For each (x_1, y_1) the operator $\mathcal{B}^{x_1,y_1}$ extends to a bounded operator on $L^2(\mathbb{R})$ with norm bounded independently of (x_1, y_1).*

Proof. For $\epsilon, \epsilon' \in \{\pm 1\}$, define

$$E^{\epsilon}_{x_1 y_1} = \{(s,t) : \epsilon(s + t + p'(x_1 + y_1)) \geq 0\}.$$

One computes that

$$2i\mathcal{B}^{x_1,y_1} g(s) = \sum_{\epsilon'=\pm 1} \sum_{\epsilon=\pm 1} \epsilon' e^{i\rho_\epsilon(s)} \int \chi_{E^{\epsilon}_{x_1,y_1}}(s,t) g(t) e^{i\sigma_\epsilon(t)} \frac{dt}{s-t},$$

where

$$\rho_\epsilon(s) \equiv \rho_{\epsilon,x_1,y_1}(s) = (x_1 - y_1)s^2 - sp(x_1 + y_1) + \epsilon\beta(x_1 - y_1)(s^2 + p'(x_1 + y_1)s)$$
$$\sigma_\epsilon(t) \equiv \sigma_{\epsilon,x_1,y_1}(t) = (x_1 - y_1)t^2 + tp(x_1 + y_1) - \epsilon\beta(x_1 - y_1)(t^2 + p'(x_1 + y_1)t).$$

The uniform boundedness of $\mathcal{B}^{x_1 y_1}$ on $L^2(\mathbb{R})$ follows from the boundedness of Hilbert transforms and Hilbert integrals. $\square$

Lemma 3.3. *Let $\mathcal{E}$ be an operator bounded on $L^2(\mathbb{R})$, with nonnegative kernel $k(s,t)$. Let*

$$\mathcal{S}f(x) = \int k(x_1, y_1) |\mathcal{B}^{x_1,y_1}[f(y_1, \cdot)](x_2)| dy_1.$$

Then $\mathcal{S}$ is bounded on $L^2(\mathbb{R}^2)$ with operator norm $\lesssim \|\mathcal{E}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})}$.

Proof. By Lemma 3.2 we have $\|\mathcal{B}^{x_1,y_1}\| \leq C_0$ and therefore

$$\|\mathcal{S}f\| \leq \Big(\int \Big(\int k(x_1, y_1) \|\mathcal{B}^{x_1 y_1}[f(y_1, \cdot)]\| dy_1\Big)^2 dx_1\Big)^{1/2}$$
$$\leq C_0 \Big(\int \Big(\int k(x_1, y_1) \Big(\int |f(y_1, x_2)|^2 dx_2\Big)^{1/2} dy_1\Big)^2 dx_1\Big)^{1/2},$$

and the result follows from the assumed L^2 boundedness of the operator $\mathcal{E}$ acting on the function $y_1 \mapsto \|f(y_1, \cdot)\|_{L^2(\mathbb{R})}$. $\square$

Lemma 3.4. *The operator $\mathcal{O}^R - \mathcal{T}^R$ is bounded on $L^2(\mathbb{R}^2)$ with an admissible operator norm uniformly in R.*

Proof. Let $E_R = [-40, 40] \cup [R/4, 4R] \cup [-4R, -R/4]$ and $k_R(s) = \chi_{E_R}(s)\, \beta(s)^{-1}$. Observe that the $L^1(\mathbb{R})$ norm of k_R is uniformly bounded in R. Note that

$$|\mathcal{O}^R f(x) - \mathcal{T}^R f(x)| \lesssim \int k_R(x_1 - y_1)\, \big|\mathcal{B}^{x_1 y_1}[f(y_1, \cdot)](x_2)\big| dy_1$$

so that the assertion follows from Lemma 3.3. $\square$

By Lemma 3.4 it suffices to show the bound

$$\|\mathcal{T}^R\| = O(\log\log R) \tag{3.10}$$

for large R. The next four sections will be devoted to the proof of (3.10). The argument relies on a crucial cancellation property for the affine case, where $p(x) = ax + b$, for which one obtains the bound $\|\mathcal{T}^R\| = O(1)$. This will be carried out in Section 4. The general case involves an approximation by operators which share the properties of the affine case; for various remainder terms one uses the oscillatory properties of the phase function and Hilbert integral arguments. The basic decomposition describing the remainder terms and relevant orthogonality arguments is introduced in Section 5; here we state several propositions containing estimates for the constituents in the basic decomposition and deduce the main estimate (3.10). Section 6 contains a few auxiliary facts and Section 7 contains the proof of the propositions.

4. Boundedness for affine polynomials

Let I be a set of pairs (j_1, j_2) with the property that $j_1, j_2 \in \mathbb{Z}$ and $j_1 \geq 10$. Define

$$\mathcal{T}f(x) = \sum_{j=(j_1,j_2)\in I} 2^{-j_1-j_2} \int e^{i\gamma\Psi(x,y)} \sin\theta(x,y)\chi_j(x-y)f(y)\,dy \tag{4.1}$$

Theorem 4.1. *Assume that* $\alpha_0, \alpha_1 \in \mathbb{R}$, *and that*

$$p(s) = \alpha_1 s + \alpha_0.$$

Suppose that $1/2 \leq |\gamma| \leq \Gamma$. *Then the operator* $\mathcal{T}$ *extends to a bounded operator on* $L^2(\mathbb{R}^2)$, *and*

$$\|\mathcal{T}\| \leq C_\Gamma,$$

where C_Γ *does not depend on* I, α_0, *or* α_1.

Proof. We have now

$$\begin{aligned} \Psi(x,y) &= (x_1 - y_1)(x_2^2 + y_2^2) - (x_2 - y_2)(\alpha_1(x_1 + y_1) + \alpha_0) \\ \theta(x,y) &= \beta(x_1 - y_1)|x_2 + y_2 + \alpha_1|(x_2 - y_2) \end{aligned} \tag{4.2}$$

and, setting $A(x) = \frac{\alpha_1^2}{2}x_1 - 2\alpha_1 x_1 x_2 - \alpha_0 x_2$, we compute that

$$\Psi(x_1, x_2 - \tfrac{\alpha_1}{2}, y_1, y_2 - \tfrac{\alpha_1}{2}) = (x_1 - y_1)(x_2^2 + y_2^2) + A(x) - A(y);$$

moreover

$$\theta(x_1, x_2 - \tfrac{\alpha_1}{2}, y_1, y_2 - \tfrac{\alpha_1}{2}) = \beta(x_1 - y_1)|x_2 + y_2|(x_2 - y_2).$$

From (4.1) and (4.2), we see that we can reduce matters to the case where $p = 0$, after a translation in the x_2 variable and a conjugation with a multiplication operator of norm 1. Therefore we shall now work with (4.2) where $\alpha_0 = \alpha_1 = 0$, and consider the integral operator $\mathcal{K}$ with kernel

$$K = \sum_{j\in I} K_j,$$

where

$$K_j(x,y) = 2^{-j_1-j_2} e^{i\gamma(x_2^2+y_2^2)(x_1-y_1)} \sin(\beta(x_1 - y_1)|x_2^2 - y_2^2|)\widetilde{\chi}_j(x-y);$$

see formula (3.6). For $\xi_1 \in \mathbb{R}$, let

$$S_{\xi_1} g(x_2) = \sum_{j_2} \int g(y_2) h(\xi_1, x_2, y_2, j_2) 2^{-j_2} dy_2,$$

where

$$h(\xi_1, x_2, y_2, j_2) = \widetilde{\chi}_{2,j_2}(x_2 - y_2) \sum_{j_1:(j_1,j_2)\in I} h_{j_1}(\xi_1, x_2, y_2),$$

with

$$h_{j_1}(\xi_1, x_2, y_2) = 2^{-j_1} \int e^{i\sigma\gamma(x_2^2+y_2^2)-i\sigma\xi_1} \sin(\beta(\sigma)|x_2^2 - y_2^2|)\chi_{1,j_1}(\sigma)d\sigma.$$

Then

$$\widetilde{\mathcal{T}f}(\xi_1, x_2) = S_{\xi_1}[\widetilde{f}(\xi_1, \cdot)](x_2),$$

where $\widetilde{f}$ denotes the Fourier transform of f with respect to the first variable. Thus it suffices to fix ξ_1 and show that S_{ξ_1} is bounded on $L^2(\mathbb{R})$ uniformly in ξ_1.

Lemma 4.2.

(i) There is a constant C so that

$$|h(\xi_1, x_2, y_2, j_2)| \le C \tag{4.3}$$

for all ξ_1, x_2, y_2, j_2. Moreover

$$h(\xi_1, x_2, y_2, j_2) = 0 \quad \text{if } |x_2 - y_2| \notin [2^{j_2-2}, 2^{j_2+2}]. \tag{4.4}$$

(ii) For each j_1

$$|h_{j_1}(\xi_1, x_2, y_2)| \lesssim 2^{j_1}|x_2^2 - y_2^2|. \tag{4.5}$$

(iii) Suppose that $|\xi_1 - \gamma(x_2^2 + y_2^2)| \ge 2|x_2^2 - y_2^2|$. Then

$$|h_{j_1}(\xi_1, x_2, y_2)| \le C_N (2^{j_1}|\xi_1 - \gamma(x_2^2 + y_2^2)|)^{-N}. \tag{4.6}$$

Proof. The assertion (ii) follows immediately from the inequality $|\sin\alpha| \le |\alpha|$. Moreover (4.4) is immediate from the definitions. In what follows we shall use simple properties of β stated in (6.1), (6.2) below.

We now prove the uniform boundedness of h. Since χ_{1,j_1} is an even function,

$$\begin{aligned} h_{j_1}(\xi_1, x_2, y_2) &= 2^{-j_1+1}\int_{\sigma>0} \cos(\sigma(\xi_1 - \gamma(x_2^2 + y_2^2)))\sin(\beta(\sigma)|x_2^2 - y_2^2|)\chi_{1,j_1}(\sigma)d\sigma \\ &= h_{j_1}^+(\xi_1, x_2, y_2) - h_{j_1}^-(\xi_1, x_2, y_2), \end{aligned}$$

where

$$h_{j_1}^{\pm}(\xi_1, x_2, y_2) = 2^{-j_1}\int_{\sigma>0} \sin(\phi^{\pm}(\sigma; \xi_1, x_2, y_2))\chi_{1,j_1}(\sigma)d\sigma,$$

with

$$\phi^{\pm}(\sigma; \xi_1, x_2, y_2) = \sigma(\xi_1 - \gamma(x_2^2 + y_2^2)) \pm \beta(\sigma)|x_2^2 - y_2^2|.$$

Observe that

$$\begin{aligned} (\phi^{\pm})'(\sigma) &= \xi_1 - \gamma(x_2^2 + y_2^2) \pm \beta'(\sigma)|x_2^2 - y_2^2| \\ (\phi^{\pm})''(\sigma) &= \pm\beta''(\sigma)|x_2^2 - y_2^2|, \end{aligned}$$

so that

$$|(\phi^{\pm})''(\sigma)| \approx 2^{-3j_1}|x_2^2 - y_2^2|$$

in the support of χ_{1,j_1} (see (6.1) below). Using van der Corput's Lemma ([33, p. 334]), we obtain the inequality

$$|h_{j_1}^{\pm}(\xi_1, x_2, y_2)| \le C 2^{j_1/2}|x_2^2 - y_2^2|^{-1/2}. \tag{4.7}$$

Now since $\sigma > 0$, we have in view of (6.2.1) below

$$\sin\phi^{\pm}(\sigma) = \sin\big(\sigma(\xi_1 - \gamma(x_2^2 + y_2^2) \pm \frac{1}{2}|x_2^2 - y_2^2|)\big) + O(|x_2^2 - y_2^2|/\sigma),$$

so that

$$h_{j_1}^{\pm}(\xi_1,x_2,y_2)=2^{-j_1}\int_{\sigma>0}\sin\big(\sigma(\xi_1-\gamma(x_2^2+y_2^2)\pm\frac{1}{2}|x_2^2-y_2^2|)\big)\chi_{1,j_1}(\sigma)d\sigma \\ +r_{j_1}^{\pm}(\xi_1,x_2,y_2), \tag{4.8}$$

where the error terms $r_{j_1}^{\pm}$ satisfy the estimate

$$|r_{j_1}^{\pm}(\xi_1,x_2,y_2)|\lesssim 2^{-j_1}|x_2^2-y_2^2|. \tag{4.9}$$

Concerning the integral in (4.8), observe that

$$\Big|\sum_{j_1\in\mathcal{E}}\int_{\sigma>0}\sin(A\sigma)2^{-j_1}\chi_{1,j_1}(\sigma)d\sigma\Big|\le C, \tag{4.10}$$

where the sum is over any finite set $\mathcal{E}$ consisting of positive j_1; the constant C can be chosen independently of $\mathcal{E}$ and of A. To see this we use the inequality $|\sin\alpha|\le|\alpha|$ for the terms with $A2^{j_1}\le 1$ and integration by parts for the terms with $A2^{j_1}>1$. From (4.10),

$$\Big|\sum_{j_1\in\mathcal{E}}\big(h_{j_1}^{\pm}(\xi_1,x_2,y_2)-r_{j_1}^{\pm}(\xi_1,x_2,y_2)\big)\Big|\le C, \tag{4.11}$$

where $\mathcal{E}$ is again any set of positive indices and the bound is uniform in ξ_1,x_2,y_2,j_2.

Now an application of formulae (4.7), (4.8), (4.9) and (4.11) shows that h is uniformly bounded. Finally, the estimate (iii) follows by integration by parts, using the lower bound

$$|(\phi^{\pm})'(\sigma)|\ge\frac{1}{2}|\xi_1-\gamma(x_2^2+y_2^2)|,$$

in the present case of assertion (iii), see formula (6.2.2) below. Moreover, if $\nu\ge 2$, then

$$|\partial_\sigma^\nu\phi^{\pm}(\sigma)|=|\beta^{(\nu)}(\sigma)||x_2^2-y_2^2|\lesssim(1+|\sigma|)^{-|\nu|-1}|x_2^2-y_2^2|,$$

which is an acceptable upper bound in (iii). □

In what follows, ξ_1 will be fixed, and we shall not always indicate the dependence of the operators on ξ_1. For $M\in\mathbb{Z}$, let

$$\mathcal{S}_{j_2}^M g(x_2)=\eta(2^{-M}x_2)2^{-j_2}\int g(y_2)h(\xi_1,x_2,y_2,j_2)dy_2.$$

Let C_0 be an integer with $2^{C_0-100}\ge\Gamma$. We split

$$\mathcal{S}_{\xi_1}=\mathcal{S}+\sum_{\substack{(j_2,M)\\ M\le j_2+C_0}}\mathcal{S}_{j_2}^M.$$

It is easy to see using the uniform boundedness of h and the definition of the cut-off functions that

$$\sum_{\substack{(j_2,M)\\ M\le j_2+C_0}}|\mathcal{S}_{j_2}^M g(x_2)|\lesssim\int_{|x_2|\le 2^{C_0+2}|x_2-y_2|}\frac{|g(y_2)|}{|x_2-y_2|}dy_2. \tag{4.12}$$

The integral on the right hand side in (4.12) is a standard Hilbert integral and therefore defines a bounded operator on $L^2(\mathbb{R})$ (see [32, p. 271]).

We let

$$\mathcal{S}^M = \sum_{j_2 < M - C_0} \mathcal{S}^M_{j_2}$$

and the kernel $S^M(x_2, y_2)$ is supported where $2^{M-1} \le |x_2| \le 2^{M+1}$, $2^{M-2} \le |y_2| \le 2^{M+2}$. Therefore the almost orthogonality property

$$\|\mathcal{S}\| \lesssim \sup_M \|\mathcal{S}^M\| \tag{4.13}$$

holds. Thus it suffices to prove a uniform estimate for the operators $\mathcal{S}^M$. We split $\mathcal{S}^M = \mathcal{P}^M + (\mathcal{S}^M - \mathcal{P}^M)$, where

$$\mathcal{P}^M g(x_2) = \sum_{j_2 < M - C_0} \eta_0(2^{-M-j_2-C_0+10}(\xi_1 - 2\gamma x_2^2))\mathcal{S}^M_{j_2} g(x_2).$$

We first show that the operators $\mathcal{P}^M$ are uniformly bounded. Since $j_2 \le M - C_0$, we observe that the conditions $|2^{-M-j_2-C_0+10}(\xi_1 - 2\gamma x_2^2))| \le 1$ and $2^{M-1} \le |x_2| \le 2^{M+1}$ imply that $\xi_1(2\gamma)^{-1} > 0$ and $\xi_1(2\gamma)^{-1} \approx 2^{2M}$, and therefore

$$\big| |x_2| - (\xi_1/2\gamma)^{1/2} \big| \le C_1 |x_2 - y_2|.$$

Consequently

$$|\mathcal{P}^M g(x_2)| \lesssim \int_{||x_2| - (|\xi_1/2\gamma|)^{1/2}| \le C_1|x_2 - y_2|} \frac{|g(y_2)|}{|x_2 - y_2|} dy_2.$$

The right hand side is a sum of two operators, each of them a Hilbert integral operator composed with translation operators. Therefore it defines a bounded operator on $L^2(\mathbb{R})$ and

$$\|\mathcal{P}^M\| = O(1).$$

Next we consider the operator $\mathcal{S}^M - \mathcal{P}^M$ which we split as

$$\begin{aligned}
[\mathcal{S}^M - \mathcal{P}^M]g(x_2) &= \sum_{j_2 < M - C_0} (1 - \eta_0(2^{-M-j_2-C_0+10}(\xi_1 - 2\gamma x_2^2)))\mathcal{S}^M_{j_2} g(x_2) \\
&= \sum_{r \in \mathbb{Z}} \eta(2^{-r}(\xi_1 - 2\gamma x_2^2)) \sum_{\substack{j_2 < r - M - C_0 + 10 \\ j_2 < M - C_0}} \mathcal{S}^M_{j_2} g(x_2) \\
&= \sum_{r \in \mathbb{Z}} \mathcal{Q}^M_r g(x_2),
\end{aligned}$$

say. Since $2^{C_0} \ge 2^{100}\Gamma$, we have $|2\gamma(x_2^2 - y_2^2)| \le 2^{M+j_2+3}|\Gamma| \le 2^{r-C_0+13}|\Gamma| \le 2^{r-10}$, and hence

$$|\xi_1 - 2\gamma y_2^2| = |\xi_1 - 2\gamma x_2^2 + 2\gamma(x_2^2 - y_2^2)| \approx 2^r. \tag{4.14}$$

Thus $|\xi_1 - 2\gamma x_2^2| \in (2^{r-1}, 2^{r+1})$, which implies that $|\xi_1 - 2\gamma y_2^2| \approx 2^r$ and we can deduce the almost orthogonality property

$$\|\mathcal{S}^M - \mathcal{P}^M\| \lesssim \sup_r \|\mathcal{Q}^M_r\|. \tag{4.15}$$

Now, analogously to (4.14), we also have

$$|\xi_1 - \gamma(x_2^2 + y_2^2)| \approx 2^r. \tag{4.16}$$

By Lemma 4.2 (ii) and (iii),

$$|h_{j_1}(\xi_1, x_2, y_2)| \lesssim \min\{2^{-j_1-r}, 2^{j_1+j_2+M}\}$$

if $|x_2 - y_2| \approx 2^{j_2}$ and $|x_2| \approx |y_2| \approx 2^M$, and $|\xi_1 - \gamma(x_2^2 + y_2^2)| \approx 2^r \gg 2^{M+j_2}$. Therefore

$$|h(\xi_1, x_2, y_2, j_2)| \lesssim 2^{(M+j_2-r)/2},$$

and it follows that

$$\|\mathcal{Q}_r^M\| \lesssim \sum_{M+j_2 \le r} 2^{(M+j_2-r)/2} \le C.$$

This now implies the uniform boundedness of the operators $\mathcal{S}^M - \mathcal{P}^M$. Together with the L^2 boundedness of $\mathcal{P}^M$ and the orthogonality property of the operators $\mathcal{S}^M$ this completes the proof of Theorem 4.1. □

5. Basic decompositions and outline of the proof for $n \geq 2$

We shall now assume that $n \geq 2$ and that p is a nonaffine polynomial of degree $\leq n$. Since we are estimating the operator $\mathcal{T}^R$ we shall assume that sums in j are always taken over subsets of $\{(j_1, j_2) : 10 < j_1 \leq \log R\}$.

We begin by refining the dyadic decomposition from Section 3. Using the cut-off functions η_0 and η as defined in Section 3 we set

$$a_m(\sigma) = \prod_{\nu=2}^{\deg(p)-1} \eta_0\big(2^{m+10}\frac{p^{(\nu+1)}(\sigma)}{p^{(\nu)}(\sigma)}\big) \tag{5.1}$$

if $\deg(p) \geq 3$ and $a_m(\sigma) = 1$ if $\deg(p) = 2$. Next,

$$b_m(X_1, X_2) = \eta_0\big(2^{m+10}\frac{p''(X_1)}{p'(X_1) + X_2}\big). \tag{5.2}$$

Moreover, let

$$h_l(X_1, X_2) = \eta_0(2^{-l-10}(X_2 + p'(X_1))), \tag{5.3.1}$$

$$h_{l,r}(X_1, X_2) = \eta(2^{-l+r-10}(X_2 + p'(X_1))), \tag{5.3.2}$$

so that $h_l = \sum_{r>0} h_{l,r}$ a.e.

Now let $T_j(x, y)$ be as in (3.7); our basic splitting (assuming $j_1 > 10$) is

$$T_j = H_j + U_j + W_j + \sum_{r>0} V_j^r, \tag{5.4}$$

where

$$H_j(x,y) = T_j(x,y)(1 - a_{j_1}(x_1+y_1)), \tag{5.5.1}$$

$$U_j(x,y) = T_j(x,y)a_{j_1}(x_1+y_1)(1 - b_{j_1}(x+y)), \tag{5.5.2}$$

$$V_j^r(x,y) = T_j(x,y)a_{j_1}(x_1+y_1)b_{j_1}(x+y)h_{j_2,r}(x+y), \tag{5.5.3}$$

$$W_j(x,y) = T_j(x,y)a_{j_1}(x_1+y_1)b_{j_1}(x+y)(1 - h_{j_2}(x+y)). \tag{5.5.4}$$

Let $\mathcal{H}_j$, $\mathcal{U}_j$, $\mathcal{V}_j^r$, $\mathcal{W}_j$ be the corresponding operators. Let $\mathcal{H}$, $\mathcal{U}$, $\mathcal{V}^r$, $\mathcal{W}$ denote the operators $\sum_j \mathcal{H}_j$, $\sum_j \mathcal{U}_j$, $\sum_j \mathcal{V}_j^r$, and $\sum_j \mathcal{W}_j$.

We shall also use the notation

$$u_j(x,y) = a_{j_1}(x_1+y_1)(1 - b_{j_1}(x+y)), \tag{5.6.1}$$

$$v_j^r(x,y) = a_{j_1}(x_1+y_1)b_{j_1}(x+y)h_{j_2,r}(x+y), \tag{5.6.2}$$

$$w_j(x,y) = a_{j_1}(x_1+y_1)b_{j_1}(x+y)(1 - h_{j_2}(x+y)). \tag{5.6.3}$$

Proposition 5.1. *The operator $\mathcal{H}$ is bounded on $L^2(\mathbb{R}^2)$.*

Proposition 5.2. *Let $\mathcal{U}_j^L$ be the operator with kernel U_j^L given by*

$$U_j^L(x,y) = U_j(x,y)\eta(2^{-L}p''(2x_1)),$$

and let $\mathcal{U}^L = \sum \mathcal{U}_j^L$. Then

(i)

$$\|\mathcal{U}\| \lesssim \sup_L \|\mathcal{U}^L\|. \tag{5.7}$$

(ii)

$$\|\mathcal{U}_j^L\| \lesssim \min\{2^{L+2j_1+j_2}, 2^{-(L+2j_1+j_2)/4}\}. \tag{5.8}$$

(iii)

$$\|(\mathcal{U}_j^L)^*\mathcal{U}_k^L\| + \|\mathcal{U}_j^L(\mathcal{U}_k^L)^*\| \lesssim 2^{-|j_1-k_1|/2}. \tag{5.9}$$

Proposition 5.3.

(i)

$$\|(\mathcal{V}_j^r)^*\mathcal{V}_k^r\| + \|\mathcal{V}_j^r(\mathcal{V}_k^r)^*\| \lesssim 2^{-r-|j_2-k_2|/2}. \tag{5.10}$$

(ii)

$$\|\mathcal{V}_j^r\| \lesssim \min\{2^{2j_2+j_1-r}, 2^{r/2-(2j_2+j_1)/4}\}. \tag{5.11}$$

Proposition 5.4. *For $M \in \mathbb{Z}$, $L \in \mathbb{Z}$, let $\mathcal{W}_j^{M,L}$ be the operator with kernel*

$$W_j^{M,L}(x,y) = W_j(x,y)\eta(2^{-M}(2x_2 + p'(2x_1)))\eta(2^{-L}p''(2x_1)), \tag{5.12}$$

and let $\mathcal{W}^{M,L} = \sum_j \mathcal{W}_j^{M,L}$. Then

(i)

$$\|\mathcal{W}\| \lesssim \sup_{M,L} \|\mathcal{W}^{M,L}\|. \tag{5.13}$$

(ii)

$$\|\mathcal{W}_j^{M,L}\| \lesssim \min\{2^{M+j_1+j_2}, 2^{-(M+j_1+j_2)/4}\}. \tag{5.14}$$

(iii) The estimate (5.14) also holds if $W_j^{M,L}(x,y)$ is replaced by $W_j^{M,L}(x,y)\rho_j(x,y)$ where ρ_j satisfies $\partial_x^\alpha \rho_j$, $\partial_y^\alpha \rho_j = O(2^{-j_1|\alpha_1|-j_2|\alpha_2|})$, for $\alpha_1, \alpha_2 \in \{0,1\}$.

The previous propositions are enough to obtain a uniform bound on the operators $\mathcal{U}$ and $\mathcal{V}$. For $\mathcal{W}$, an analogue of the crucial orthogonality properties (5.9) and (5.10) is missing, and we shall instead use an approximation by operators treated in Section 4.

Proposition 5.5. *Suppose that $m \geq 0$. Fix $M \in \mathbb{Z}$ and $L \in \mathbb{Z}$, and let I be a set of integer pairs $j = (j_1, j_2)$ satisfying*

$$\begin{aligned} M - L - m(1 + \frac{1}{2n}) < j_1 \leq M - L - m \\ L + 2j_1 + j_2 \leq 0 \\ 10 \leq j_1 < \log R. \end{aligned} \tag{5.15}$$

Let

$$\mathcal{Z}^I = \sum_{j \in I} \mathcal{W}_j^{M,L}.$$

Then $\mathcal{Z}^I$ is bounded on $L^2(\mathbb{R}^2)$ and

$$\|\mathcal{Z}^I\|_{L^2 \to L^2} \leq C, \tag{5.16}$$

where the admissible constant C is independent of I, L, M, m, R.

Taking Propositions 5.1–5.5 for granted, we are now able to give a proof of the main theorem.

Proof of Theorem 3.1. By the discussion in Section 3 it suffices to prove the estimate (3.10). In view of Proposition 5.1, we have to bound $\mathcal{U}$, $\mathcal{V}$ and $\mathcal{W}$. In order to bound $\mathcal{U}$, it is sufficient to obtain a uniform bound for the operators $\mathcal{U}^L$, by (5.7). Let

$$\mathcal{U}^{L,\ell} = \sum_{L+2j_1+j_2=\ell} \mathcal{U}_j^L.$$

Suppose $L + 2j_1 + j_2 = \ell$, $L + 2k_1 + k_2 = \ell$ and $|j_1 - k_1| = s$; then by (5.8) and (5.9),

$$\|(\mathcal{U}_j^L)^*\mathcal{U}_k^L\| + \|\mathcal{U}_j^L(\mathcal{U}_k^L)^*\| \lesssim \min\{2^{2\ell}, 2^{-\ell/2}, 2^{-s/2}\},$$

and by the Cotlar–Stein Lemma ([33, p. 280]) it follows that

$$\mathcal{U}^{L,\ell} \lesssim \sum_{s=0}^{\infty} \min\{2^{\ell}, 2^{-\ell/4}, 2^{-s/4}\} \lesssim (1+|\ell|)\min\{2^{\ell}, 2^{-\ell/4}\}.$$

Summing over ℓ yields the desired uniform bound for $\mathcal{U}^L$ and thus the boundedness of $\mathcal{U}$.

The operator $\mathcal{V}^r$ is handled similarly. Now let $\mathcal{V}^{\ell,r} = \sum_{2j_2+j_1=\ell} \mathcal{V}^r_j$. From Proposition 5.3, we have

$$\|(\mathcal{V}^r_j)^*\mathcal{V}^r_k\| + \|(\mathcal{V}^r_j(\mathcal{V}^r_k)^*\| \lesssim \min\{2^{2\ell-2r}, 2^{r-\ell/2}, 2^{-r-s/2}\}$$

if $2j_2+j_1 = 2k_2+k_1 = \ell, |j_2-k_2| = s$, and we obtain from the Cotlar–Stein Lemma that

$$\|\mathcal{V}^{\ell,r}\| \lesssim \sum_{s\ge 0} \min\{2^{\ell-r}, 2^{r/2-\ell/4}, 2^{-r/2-s/4}\}$$

and thus

$$\begin{aligned}\|\mathcal{V}\| &\le \sum_{r>0}\sum_{\ell\in\mathbb{Z}} \|\mathcal{V}^{\ell,r}\| \\ &\lesssim \sum_{r>0}\sum_{\ell\le 0} 2^{-|\ell|-r}(1+|\ell|+r) + \sum_{r>0}\sum_{0\le\ell\le 4r} 2^{-r/2} \\ &\quad + \sum_{r>0}\sum_{\ell>4r}\Big(2^{-r/2-(\ell-4r)/4} + 2^{-r/2}2^{r-\ell/4}(\ell-4r)\Big),\end{aligned}$$

and $\mathcal{V}$ is easily seen to be bounded on L^2.

Now we turn to the operator $\mathcal{W}$. By (5.13) it suffices to obtain a uniform bound for $\mathcal{W}^{M,L}$. We note that $\mathcal{W}^{M,L}_j = 0$ if $L+j_1 \ge M$. Therefore by (5.14)

$$\sum_{L+2j_1+j_2\ge 0} \|\mathcal{W}^{M,L}_j\| \lesssim \sum_{\substack{L+2j_1+j_2\ge 0\\ L+j_1\le M}} 2^{-(M+j_1+j_2)/4} \le C.$$

For sums of terms $\mathcal{W}^{M,L}_j$ which satisfy $L+2j_1+j_2<0$ we use Proposition 5.5. For $s=1,2,\dots,$ let

$$I^{M,L}_{s,R} = \{j : M-L-(\tfrac{2n+1}{2n})^s < j_1 \le M-L-(\tfrac{2n+1}{2n})^{s-1}, L+2j_1+j_2<0, 10<j_1\le \log(10+R)\}.$$

Then from Proposition 5.5,

$$\Big\|\sum_{j\in I^{M,L}_{s,R}} \mathcal{W}^{M,L}_j\Big\| \le C,$$

uniformly in s, R and M, L. Now for fixed M, L, R the sets $I^{M,L}_{s,R}$ are nonempty for no more than $C_0 \log\log(10+R)$ choices of s; here C_0 is admissible. Summing over s we see that $\|\mathcal{W}^{M,L}\| = O(\log\log(10+R))$, with an admissible constant, and by (5.13) we obtain the same bound for $\mathcal{W}$. □

6. Auxiliary Lemmas

We first collect formulae for the derivatives of β, Ψ and θ.

Lemma 6.1. *(i)*

$$\beta'(s) = \frac{s}{4\beta(s)}, \qquad \beta''(s) = \frac{1}{4[\beta(s)]^3} \tag{6.1}$$

and

$$\left|\beta(s) - \frac{|s|}{2}\right| = \frac{2}{2\beta(s) + |s|} \le \frac{1}{|s|} \tag{6.2.1}$$

$$\left|\beta'(s) - \frac{1}{2}\operatorname{sign}(s)\right| \le \frac{4}{4+s^2}. \tag{6.2.2}$$

(ii) Let $\Xi(x,y) = x_2 + y_2 + p'(x_1+y_1)$. *Suppose that* $\Xi(x,y) \neq 0$. *Then*

$$\Psi_{x_1}(x,y) = 2x_2^2 - (x_2 - y_2)\Xi(x,y) \tag{6.3}$$

$$\Psi_{x_2}(x,y) = 2x_2(x_1 - y_1) - p(x_1+y_1) \tag{6.4}$$

$$\theta_{x_1}(x,y) = (x_2 - y_2)\big[\beta'(x_1-y_1)|\Xi(x,y)| + \beta(x_1-y_1)p''(x_1+y_1)\operatorname{sign}(\Xi(x,y))\big] \tag{6.5}$$

$$\theta_{x_2}(x,y) = \beta(x_1-y_1)(2x_2 + p'(x_1+y_1))\operatorname{sign}(\Xi(x,y)). \tag{6.6}$$

(iii)

$$\Psi_{x_1y_1}(x,y) = -(x_2-y_2)p''(x_1+y_1) \tag{6.7}$$

$$\theta_{x_1y_1}(x,y) = (x_2-y_2)\big[\beta(x_1-y_1)p'''(x_1+y_1)\operatorname{sign}(\Xi(x,y)) - \beta''(x_1-y_1)|\Xi(x,y)|\big]. \tag{6.8}$$

(iv)

$$\Psi_{x_1y_2}(x,y) = 2y_2 + p'(x_1+y_1) \tag{6.9}$$

$$\Psi_{x_2y_1}(x,y) = -(2x_2 + p'(x_1+y_1)), \tag{6.10}$$

$$\begin{aligned}\theta_{x_1y_2}(x,y) &= \beta'(x_1-y_1)\big[(x_2-y_2)\operatorname{sign}(\Xi) - |\Xi(x,y)|\big] - \beta(x_1-y_1)p''(x_1+y_1)\operatorname{sign}(\Xi)\\ &= \big[-\beta(x_1-y_1)p''(x_1+y_1) - \beta'(x_1-y_1)(2y_2 + p'(x_1+y_1))\big]\operatorname{sign}(\Xi)\end{aligned} \tag{6.11}$$

$$\begin{aligned}\theta_{x_2y_1}(x,y) &= -\beta'(x_1-y_1)\big[(x_2-y_2)\operatorname{sign}(\Xi) + |\Xi(x,y)|\big] + \beta(x_1-y_1)p''(x_1+y_1)\operatorname{sign}(\Xi)\\ &= \big[\beta(x_1-y_1)p''(x_1+y_1) - \beta'(x_1-y_1)(2x_2 + p'(x_1+y_1))\big]\operatorname{sign}(\Xi).\end{aligned} \tag{6.12}$$

(v)

$$\Psi_{x_2x_2y_1}(x,y) = -2 \tag{6.13}$$

$$\theta_{x_2x_2y_1}(x,y) = -2\beta'(x_1-y_1)\operatorname{sign}(\Xi) \tag{6.14}$$

$$\Psi_{x_2y_2}(x,y) = 0 \tag{6.15}$$

$$\theta_{x_2y_2}(x,y) = 0. \tag{6.16}$$

Proof. These are straightforward computations. □

We shall now examine the properties of the cut-off functions in (5.1–5.3). For this, the following observations are essential.

Lemma 6.2. *Let P be a polynomial, let $\ell \le \deg(P)$ and let*

$$\alpha_m(\sigma) = \prod_{\nu=\ell}^{\deg(P)} \eta_0\big(2^{m+10}\frac{P^{(\nu)}(\sigma)}{P^{(\nu-1)}(\sigma)}\big).$$

(i) Suppose that $\sigma \in \operatorname{supp}\alpha_m$, and $|\sigma-\tau| \le 2^{m+7}$. Then for $\nu = \ell, \dots, \deg(P)$

$$|P^{(\nu)}(\tau) - P^{(\nu)}(\sigma)| \le \frac{1}{5}|P^{(\nu)}(\sigma)| \tag{6.17}$$

and

$$|P^{(\nu)}(\tau)| \le 2^{1-(m+4)k}|P^{(\nu-k)}(\sigma)| \quad \text{if } \nu - k \ge \ell. \tag{6.18}$$

(ii) For $r = 1, 2, 3, \dots$

$$|\alpha_m^{(r)}(\sigma)| \le C_r 2^{-rm}. \tag{6.19}$$

Proof. (i) If $\sigma = \tau$ then a slightly better estimate than (6.18) follows from the definition of α_m, and then for $|\sigma-\tau| \le 2^{m+7}$ the estimate (6.18) follows once (6.17) is proved. To see (6.17) suppose that $\sigma \in \operatorname{supp}\alpha_m$, and $|\sigma-\tau| \le 2^{m+7}$. Then a Taylor expansion yields

$$\begin{aligned} |P^{(\nu)}(\tau) - P^{(\nu)}(\sigma)| &\le \sum_{k=1}^{\deg(P)-\nu} \Big|\frac{P^{(\nu+k)}(\sigma)}{k!}(\tau-\sigma)^k\Big| \\ &\le |P^{(\nu)}(\sigma)| \sum_{k=1}^{\deg(P)-\nu} \frac{2^{-(m+10)k}2^{(m+7)k}}{k!} \\ &\le (e^{1/8}-1)|P^{(\nu)}(\sigma)| \end{aligned}$$

and $e^{1/8} - 1 \le 1/5$.

(ii) follows from multiple applications of the chain rule, and the definition of the cut-off functions. □

We now set

$$(\tilde u_j, \tilde v_j^r, \tilde w_j) = (u_j\chi_j, v_j^r\chi_j, w_j\chi_j) \tag{6.20}$$

and

$$\tilde u_j^L(x,y) = \tilde u_j(x,y)\eta(2^{-L}(p''(2x_1))) \tag{6.21.1}$$

$$\tilde w_j^{M,L}(x,y) = \tilde w_j(x,y)\eta(2^{-L}(p''(2x_1)))\eta(2^{-M}(p'(2x_1)+2x_2)) \tag{6.21.2}$$

Lemma 6.3.

For $l = 1, 2, 3, \dots$ the following holds.

(i)

$$|\partial_{x_1}^l \tilde u_j(x,y)| + |\partial_{y_1}^l \tilde u_j(x,y)| \le C_l 2^{-lj_1}. \tag{6.22}$$

(ii)

$$|\partial_{x_1}^l \tilde{u}_j^L(x,y)| + |\partial_{y_1}^l \tilde{u}_j^L(x,y)| \le C_l 2^{-lj_1}, \qquad \text{for all } j_1; \tag{6.23}$$

(iii) $\tilde{w}_j^{M,L}(x,y) = 0$ *if either* $M \le j_2$ *or* $L + j_1 \ge M$. *Moreover*

$$|\partial_{x_1}^{l_1}\partial_{x_2}^{l_2} \tilde{w}_j^{M,L}(x,y)| + |\partial_{y_1}^{l_1}\partial_{y_2}^{l_2} \tilde{w}_j^{M,L}(x,y)| \le C_l 2^{-l_1 j_1 - l_2 j_2}. \tag{6.24}$$

(iv) For all x_1, x_2, y_1, y_2

$$\int |\partial_{x_2} \tilde{v}_j^r(x,y)| dx_2 + \int |\partial_{y_2} \tilde{v}_j^r(x,y)| dy_2 \le C. \tag{6.25}$$

Proof. These are straightforward computations using the chain rule, Lemma 6.2 and the definition of the cut-off functions. For (6.25), we use the fact that the sign of η_0' changes finitely many times. □

The next lemma is used to estimate various operators of Hilbert integral type. The argument is closely related to one in [9].

Lemma 6.4. *Let P be a polynomial of degree $\le m$. Then for $\rho > 0$*

$$\iint_{|s-t|\le\rho} \left|1 - \eta_0\left(A\frac{P'(c_1 s + c_2 t)}{P(c_1 s + c_2 t)}\rho\right)\right| ds\, dt \le C \frac{Am^2}{|c_1 + c_2|}\rho^2.$$

Proof. Let $\kappa_1 < \cdots < \kappa_\ell$ be the real parts of the zeroes of P. For $\nu = 1, \dots, \ell - 1$, let $\mu_\nu = (\kappa_\nu + \kappa_{\nu+1})/2$. Let $I_1 = (-\infty, \mu_1)$, $I_\nu = (\mu_{\nu-1}, \mu_\nu)$, $2 \le \nu \le \ell - 1$, and $I_\ell = (\mu_{\ell-1}, \infty)$. Then

$$\left|\frac{P'(\sigma)}{P(\sigma)}\right| \le \frac{m}{|\sigma - \kappa_\nu|}, \text{ for } \sigma \in I_\nu.$$

Therefore the set

$$\{(s,t) : |s - t| \le \rho, A\frac{|P'(c_1 s + c_2 t)|}{|P(c_1 s + c_2 t)|}\rho \ge 1/2\}$$

is contained in

$$\bigcup_{\nu=0}^{\ell} \{(s,t) : (c_1 s + c_2 t) \in I_\nu;\ |c_1 s + c_2 t - \kappa_\nu| \le 2mA\rho;\ |s - t| \le \rho\}$$

which is easily seen to be of measure $O(\rho^2)$; in particular one may check the asserted dependence on c_1, c_2. □

Remark. We shall use this lemma just for the regular case where $(c_1, c_2) = (1, 1)$.

7. Proofs of Propositions 5.1-5.5

7.1. Proof of Proposition 5.1.

We may assume that p is a polynomial of degree at least three, since otherwise $\mathcal{H} = 0$. For $2 \le \nu \le n - 1$, let

$$Q_m^\nu(s) = \{t : |s - t| \le 2^{m+1}, 2^{m+12}|p^{(\nu+1)}(s + t)| \ge |p^{(\nu)}(s + t)|\},$$

and for $g \in L^2(\mathbb{R})$,

$$\mathcal{E}_m^\nu g(s) = 2^{-m} \int_{Q_m^\nu(s)} g(t)\, dt$$

and $\mathcal{E}^\nu = \sum_m \mathcal{E}_m^\nu$. One can use an argument in [9] to show that $\mathcal{E}^\nu$ is bounded on $L^2(\mathbb{R})$. Alternatively, we use an almost orthogonality argument based on Lemma 6.4. Specifically, denote by $k_{lm}(w,z)$ the kernel of $(\mathcal{E}_l^\nu)^*\mathcal{E}_m^\nu$. Since $((\mathcal{E}_l^\nu)^*\mathcal{E}_m^\nu)^* = (\mathcal{E}_m^\nu)^*\mathcal{E}_l^\nu$ we may assume that $l \le m$. Then, for fixed z,

$$\int |k_{lm}(w,z)| dw \lesssim 2^{-m-l} \Big|\{(w,s) : |s-w| \le 2^{l+1}, \frac{|p^{(\nu+1)}(s+w)| 2^{l+1}}{|p^{(\nu)}(s+w)|} \ge \frac{1}{2}\}\Big| \lesssim 2^{l-m},$$

by Lemma 6.4, and since also $\int |k_{lm}(w,z)| dz = O(1)$ for all w we see from Schur's test that $\|(\mathcal{E}_l^\nu)^*\mathcal{E}_m^\nu\| = O(2^{-|m-l|/2})$. By the symmetry of the operators $\mathcal{E}_l^\nu$, one also gets $\|\mathcal{E}_l^\nu(\mathcal{E}_m^\nu)^*\| = O(2^{-|m-l|/2})$ and the Cotlar–Stein Lemma shows the L^2 boundedness of $\mathcal{E}^\nu$. Now

$$|\mathcal{H}f(x)| \lesssim \sum_{\nu=2}^{n-1} \sum_{j_1} 2^{-j_1} \int_{Q_{j_1}^\nu(x_1)} \big|\mathcal{B}^{x_1,y_1}[f(y_1,\cdot)](x_2)\big| dy_1,$$

where $\mathcal{B}^{x_1,y_1}$ is as in (3.9), and by Lemma 3.3 it follows that $\mathcal{H}$ is bounded. □

7.2. Proof of Proposition 5.2.

Part (i) follows from Lemma 6.2 above. Indeed suppose that $a_{j_1}(x_1+y_1) \ne 0$, $|x_1 - y_1| \le 2^{j_1+1}$ and $2^{L-1} \le |p''(2x_1)| \le 2^{L+1}$. Then from (6.17)

$$|p''(2x_1) - p''(x_1+y_1)| \le \frac{1}{5}|p''(2x_1)| \le \frac{4}{5} 2^{L-1},$$

and similarly $|p''(2y_1) - p''(2x_1)| \le \frac{4}{5}2^{L-1}$. Hence if $(x,y) \in \operatorname{supp} U_j^L$ then both $p''(2x_1)$ and $p''(2y_1)$ lie in the interval $(2^{L-4}, 2^{L+2})$. This clearly implies that the operators $\mathcal{U}^L$ are almost orthogonal (in fact $(\mathcal{U}^L)^*\mathcal{U}^{L'} = 0$ and $\mathcal{U}^L(\mathcal{U}^{L'})^* = 0$ if $|L - L'| \ge 10$).

Assuming that $L + 2j_1 + j_2 \le 0$, the estimate (5.8) follows from the definition of $1 - b_{j_1}$ and the inequality $|\sin a| \le |a|$.

Now assume that $L + 2j_1 + j_2 \ge 1$ and write the sine as the sum of two complex exponentials. Then we have to estimate operators $\mathcal{R}_j^{\epsilon,L}$ with kernels

$$R_j^{\epsilon,L}(x,y) = 2^{-j_1-j_2}\chi_j(x-y) u_j^L(x,y) e^{i\gamma\Psi(x,y)+\epsilon\theta(x,y)}, \tag{7.2.1}$$

where $\epsilon = \pm 1$ and $u_j^L(x,y) = u_j(x,y)\eta(2^{-L}p''(2x_1))$.

Let

$$R_j^{\epsilon,L,x_2,y_2}(x_1,y_1) = 2^{j_2} R_j^{\epsilon,L}(x_1,x_2,y_1,y_2), \tag{7.2.2}$$

and denote by $\mathcal{R}_j^{\epsilon,L,x_2,y_2}$ the corresponding operator acting on functions in $L^2(\mathbb{R})$.

Let $\Phi \equiv \Phi^\epsilon = \gamma\Psi + \epsilon\theta$. We note that

$$|\Phi_{x_1}(x_1,y_1,x_2,y_2) - \Phi_{x_1}(x_1,z_1,x_2,y_2)| \approx 2^{L+j_2}|y_1 - z_1|.$$

This follows since by (6.7) and (6.17),

$$|\Phi_{x_1y_1}(x_1,\tilde{z}_1,x_2,y_2)| \approx 2^{L+j_2}$$

if $\tilde{z}_1$ is between z_1 and y_1. The derivative $\Phi_{x_1}(x_1, y_1, x_2, y_2) - \Phi_{x_1}(x_1, z_1, x_2, y_2)$ has only a bounded number of sign changes and we may use van der Corput's Lemma to see that the kernel $K_j(y_1, z_1)$ of $(\mathcal{R}_j^{\epsilon,L,x_2,y_2})^* \mathcal{R}_j^{\epsilon,L,x_2,y_2}$ satisfies the estimate

$$|K_j(y_1, z_1)| \lesssim 2^{-j_1}(1 + 2^{L+j_2+j_1}|y_1 - z_1|)^{-1}.$$

Hence it follows from Schur's test that

$$\|(\mathcal{R}_j^{\epsilon,L,x_2,y_2})^* \mathcal{R}_j^{\epsilon,L,x_2,y_2}\| \lesssim (L + 2j_1 + j_2)2^{-L-2j_1-j_2} \lesssim 2^{-(L+2j_1+j_2)/2},$$

uniformly in x_2, y_2. Consequently $\mathcal{R}_j^{\epsilon,L,x_2,y_2}$ is bounded on $L^2(\mathbb{R})$, with operator norm of order at most $2^{-(L+2j_1+j_2)/4}$, and by an averaging argument (see the proof of Lemma 3.3) it follows that

$$\|\mathcal{R}_j^{\epsilon,L}\| \lesssim 2^{-(L+2j_1+j_2)/4}$$

and also that the same bound holds for $\mathcal{U}_j^L$.

The orthogonality property (5.9) follows again from the argument in Lemma 6.4. We now give the proof for $(\mathcal{U}_j^L)^* \mathcal{U}_k^L$, and without loss of generality, we may assume that $k_1 \leq j_1$.

Let $K_{jk}(y, z)$ be the kernel of $(\mathcal{U}_j^L)^* \mathcal{U}_k^L$; with $k_1 \leq j_1$. Now for every (x_2, z_2), let

$$E_{x_2,z_2}^{k_1} = \{(x_1, z_1) : |x_1 - z_1| \leq 2^{k_1+1},\ |p''(x_1 + z_1)| \geq 2^{-k_1-100}|p'(x_1 + z_1) + x_2 + z_2|\}.$$

By Lemma 6.4, the measure of $E_{x_2,z_2}^{k_1}$ is $O(2^{2k_1})$. Therefore

$$\iint |u_j^L(x, y)u_k^L(x, z)\chi_j(x - y)\chi_k(x - z)|dx_1 dz_1 \leq \iint_{E_{x_2,z_2}^{k_1}} dx_1 dz_1 \lesssim 2^{2k_1}.$$

This yields

$$\sup_y \int |K_{jk}(y, z)|dz \lesssim 2^{k_1-j_1},$$

and together with the obvious estimate

$$\sup_z \int |K_{jk}(y, z)|dy \lesssim C,$$

this implies that $\|(\mathcal{U}_j^L)^* \mathcal{U}_k^L\| = O(2^{(k_1-j_1)/2})$, if $k_1 \leq j_1$.

For the estimation of $\|\mathcal{U}_j^L(\mathcal{U}_k^L)^*\|$, one uses that also $|p''(2y_1)| \approx |p''(2x_1)| \approx 2^L$ on the support of the amplitudes (as pointed out above); the argument is then the same as for $(\mathcal{U}_j^L)^* \mathcal{U}_k^L$. $\square$

7.3. Proof of Proposition 5.3.

We first show the bounds asserted for $(\mathcal{V}_j^r)^* \mathcal{V}_k^r$ in (5.10). Since $\|(\mathcal{V}_j^r)^* \mathcal{V}_k^r\| = \|(\mathcal{V}_k^r)^* \mathcal{V}_j^r\|$, it suffices to consider the case where $k_2 \leq j_2$. Observe that the kernel K_{jk} of $(\mathcal{V}_j^r)^* \mathcal{V}_k^r$ is given by

$$K_{jk}(y, z) = 2^{-j_1-k_1-j_2-k_2} \int e^{i(\Psi(x,z)-\Psi(x,y))} \sin\theta(x, z) \sin\theta(x, y) v_j^r(x, y) v_k^r(x, z) \chi_j(x - y)\chi_k(x - z)dx.$$

For fixed x_1 and z_1, we estimate

$$\tag{7.3.1} \begin{aligned} &\iint |v_j^r(x,y)v_k^r(x,z)\chi_j(x-y)\chi_k(x-z)|dx_2dz_2 \\ &\leq \iint_{\substack{|p'(x_1+z_1)+x_2+z_2|\leq 2^{k_2+10-r} \\ |x_2-z_2|\leq 2^{k_2+1}}} dx_2dz_2 \lesssim 2^{2k_2-r}, \end{aligned}$$

and therefore

$$\int |K_{jk}(y,z_1,z_2)|dz_2 \lesssim \min\{2^{j_1},2^{k_1}\}2^{2k_2-r}2^{-j_1-j_2-k_1-k_2}.$$

Now K_{jk} is supported where $|y_1-z_1|\leq \max\{2^{j_1+2},2^{k_1+2}\}$, and so

$$\sup_y \int |K_{jk}(y,z)|dz \lesssim 2^{k_2-j_2-r}.$$

If we reverse the role of y and z in (7.3.1), we have to use the less favorable bound

$$\tag{7.3.2} \begin{aligned} &\iint |v_j^r(x,y)v_k^r(x,z)\chi_j(x-y)\chi_k(x-z)|dx_2dy_2 \\ &\leq \iint_{\substack{|p'(x_1+z_1)+x_2+z_2|\leq 2^{k_2+10-r} \\ |x_2-y_2|\leq 2^{j_2+1}}} dx_2dy_2 \lesssim 2^{j_2+k_2-r}, \end{aligned}$$

and we obtain

$$\sup_z \int |K_{jk}(y,z)|dy \lesssim C2^{-r}.$$

Taking the geometric mean and applying Schur's test, it follows that

$$\tag{7.3.3} \|(\mathcal{V}_j^r)^*\mathcal{V}_k^r\| = O(2^{-r-|k_2-j_2|/2}).$$

By the symmetry of $\mathcal{V}_j^r$ we obtain the same bound for $\|\mathcal{V}_j^r(\mathcal{V}_k^r)^*\|$.

We now turn to the assertion (ii). To obtain the bound $\|\mathcal{V}_j^r\| = O(2^{2j_2+j_1-r})$ we just use Schur's lemma and invoke the estimate $|\sin a|\leq |a|$ and the support property of $h_{j_2,r}$.

It remains to prove that $\|c\mathcal{V}_j^r\| = O(2^{r/2-(2j_2+j_1)/4})$. Take $\epsilon,\epsilon'\in\{\pm 1\}$, and define

$$\tag{7.3.4} \Gamma_{\epsilon,\epsilon'} = \{(x,y) : \operatorname{sign}\gamma = \epsilon'\epsilon \operatorname{sign}\beta'(x_1-y_1)\ \operatorname{sign}(x_2+y_2+p'(x_1+y_1))\}.$$

Let $\chi_{\epsilon,\epsilon'}$ be the characteristic function of $\Gamma_{\epsilon,\epsilon'}$, and let

$$\tag{7.3.5} V_j^{r,\epsilon,\epsilon'}(x,y) = 2^{-j_1-j_2}\chi_j(x-y)e^{i(\gamma\Psi(x,y)+\epsilon\theta(x,y))}v_j^r(x,y)\chi_{\epsilon,\epsilon'}(x,y),$$

so that

$$V_j^r = \sum_{\epsilon,\epsilon'\in\{-1,1\}} \frac{\epsilon}{2i}V_j^{r,\epsilon,\epsilon'}.$$

It clearly suffices to prove that

$$\tag{7.3.6} \|\mathcal{V}_j^{r,\epsilon,\epsilon'}\| = O(2^{r/2-(2j_2+j_1)/4}).$$

The kernel $K_j(y,z)$ of $(\mathcal{V}_j^{r,\epsilon,\epsilon'})^*\mathcal{V}_j^{r,\epsilon,\epsilon'}$ is given by

$$K_j(y,z) = 2^{-2j_1-2j_2} \int\int_{E(y,z,x_1)} e^{i(\Phi(x,z)-\Phi(x,y))} v_j^r(x,y) v_j^r(x,z)\chi_j(x-y)\chi_j(x-z)\, dx_2\, dx_1,$$

where

$$E(y,z,x_1) = \{x_2 : (x_1,x_2,y) \in \Gamma_{\epsilon,\epsilon'}, (x_1,x_2,z) \in \Gamma_{\epsilon,\epsilon'}\}.$$

Clearly $E(y,z,x_1)$ is the union of no more than 16 intervals. We note that

$$|\Phi_{x_2x_2}(x,z) - \Phi_{x_2x_2}(x,y))| \approx |y_1 - z_1|. \tag{7.3.7}$$

To see this, apply the mean value theorem and observe that $\Psi_{x_2x_2y_1} = -2$ and

$$\theta_{x_2x_2y_1} = -\epsilon'\epsilon\,\mathrm{sign}\,\gamma + o_{j_1},$$

where $|o_{j_1}| \le 2^{-2j_1}$. Thus, since $|\gamma| \ge 1$ and $j_1 \ge 10$ we see that $|\Phi_{x_2x_2y_1}| \approx 2$. Hence we can use (7.3.7) to apply van der Corput's lemma on each of the connected components of $E(y,z,x_1)$. Taking into account the bound (6.25), we see that

$$\Big|\int_{E(y,z,x_1)} e^{i(\Phi(x,z)-\Phi(x,y))} v_j^r(x,y)v_j^r(x,z)\chi_j(x-y)\chi_j(x-z)dx_2\Big| \lesssim 2^r|y_1-z_1|^{-1/2}, \tag{7.3.8}$$

uniformly in x_1, y_2 and z_2. From (7.3.8), it follows that $|K_j(y,z)|$ is dominated by $2^{r-j_1-2j_2}|y_1 - z_1|^{-1/2}$, and of course it is supported where $|y_1 - z_1| \le 2^{j_1+1}$, $|y_2 - z_2| \le 2^{j_2+1}$. We apply Schur's test and deduce that

$$\|(\mathcal{V}_j^{r,\epsilon,\epsilon'})^*\mathcal{V}_j^{r,\epsilon,\epsilon'}\| \lesssim 2^{r-(2j_2+j_1)/2};$$

hence we get the bound (7.3.6) and consequently the bound $\|\mathcal{V}_j^r\| \lesssim 2^{r/2-(2j_2+j_1)/4}$. □

7.4. Proof of Proposition 5.4.

Part (i) follows in view of the localization of the amplitude. Suppose that $(x,y) \in \mathrm{supp}\, W_j^{M,L}$ and $\chi_j(x,y) \neq 0$. Then $2^{M-1} \le |2x_2 + p'(2x_1)| \le 2^{M+1}$ and since $|p'(2y_1) - p'(2x_1)| \le 2^{j_1+L+2}$ from Lemma 6.2, we have

$$2y_2 + p'(2y_1) \in [2^{M-1} - 2^{j_2+2} - 2^{j_1+L+2}, 2^{M+1} + 2^{j_2+2} + 2^{j_1+L+2}];$$

moreover the quantity $x_2 + y_2 + p'(x_1 + y_1)$ is also contained in this interval. Since $j_2 \le M - 10$ and $L + j_1 \le M - 10$, we see that

$$2^{M-2} \le |x_2 + y_2 + p'(x_1 + y_1)| \le 2^{M+2} \tag{7.4.1}$$

$$2^{M-2} \le |2y_2 + p'(2y_1)| \le 2^{M+2}. \tag{7.4.2}$$

Furthermore $|x_1 - y_1| \le 2^{j_1+1}$, $2^{L-1} \le |p''(2x_1)| \le 2^{L+1}$, and so Lemma 6.2 yields $|p''(2y_1)| \approx |p''(2x_1)| \approx 2^L$. Therefore we can conclude that the operators $\mathcal{W}^{M,L}$ are almost orthogonal; specifically $(\mathcal{W}^{M,L})^*\mathcal{W}^{M',L'} = 0$ and $\mathcal{W}^{M,L}(\mathcal{W}^{M',L'})^* = 0$ if either $|M - M'| \ge 10$ or $|L - L'| \ge 10$. This implies (5.13). If $M + j_1 + j_2 \le 0$ the estimate (5.14) follows from the fact that $\sin a = O(|a|)$.

We now assume that $M + j_1 + j_2 \ge 0$. For $\epsilon, \epsilon' \in \{\pm 1\}$, let $\chi_{\epsilon,\epsilon'}$ be the characteristic function of the set $\Gamma_{\epsilon,\epsilon'}$, defined in (7.3.4). Fix $L, M, \epsilon, \epsilon'$ and let

$$\omega_j(x,y) \equiv \omega_j^{M,L,\epsilon,\epsilon'}(x,y) := \widetilde{w}_j^{M,L}(x,y)\chi_{\epsilon,\epsilon'}(x,y)$$

(see (6.21.2)), and let $\mathcal{W}_j^{M,L,\epsilon,\epsilon'}$ be the integral operator with kernel

$$W_j^{M,L,\epsilon,\epsilon'}(x,y) = 2^{-j_1-j_2}\omega_j^{M,L,\epsilon,\epsilon'}(x,y)e^{i(\gamma\Psi(x,y)+\epsilon\theta(x,y))}.$$

Multiplication with the characteristic function $\chi_{\epsilon,\epsilon'}$ does not introduce additional singularities in view of the localization of the symbol w_j; in fact, we have the estimates

$$|\partial_x^\alpha \partial_y^\beta \omega_j(x,y)| \le C_{\alpha,\beta} 2^{-j_1(\alpha_1+\beta_1)} 2^{-j_2(\alpha_2+\beta_2)}. \tag{7.4.3}$$

The kernel $K_j(y,z)$ of $(\mathcal{W}_j^{M,L,\epsilon,\epsilon'})^* \mathcal{W}_j^{M,L,\epsilon,\epsilon'}$ is given by

$$K_j(y,z) = 2^{-2j_1-2j_2} \int e^{i(\Phi(x,z)-\Phi(x,y))} \omega_j(x,y)\omega_j(x,z)\chi_j(x-y)\chi_j(x-z)dx.$$

In view of our assumptions that $|\gamma| \ge 1$ and $j_1 \ge 10$, we see that

$$|\Phi_{x_2 y_1}| \approx |2x_2 + p'(x_1+y_1)| \approx 2^M,$$

and also that $\Phi_{x_2 y_2} \equiv 0$. Hence

$$|\Phi_{x_2}(x,z) - \Phi_{x_2}(x,y)| \approx 2^M |y_1 - z_1|.$$

Applying van der Corput's Lemma,

$$|K_j(y,z)| \lesssim 2^{-j_1-j_2}(1 + 2^{M+j_2}|y_1-z_1|)^{-1},$$

and since K_j is supported where $|y_2 - z_2| \le 2^{j_2+1}$, we have

$$\sup_y \int |K_j(y,z)|dz + \sup_z \int |K_j(y,z)|dy \lesssim (M+j_2+j_1)2^{-M-j_2-j_1}.$$

By Schur's test,

$$\|\mathcal{W}_j^{M,L,\epsilon,\epsilon'}\| \lesssim (M+j_2+j_1)^{1/2} 2^{-(M+j_2+j_1)},$$

and this completes the proof of (5.14).

Since we have only used property (7.4.3) our argument proves the assertion (iii) as well. □

7.5. Proof of Proposition 5.5.

Fix I and let $\mathcal{Z} \equiv \mathcal{Z}^I$. Choose $\zeta \in C_0^\infty(\mathbb{R})$, supported in $(-1,1)$ and with the property that $\sum_{\nu\in\mathbb{Z}} \zeta(s-\nu) = 1$. Define the operator $\mathcal{Z}_\nu \equiv \mathcal{Z}_\nu^I$ with kernel

$$Z_\nu(x,y) = \zeta(2^{L+m-M+10}x_1 - \nu)Z^I(x,y).$$

In view of the localization $|x_1 - y_1| \le 2^{M-L-m+1}$ we see that the operators $\mathcal{Z}_\nu$ are almost orthogonal; i.e., $(\mathcal{Z}_\nu)^*\mathcal{Z}_{\nu'} = 0$ and $\mathcal{Z}_\nu(\mathcal{Z}_{\nu'})^* = 0$ if $|\nu - \nu'| \ge 100$. Therefore

$$\|\mathcal{Z}\| \lesssim \sup_\nu \|\mathcal{Z}_\nu\|. \tag{7.5.1}$$

It hence suffices to prove a uniform estimate for the operators $\mathcal{Z}_\nu$. We wish to approximate the phase functions Ψ and θ by affine functions in the first variable.

We may suppose there is a point c_ν such that

(7.5.2) $$\eta(2^{-L}p''(2c_\nu)) \neq 0 \text{ and } |c_\nu - 2^{M-L-m}\nu| \leq 2^{M-L-m-9},$$

for if not then $\mathcal{Z}_\nu \equiv 0$. Define

$$\Psi_\nu(x,y) = (x_1 - y_1)(x_2^2 + y_2^2) - (x_2 - y_2)p(2c_\nu) - (x_2 - y_2)p'(2c_\nu)(x_1 + y_1 - 2c_\nu)$$
$$\theta_\nu(x,y) = \beta(x_1 - y_1)(x_2 - y_2)|x_2 + y_2 + p'(2c_\nu)|.$$

Now $\mathcal{Z}_\nu = \sum_{j\in A} \mathcal{Z}_{\nu,j}$ where in the sum only those j_1 come up which satisfy

$$M - L - \frac{2n+1}{2n}m < j_1 \leq M - L - m,$$

and the kernel of $\mathcal{Z}_{\nu,j}$ is defined by

(7.5.3) $$Z_{\nu,j}(x,y) = 2^{-j_1-j_2}e^{i\gamma\Psi(x,y)}\sin\theta(x,y)\chi_j(x-y)$$
$$\times\, a_{j_1}(x_1+y_1)b_{j_1}(x+y)\eta(2^{-L}p''(2x_1))\eta(2^{-M}(2x_2 + p'(2x_1)))\zeta(2^{L+m-M+10}x_1 - \nu).$$

Since we assume that $L + j_1 \ll M$ the function $b_{j_1}(x+y)$ can be omitted in (7.5.3); it is equal to 1 on the support of the other cut-off functions. Let

$$z_j^{L,M,\nu}(x,y) = \chi_j(x-y)\eta(2^{-L}p''(2x_1))\eta(2^{-M}(2x_2 + p'(2x_1)))\zeta(2^{L+m-M+10}x_1 - \nu).$$

We split $Z_{\nu,j}(x,y)$ as $\sum_{i=1}^3 Z_{\nu,j}^i(x,y)$, where

(7.5.4)
$$Z_{\nu,j}^1(x,y) = 2^{-j_1-j_2}\big(e^{i\gamma\Psi(x,y)}\sin\theta(x,y) - e^{i\gamma\Psi_\nu(x,y)}\sin\theta_\nu(x,y)\big)z_j^{L,M,\nu}(x,y)a_{j_1}(x_1+y_1),$$

(7.5.5)
$$Z_{\nu,j}^2(x,y) = 2^{-j_1-j_2}e^{i\gamma\Psi_\nu(x,y)}\sin\theta_\nu(x,y)z_j^{L,M,\nu}(x,y)\big(a_{j_1}(x_1+y_1) - 1\big),$$

(7.5.6)
$$Z_{\nu,j}^3(x,y) = 2^{-j_1-j_2}e^{i\gamma\Psi_\nu(x,y)}\sin\theta_\nu(x,y)z_j^{L,M,\nu}(x,y),$$

and form operators $\mathcal{Z}_\nu^i = \sum_{j\in A}\mathcal{Z}_{\nu,j}^i$ where $\mathcal{Z}_{\nu,j}^i$ has kernel $Z_{\nu,j}^i$.

The operator $\mathcal{Z}_\nu^2$ is handled by the argument in the proof of Proposition 5.1, with only notational changes.

The operator $\mathcal{Z}_\nu^3$ represents the main term. Note however that $\mathcal{Z}_\nu^3 f(x) = g(x)\widetilde{\mathcal{Z}}_{\nu,3}f(x)$, where g is a bounded function, and $\widetilde{\mathcal{Z}}_{\nu,3}$ is an operator which is already shown to be bounded by Theorem 4.1. Thus $\|\mathcal{Z}_\nu^3\| = O(1)$.

It remains to estimate the kernel $Z_{\nu,j}^1$. Suppose that $a_{j_1}(x_1+y_1)z_j^{L,M,\nu}(x,y) \neq 0$. Then

$$|\Psi(x,y) - \Psi_\nu(x,y)| \leq |p(x_1+y_1) - p(2c_\nu) - p'(2c_\nu)(x_1+y_1-2c_\nu)||x_2 - y_2|$$
$$\leq |x_2 - y_2|\sum_{l=2}^{n}\frac{|p^{(l)}(2c_\nu)|}{l!}|x_1+y_1-2c_\nu|^l$$
$$\leq |x_2 - y_2|\sum_{l=2}^{n}\Big|\sum_{s=0}^{n-l}p^{(l+s)}(2x_1)\frac{(2c_\nu - 2x_1)^s}{s!}\Big|\frac{|x_1+y_1-2c_\nu|^l}{l!}$$
$$\lesssim 2^{j_2}\sum_{l=2}^{n}2^{L-(l-2)(j_1+10)}2^{(M-L-m+1)l}$$
$$\lesssim 2^{L+2j_1+j_2}\sum_{l=2}^{n}2^{(M-L-m+1-j_1)l}$$

(7.5.7) $$\lesssim 2^{L+2j_1+j_2+\frac{m}{2}},$$

since we assumed that $M - L - m - j_1 \le m/2n$. Similarly

$$\begin{aligned} |\theta(x,y) - \theta_\nu(x,y)| &\le \beta(x_1 - y_1)|x_2 - y_2||p'(2x_1) - p'(2c_\nu)| \\ &\lesssim 2^{j_1+j_2} \sum_{l=2}^{n} \frac{|p^{(l)}(2x_1)|}{(l-1)!}|2x_1 - 2c_\nu|^{l-1} \\ &\lesssim 2^{j_1+j_2} \sum_{l=2}^{n} 2^{L-(l-2)(j_1+10)} 2^{(M-L-m+1)(l-1)} \\ &\lesssim 2^{L+2j_1+j_2+\frac{m}{2}}. \end{aligned} \tag{7.5.8}$$

Moreover

$$|2x_2 + p'(2x_1)| \approx |2y_2 + p'(2y_1)| \approx 2^M, \tag{7.5.9}$$

and then

$$|2x_2 + p'(2c_\nu)| \approx |2y_2 + p'(2c_\nu)| \approx 2^M \tag{7.5.10}$$

because $j_2 \ll M$ and

$$\begin{aligned} |p'(2x_1) - p'(2c_\nu)| &\le \sum_{l=2}^{n} \frac{|p^{(l)}(2c_\nu)|}{(l-1)!}|2x_1 - 2c_\nu|^{l-1} \\ &\le \sum_{l=2}^{n} 2^{L-j_1(l-2)} 2^{(M-L-m)(l-1)} \\ &\le 2^{L+j_1+\frac{m}{2}+2} \le 2^{M-\frac{m}{2}+2}. \end{aligned}$$

Similarly $|p'(2y_1) - p'(2c_\nu)| \le 2^{M-\frac{m}{2}+2}$.

From (7.5.7) and (7.5.8), it follows that

$$\left| e^{i\gamma\Psi(x,y)} \sin\theta(x,y) - e^{i\gamma\Psi_\nu(x,y)} \sin\theta_\nu(x,y) \right| \lesssim 2^{L+2j_1+j_2+\frac{m}{2}}$$

for the relevant values of (x, y) in (7.5.4), and Schur's test yields

$$\|\mathcal{Z}^1_{\nu,j}\| \lesssim 2^{L+2j_1+j_2+\frac{m}{2}}.$$

By (7.5.9) and (7.5.10), we may apply either Proposition 5.4 with the polynomial p or with the affine polynomial $p(c_\nu) + p'(c_\nu)(s - c_\nu)$, and the suitable choice of ρ_j in part (iii) of Proposition 5.4. This leads to

$$\|\mathcal{Z}^1_{\nu,j}\| \lesssim \min\{2^{-(M+j_1+j_2)/4}, 2^{M+j_1+j_2}\} \tag{7.5.12}$$

if $j_1 \le M - L$. We obtain

$$\sum_{\substack{j\in I \\ M+j_1+j_2=\ell}} \|\mathcal{Z}^1_{\nu,j}\| \lesssim \sum_{\substack{j_1 \le M-L-m \\ L+2j_1+j_2 \le 0 \\ M+j_1+j_2=\ell}} \min\{2^{L+2j_1+j_2+\frac{m}{2}}, 2^{-(M+j_1+j_2)/4}\} \lesssim \min\{2^{-\frac{\ell}{4}}, 2^{\ell-\frac{m}{2}}\}.$$

Summing over ℓ demonstrates the boundedness of the operator $\mathcal{Z}^1_\nu$.

We have shown that $\mathcal{Z}^I_\nu = \sum_{i=1}^3 \mathcal{Z}^i_\nu$ is bounded with operator norm uniformly in M, L, m, I, ν. The assertion of the theorem now follows from (7.5.1). □

8. Failure of weak amenability for Lie groups

Suppose that G is a connected Lie group, with Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ decomposes as $\mathfrak{s} \oplus \mathfrak{r}$, where $\mathfrak{s}$ is a semisimple subalgebra and $\mathfrak{r}$ is the maximal solvable ideal of $\mathfrak{g}$. We may write $\mathfrak{s}$ as a sum of simple ideals:

$$\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m. \tag{8.1}$$

Denote by R, S and S_i the analytic subgroups of G corresponding to $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{s}_i$. Then R is closed, but S and S_i need not be. Further, $G = SR$, but this need not be a semidirect product, as $S \cap R$ may be nontrivial. To do analysis on G, we need S to be closed and the product SR to be semidirect. Our first result enables us to work in this better environment by passing to a finite covering group.

Proposition 8.1. *Let G, R, S and S_i be as described above, and suppose that S has finite center. Then G has a finite covering group $G^\natural$ which has closed connected subgroups $R^\natural$, $S^\natural$ and $S_i^\natural$, whose Lie algebras are $\mathfrak{r}$, $\mathfrak{s}$ and $\mathfrak{s}_i$, such that $R^\natural$ is normal and solvable, $S^\natural$ is the direct product of the simple Lie groups $S_i^\natural$, each of which has finite center, and $R^\natural \cap S^\natural = \{e\}$; thus $G^\natural$ is the semidirect product $S^\natural \ltimes R^\natural$.*

Here $G^\natural$ is said to be a finite covering group if G is isomorphic to $G^\natural/Z$ where Z is a finite normal subgroup of $G^\natural$.

We leave the proof of Proposition 8.1 until later. Observe that $\Lambda(G) = \Lambda(G^\natural)$, by 1.2.4, (ii); moreover by 1.2.4, (i), $G^\natural$ has a multiplier bounded approximate unit if G has one. Thus to compute $\Lambda(G)$, we may and shall henceforth assume that G, R, S and S_i have the properties of $G^\natural, R^\natural, S^\natural$ and $S_i^\natural$ in Proposition 8.1.

Now to prove the theorem, observe that if the factors S_i making up S are all either compact (when $i \in I$, say) or of real rank one and commute with R (when $i \in J$, say) then we may write G as a direct product:

$$G = \big(\prod_{i\in J} S_i\big) \times \big(\big(\prod_{i\in I} S_i\big) \ltimes R\big). \tag{8.2}$$

The second factor is amenable, so $\Lambda((\prod_{i\in I} S_i) \ltimes R) = 1$, and hence

$$\Lambda(G) = \prod_{i\in J} \Lambda(S_i) = \prod_{i=1}^{m} \Lambda(S_i), \tag{8.3}$$

by 1.2.1(iv).

On the other hand, if any S_i, $i \in J$, is of real rank at least two, then $\Lambda(G) \geq \Lambda(S_i) = +\infty$ (see 1.2.1(iii)); moreover the proof in [14] and [10] that $\Lambda(S_i) = \infty$ in combination with (1.2.4 (i)) shows that S_i and therefore G does not have multiplier bounded approximate units.

The remaining case to consider is when there is a factor S_i of real rank one which does not centralize R. The following result contains the structural information needed to reduce to known cases.

Proposition 8.2. *Suppose that the connected Lie group G is a semidirect product of the form $S \ltimes R$, where S is closed, connected, semisimple and has finite center, and R is closed, connected and solvable, and suppose that a noncompact factor S_i of S does not centralize R. Then G contains a closed subgroup G_0 with a compact normal subgroup K_0 such that G_0/K_0, or a double cover of G_0/K_0, is isomorphic to $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^n$ (where $n \geq 2$) or to $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^n$ (where $n \geq 1$).*

Thus under the assumptions of Proposition 8.2. it follows that G_0/K_0 does not admit multiplier bounded approximate units, by the calculations of [9] for the groups $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^n$ and of this paper for the groups $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathrm{H}^n$. Thus by (1.2.1), (iii) and (1.2.4), (i) the group G does not have multiplier bounded approximate units and in particular we have $\Lambda(G) = \infty$.

It remains to prove Propositions 8.1 and 8.2.

Proof of Proposition 8.1. To every Lie algebra $\mathfrak{a}$, we may associate a unique connected, simply connected Lie group A with Lie algebra $\mathfrak{a}$. Every connected Lie group A' with Lie algebra $\mathfrak{a}$ is a quotient of A by a discrete normal, and hence central, subgroup D of A. For these facts, and much more, about the structure of Lie algebras and Lie groups, see, for instance, [11] or [21], [34]. Consequently, we will be interested in the structure of the center of a connected, simply connected Lie group.

Let $G^\sharp$ be the simply connected covering group of G, and let $R^\sharp$ and $S^\sharp$ be the subgroups of $G^\sharp$ corresponding to $\mathfrak{r}$ and $\mathfrak{s}$. Then $R^\sharp$ and $S^\sharp$ are both closed in $G^\sharp$; further, $R^\sharp$ is normal in $G^\sharp$ and $S^\sharp \cap R^\sharp = \{e\}$, so that $G^\sharp = S^\sharp \ltimes R^\sharp$ ([34], Thm. 3.18.13). Consequently, the center $Z(G^\sharp)$ of $G^\sharp$ may be written as a direct product: $Z(G^\sharp) = Z_S \times Z_R$, where Z_R is the subgroup of the center $Z(R^\sharp)$ of $R^\sharp$ of elements which commute with $S^\sharp$, and Z_S is the subgroup of the center $Z(S^\sharp)$ of $S^\sharp$ of elements commuting with $R^\sharp$. Let $S_i^\sharp$ be the subgroup of $G^\sharp$ corresponding to $\mathfrak{s}_i$. Then $S_i^\sharp$ is closed in $S^\sharp$, and hence in $G^\sharp$; further, $S_i^\sharp$ is simply connected and normal in $S^\sharp$, so that $S^\sharp$ is a direct product of the factors $S_i^\sharp$, and $Z(S^\sharp)$ is the direct product of the centers $Z(S_i^\sharp)$ of the $S_i^\sharp$ ([34], Thm. 3.18.1).

The group G, being a quotient of $G^\sharp$, is of the form $G^\sharp/D$, where D is a discrete subgroup of $Z_S \times Z_R$. Set

$$D_0 = \prod_{i=1}^{m} [D \cap S_i^\sharp] \times [D \cap R^\sharp].$$

We need an auxiliary result.

Lemma 8.1.1.

(i) D_0 is of finite index in D.

(ii) Each $D \cap S_i^\sharp$ is of finite index in the center $Z(S_i^\sharp)$ of $S_i^\sharp$.

Taking this for granted the group G/D_0 is a finite covering of G/D, and has the required properties as it is isomorphic to

$$\prod_{i=1}^{m} [S_i^\sharp/(D \cap S_i^\sharp)] \ltimes [R^\sharp/(D \cap R^\sharp)].$$

We take $S_i^\natural$ to be $S_i^\sharp/(D \cap S_i^\sharp)$ and $R^\natural$ to be $R^\sharp/(D \cap R^\sharp)$. Then $R^\natural$ is closed, normal and solvable, and $S_i^\natural$ is closed, simple and have finite center. The center of $S^\natural$ is the product of the centers of the groups $S_i^\natural$, and is also be finite. Finally, $S^\natural \cap R^\natural$ is trivial, where $S^\natural = \prod_{i=1}^m S_i^\natural$.

It remains to give the

Proof of Lemma 8.1.1. First, we claim that Z_S is of finite index in $Z(S^\sharp)$. Indeed, the adjoint action $\mathrm{Ad}_\mathfrak{r}$ of $S^\sharp$ on $\mathfrak{r}$ is a linear representation of $S^\sharp$, and the image $\mathrm{Ad}_\mathfrak{r} S^\sharp$ of $S^\sharp$ in $\mathrm{SL}(\mathfrak{r})$ is a closed semisimple subgroup of $\mathrm{SL}(\mathfrak{r})$; the center C of this subgroup is finite [21, Prop. 7.9] Moreover, $\mathrm{Ad}_\mathfrak{r}(Z(S^\sharp))$ is contained in C, so that $\mathrm{Ad}_\mathfrak{r}(Z(S^\sharp))$ is finite. The group $R^\sharp$ is generated by arbitrarily small neighbourhoods of the identity, so an element of $S^\sharp$ centralizes $R^\sharp$ if and only if it centralizes small neighbourhoods of the identity, and hence if and only if it acts trivially on $\mathfrak{r}$ by the adjoint action. Then Z_S is the kernel of the adjoint map of $Z(S^\sharp)$ into $\mathrm{SL}(\mathfrak{r})$. In conclusion, $\mathrm{Ad}_\mathfrak{r}(Z(S^\sharp))$ is isomorphic to $Z(S^\sharp)/Z_S$, so Z_S is indeed of finite index in $Z(S^\sharp)$.

Next, note that $D \cap S^\sharp = D \cap Z(S^\sharp) = D \cap Z_S$, since $D \subseteq Z_S \times Z_R$. We claim that $D \cap S^\sharp$ is of finite index in $Z(S^\sharp)$. Indeed, S is isomorphic to $S^\sharp/D \cap S^\sharp$, and its center is isomorphic to

$Z(S^\sharp)/D\cap S^\sharp$ (in fact, if $x\in S^\sharp\setminus Z(S^\sharp)$, then $\mathrm{Ad}(x)$ acts nontrivially on any small neighbourhood of the identity in $S^\sharp$, and hence its image in $S^\sharp/D\cap S^\sharp$ also acts nontrivially on such a neighbourhood). By hypothesis, the center of S is finite, so $D\cap S^\sharp$ is of finite index in $Z(S^\sharp)$. Since

$$Z(S_i^\sharp)/[D\cap S_i^\sharp]\simeq DZ(S_i^\sharp)/D\subseteq DZ(S^\sharp)/D\simeq Z(S^\sharp)/[D\cap S^\sharp],$$

$D\cap S_i^\sharp$ is of finite index in $Z(S_i^\sharp)$. Now write E for $\prod_{i=1}^m[D\cap S_i^\sharp]$; it follows that E is a subgroup of $Z(S^\sharp)$ of finite index which is part (ii) of the lemma.

For part (i) we shall establish that $E(D\cap R^\sharp)$ is of finite index in D. This also follows from the isomorphism theorems. Since $Z_S/(D\cap Z_S)\subset Z(S^\sharp)/(D\cap Z_S)$ is finite, and $Z_S/(D\cap Z_S)\simeq DZ_S/D$, there exist $z_1,z_2,\ldots,z_J\in Z_S$ such that

$$DZ_S=\bigcup_{j=1}^J Dz_j.$$

If $Dz_j\cap Z_R\neq\emptyset$, take $r_j\in Dz_j\cap Z_R$; otherwise take $r_j=e$. Then it is easy to check that

$$Dz_j\cap Z_R\subseteq(D\cap Z_R)r_j.$$

It follows that

$$(DZ_S)\cap Z_R=\bigcup_{j=1}^J Dz_j\cap Z_R\subseteq\bigcup_{j=1}^J(D\cap Z_R)r_j,$$

and so $(DZ_S)\cap Z_R\big/(D\cap Z_R)$ is also finite. As D is contained in the direct product Z_SZ_R, it follows that $DZ_S=Z_S((DZ_S)\cap Z_R)$, and so

$$\begin{aligned}D\big/(D\cap Z_S)(D\cap Z_R)&\subseteq DZ_S\big/(D\cap Z_S)(D\cap Z_R)\\&=Z_S((DZ_S)\cap Z_R)\big/(D\cap Z_S)(D\cap Z_R)\\&\simeq(Z_S/(D\cap Z_S))((DZ_S)\cap Z_R/D\cap Z_R),\end{aligned}$$

which is finite. Thus $(D\cap Z_S)(D\cap Z_R)$ is of finite index in D. Since E has finite index in $D\cap Z_S$, $E(D\cap Z_R)$ is of finite index in $(D\cap Z_S)(D\cap Z_R)$, and thus of finite index in D. This finishes the proof of Lemma 8.1.1 and Proposition 8.1. □

Proof of Proposition 8.2.

We recall that S_i does not centralize R so that $\mathfrak{s}_i$ does not centralize $\mathfrak{r}$. The proof now proceeds by a series of reductions.

First, we cut down the semisimple part. Take a Cartan involution θ of $\mathfrak{s}_i$, so that $\mathfrak{s}_i=\mathfrak{k}\oplus\mathfrak{p}$, where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, and take a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Then the Lie algebra $\mathfrak{s}_i$ decomposes into a sum:

$$\mathfrak{s}_i=\mathfrak{g}_0+\sum_{\alpha\in\Sigma}\mathfrak{g}_\alpha,$$

where each α is a linear functional on $\mathfrak{a}$, and $X\in\mathfrak{g}_\alpha$ if and only if $[H,X]=\alpha(H)X$ for all $H\in\mathfrak{a}$. For more on these root decompositions, see, e.g., [15], [21]. Take a nonzero element X in this $\mathfrak{g}_\alpha$, where $\alpha\neq 0$. Then $\mathrm{span}\{X,\theta X,[X,\theta X]\}$ is a subalgebra $\mathfrak{s}_0$ of $\mathfrak{s}_i$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$, and the corresponding analytic subgroup S_0 of S_i is locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, has finite center, does not centralize R, and is closed in S_i and hence in S (see [35, Lemma 1.1.5.7]).

Note also that $S_0/Z(S_0)$ is isomorphic to a matrix group ([34, Thm. 2.13.2]) and the only matrix groups locally isomorphic are $SL(2,\mathbb{R})$ and $PSL(2,\mathbb{R})$ (i.e., $SL(2,\mathbb{R})$ divided by its center).

Furthermore, $[\mathfrak{s}_0,\mathfrak{r}]\neq 0$; indeed, $\{X\in\mathfrak{s}_i : \mathrm{ad}(X)|_{\mathfrak{r}}=0\}$ is an ideal in $\mathfrak{s}_i$, which is a simple Lie algebra; and hence $\{X\in\mathfrak{s}_i : \mathrm{ad}(X)|_{\mathfrak{r}}=0\}=\{0\}$. The subgroup $S_0\ltimes R$ is closed in G.

The second reduction cuts down to the nilradical. Let N be the maximal connected normal nilpotent subgroup of G, which is automatically closed, and let $\mathfrak{n}$ be its Lie algebra. Then $N\subseteq R$ and $\mathfrak{n}\subseteq\mathfrak{r}$, and moreover, $[\mathfrak{s}_0,\mathfrak{r}]\subseteq\mathfrak{n}$ (see [34, Thm. 3.8.3]). We claim that $[\mathfrak{s}_0,\mathfrak{n}]\neq\{0\}$. If it were true that $[\mathfrak{s}_0,\mathfrak{n}]=\{0\}$, then the Jacobi identity would imply that,

$$[[X,Y],Z]=[[X,Z],Y]+[X,[Y,Z]]=0,\qquad X,Y\in\mathfrak{s}_0,\ Z\in\mathfrak{r},$$

since the inner commutators of both summands of the middle term of the equality lie in $\mathfrak{n}$, from which it would follow that $[\mathfrak{s}_0,\mathfrak{r}]=\{0\}$. Thus $[\mathfrak{s}_0,\mathfrak{n}]\neq\{0\}$. It now suffices to consider $S_0\ltimes N$, which is closed in $S_0\ltimes R$ and hence in G.

The third reduction allows us to assume that N is simply connected. Let K be the maximal compact connected central subgroup of $S_0\ltimes N$; it is contained in the nilradical N. We observe that the nilradical of $(S_0\ltimes N)/K$ is equal to $S_0\ltimes(N/K)$ and we shall show that $N':=N/K$ is simply connected.

The center $Z(N)$ of N is a connected Abelian Lie group [34, Cor. 3.6.4] and thus isomorphic to $\mathbb{R}^k\times\mathbb{T}^l$, for suitable k,l. We claim that $\mathbb{T}^l$ is central in $S_0\ltimes N$. Note that then $\mathbb{T}^l$ is also the maximal compact connected central subgroup of G (since any connected central subgroup must be in the nilradical).

We first show that $\mathbb{T}^l$ is a normal subgroup. For any fixed $g\in S_0\ltimes N$ the automorphism $\phi_g : n\mapsto gng^{-1}$ fixes the center; thus $g\mathbb{T}^lg^{-1}$ is a compact Lie subgroup of $Z(N)$ which must be $\mathbb{T}^l$; thus $\mathbb{T}^l$ is normal in G. Consider the map $g\to\phi_g|_{\mathbb{T}^l}$ which takes G into the automorphism group of $\mathbb{T}^l$. This group is discrete and since G is connected we see that $\phi_g|_{\mathbb{T}^l}$ is the identity, thus $\mathbb{T}^l$ is central in G and thus isomorphic to K.

We claim that $N':=N/K$ is simply connected. Indeed let $\widetilde{N}$ be the simply connected covering group of N; it has center $Z(\widetilde{N})=\mathbb{R}^{k+l}\supset\mathbb{Z}^l$ and $N=\widetilde{N}/\mathbb{Z}^l$. Now

$$N'=N/\mathbb{T}^l=(\widetilde{N}/\mathbb{Z}^l)/(\mathbb{R}^l/\mathbb{Z}^l)\cong\widetilde{N}/\mathbb{R}^l.$$

But $N'\cong\widetilde{N}/\mathbb{R}^l$ is simply connected by [34, Thm. 3.18.2].

In the remaining part of the proof we shall show that $S_0\ltimes N'\cong(S_0\ltimes N)/K$ has a closed subgroup G_1 which is locally isomorphic to some $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^n$ $(n\geq 2)$ or some $\mathrm{SL}(2,\mathbb{R})\ltimes\mathrm{H}^n$ $(n\geq 1)$. The desired subgroup G_0 of G is then the closed subgroup of $S_0\ltimes N$ of elements whose image in $(S_0\ltimes N)/K$ under the canonical projection lies in G_1, and the appropriate compact subgroup K_0 is the direct product of $Z(S_0\ltimes N)\cap S_0$ and K.

In order to proceed we need the following lemma.

Lemma 8.2.1. *Let $\pi_n : \mathfrak{sl}(2,\mathbb{R})\to\mathrm{End}(\mathbb{R}^n)$ be the (unique) irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension n. The space of bilinear forms $B:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ which satisfy*

$$B(\pi_n(U)V,W)+B(V,\pi_n(U)W)=0\qquad\forall U\in\mathfrak{sl}(2,\mathbb{R})\quad\forall V,W\in\mathbb{R}^n\tag{8.4}$$

is one-dimensional. These forms are symmetric or skew-symmetric as n is odd or even.

Proof. Let

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ Y=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

that is, the standard basis for $\mathfrak{sl}(2,\mathbb{R})$ satisfying the commutation relation $[X,Y]=H$, $[H,X]=2X$ and $[H,Y]=-2Y$. It is well known (see for instance [18, ch. III.8]) that there is a basis $\{E_0,\dots,E_{n-1}\}$ for $\mathbb{R}^n$ such that

$$\begin{aligned}
&\pi_n(H)E_j=(n-1-2j)E_j, \quad j=0,\dots,n-1\\
&\pi_n(X)E_j=E_{j+1}, \quad j=0\dots,n-2, \quad \pi_n(X)E_{n-1}=0\\
&\pi_n(Y)E_j=j(n-j+1)E_{j-1}, \quad j=1\dots,n-1, \quad \pi_n(Y)E_0=0.
\end{aligned}$$

Let B be a bilinear form satisfying (8.4). If $0\le i,j\le n-1$, then

$$0=B(\pi_n(H)E_i,E_j)+B(E_i,\pi_n(H)E_j)=(2n-2i-2j-2)B(E_i,E_j)$$

so that

$$B(E_i,E_j)=0 \text{ if } j\neq n-i-1. \tag{8.5}$$

Further, if $1\le j\le n-1$, then

$$B(E_j,E_{n-j-1})=B(\pi_n(X)E_{j-1},E_{n-j-1})=-B(E_{j-1},\pi_n(X)E_{n-j-1})=-B(E_{j-1},E_{n-j}),$$

whence

$$B(E_i,E_{n-i-1})=(-1)^jB(E_0,E_{n-1}), \qquad 1\le i\le n-1,$$

so that B is completely determined by $B(E_0,E_{n-1})$. In particular,

$$B(E_i,E_{n-i-1})=(-1)^{n-1}B(E_{n-i-1},E_i). \tag{8.6}$$

Thus by (8.5) and (8.6), B is symmetric if n is odd and skew-symmetric if n is even. □

Proof of Proposition 8.2, continued. We now consider $S_0\ltimes N'$ and we must produce a closed subgroup of $S_0\ltimes N'$ locally isomorphic to $\mathrm{SL}(2,\mathbb{R})\ltimes\mathbb{R}^n$ $(n\ge 2)$ or to $\mathrm{SL}(2,\mathbb{R})\ltimes\mathrm{H}^n$ $(n\ge 1)$. Let $\mathfrak{n}$ be the Lie algebra of N'; since N' is simply connected, the exponential map is a homeomorphism from $\mathfrak{n}$ to N', and subalgebras of $\mathfrak{n}$ map to closed subgroups of N'.

We define the ascending central series of $\mathfrak{n}$ inductively: let $\mathfrak{n}_0$ be $\{0\}$, and if $j\ge 1$, define $\mathfrak{n}_j$ to be $\{X\in\mathfrak{n}:[X,\mathfrak{n}]\subseteq\mathfrak{n}_{j-1}\}$. Since $\mathfrak{n}$ is nilpotent, there exists a positive integer l such that $\mathfrak{n}_l=\mathfrak{n}$, so

$$\{0\}=\mathfrak{n}_0\subset\mathfrak{n}_1\subset\cdots\subset\mathfrak{n}_l=\mathfrak{n}.$$

Choose j such that $[\mathfrak{s}_0,\mathfrak{n}_{j-1}]=\{0\}$ but $[\mathfrak{s}_0,\mathfrak{n}_j]\neq\{0\}$. Under the action of the semisimple group S_0 on $\mathfrak{n}$, the subalgebra $\mathfrak{n}_j$ splits into a sum of irreducible $\mathrm{Ad}(S_0)$ modules, not all of which are trivial. Let $\mathfrak{m}$ be a nontrivial summand in this decomposition; then $[\mathfrak{s}_0,\mathfrak{m}]=\mathfrak{m}$.

From the Jacobi identity we get

$$[[X,Y],Z]=[[X,Z],Y]+[X,[Y,Z]]=0, \qquad X\in\mathfrak{s}_0,\ Y\in\mathfrak{m} \text{ and } Z\in\mathfrak{n}_{j-1},$$

since $[X,Z]\in[\mathfrak{s}_0,\mathfrak{n}_{j-1}]=\{0\}$ and $[X,[Y,Z]]\in[\mathfrak{s}_0,\mathfrak{n}_{j-1}]=\{0\}$. It follows that $[\mathfrak{m},\mathfrak{n}_{j-1}]=0$. In particular, $[\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{n}_{j-1}$, so $[\mathfrak{m},[\mathfrak{m},\mathfrak{m}]]=\{0\}$, and $\mathfrak{m}+[\mathfrak{m},\mathfrak{m}]$ is a subalgebra of $\mathfrak{n}$. Given any linear form λ on $[\mathfrak{m},\mathfrak{m}]$, the bilinear form $B:(V,W)\mapsto\lambda[V,W]$ satisfies

$$B(\mathrm{ad}(U)V,W)+B(V,\mathrm{ad}(U)W)=\lambda([\mathrm{ad}(U)V,W]+[V,\mathrm{ad}(U)W])=\lambda(\mathrm{ad}(U)[V,W])=0$$

for all U in $\mathfrak{sl}(2,\mathbb{R})$ and all V and W in $\mathfrak{m}$. Since the space of such bilinear forms is one-dimensional, from Lemma 8.2.1, it follows that $\dim([\mathfrak{m},\mathfrak{m}])\le 1$ (in particular, if $\dim(\mathfrak{m})$ is odd, then $[\mathfrak{m},\mathfrak{m}]=\{0\}$, for the form $(V,W)\mapsto\lambda[V,W]$ is skew-symmetric). Let $m=\dim(\mathfrak{m})$. Then $\exp(\mathfrak{m}+[\mathfrak{m},\mathfrak{m}])$ is isomorphic to $\mathbb{R}^m$ if m is odd or m is even and $[\mathfrak{m},\mathfrak{m}]=0$, and isomorphic to $H^{m/2}$ if m is even and $[\mathfrak{m},\mathfrak{m}]\neq 0$. The group $S_0\ltimes\exp(\mathfrak{m}+[\mathfrak{m},\mathfrak{m}])$ is the required subgroup of $S_0\ltimes N'$. □

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Michael Cowling, School of Mathematics, University of New South Wales, Sydney NSW 2052, Australia
E-mail address: m.cowling@maths.unsw.edu.au

Brian Dorofaeff, Department of Mathematics, Sydney Grammar School, Darlinghurst, NSW 2010, Australia
E-mail address: bad@sydgram.nsw.edu.au

Andreas Seeger, Department of Mathematics, University of Wisconsin-Madison, Madison, WI 53706, USA
E-mail address: seeger@math.wisc.edu

James Wright, Department of Mathematics and Statistics, University of Edinburgh, King's Building, Mayfield Road, Edinburgh EH3 9JZ, U.K.
E-mail address: wright@maths.ed.ac.uk