SINGULAR INTEGRALS AND THE NEWTON DIAGRAM

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ABSTRACT. We examine several scalar oscillatory singular integrals involving a real-analytic phase function $\phi(s,t)$ of two real variables and illustrate how one can use the Newton diagram of ϕ to efficiently analyse these objects. We use these results to bound certain singular integral operators.

1. INTRODUCTION

Arnold conjectured and Varčenko verified that sharp asymptotics for a scalar oscillatory integral with phase function ϕ can be measured in terms of the Newton diagram of ϕ. For any smooth real-valued function $\phi \in C^\infty(\mathbb{R}^d)$ with Taylor expansion $\sum_\alpha b_\alpha x^\alpha$, the Newton diagram Π of ϕ is the unbounded polyhedron formed as the smallest closed convex set in the positive cone $\mathbb{R}^d_+$ containing

$$\bigcup_{\alpha\in\Lambda} \{x \in \mathbb{R}^d \mid x \geq \alpha\}$$

where $\Lambda = \{\alpha \in \mathbb{Z}^d_+ \mid b_\alpha \neq 0\}$ and $\alpha \leq x$ is the partial order defined by $\alpha_1 \leq x_1, \dots, \alpha_d \leq x_d$ where $\alpha = (\alpha_1, \dots, \alpha_d)$ and $x = (x_1, \dots, x_d)$. When $d = 1$ the Newton diagram is a half-line and simply encodes the smallest nonvanishing Taylor coefficient of ϕ.

In this paper we will describe an elementary method initiated in [2], [3] and [4] (see also [8], [10]) by analysing certain two dimensional oscillatory integrals of the form

$$I_\lambda(K) = \int\int e^{i\lambda\phi(s,t)} K(s,t)\chi(s,t)\, dsdt$$

for large real λ and various (possibly) singular kernels K. Here ϕ is real-analytic at the origin $(0,0)$, $\phi(0,0) = 0$, and $\chi \in C^\infty_c(\mathbb{R}^2)$. When $K \equiv 1$, the behaviour of $I_\lambda(1)$ for large λ is determined by the Newton distance β of Π, defined as the positive parameter such that $\beta\mathbf{1}$ lies on the boundary of Π (here $\mathbf{1} = (1,1)$).

The boundary of Π consists of finitely many vertices $\{V_1, \dots, V_N\}$ and compact edges $\{E_1 = \overline{V_1V_2}, \dots, E_{N-1} = \overline{V_{N-1}V_N}\}$, together with two infinite (vertical and horizontal) edges E_0 and E_N. To each edge $E_j, 0 \leq j \leq N$, we associate the

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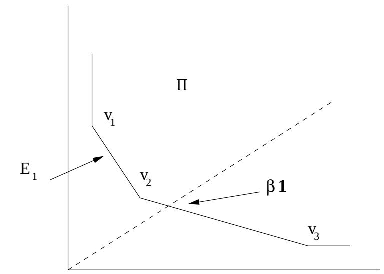

FIGURE 1.

corresponding part of the phase $\phi_{E_j}(s,t) = \sum_{\alpha \in E_j \cap \Lambda} c_\alpha s^{\alpha_1} t^{\alpha_2}$. We say that ϕ is $\mathbb{R}$-nondegenerate if for each compact edge $E_j, 1 \le j \le N-1$,

$$\nabla \phi_{E_j}(s,t) \ \neq \ 0$$

for all (s,t) with $st \neq 0$.

Theorem 1.1. *(Varčenko [15]) Let ϕ be $\mathbb{R}$-nondegenerate, real-valued and real-analytic at the origin $(0,0)$ such that $\phi(0) = \nabla\phi(0) = 0$. If $\chi \in C_c^\infty(\mathbb{R}^2)$ is supported in a sufficiently small neighbourhood of $(0,0)$ and if*

i) $\beta\mathbf{1} \notin \{V_1, \ldots, V_N\}$ or $\beta = 1$, then

$$I_\lambda(1) = c_1\, \lambda^{-1/\beta} + O(\lambda^{-(1/\beta+\epsilon)})$$

for some $\epsilon > 0$;

ii) $\beta\mathbf{1} = V_j$ for some j and $\beta > 1$, then

$$I_\lambda(1) = c_2\, \lambda^{-1/\beta} \log \lambda + O(\lambda^{-1/\beta}).$$

Here c_1 and c_2 are explicit constants depending on ϕ.

As an application of our elementary method we will give a new proof of Theorem 1.1 in section 4. The proof does not use any resolutions of singularities.

Remarks 1.2.

- Theorem 1.1 is not true without the assumption that ϕ is $\mathbb{R}$-nondegenerate since a (real-analytic) change of variables leaves $I_\lambda(1)$ unchanged but can change the Newton diagram and distance of ϕ. The $\mathbb{R}$-degenerate phase $\phi(s,t) = (s-t)^k$ with Newton distance $\beta = 1/2k$ provides a simple example. A rotation transforms this example to the $\mathbb{R}$-nondegenerate phase $\tilde{\phi}(s,t) = s^k$ with Newton distance $\tilde{\beta} = 1/k$ which is the correct decay parameter for $I_\lambda(1)$ in this case. An interesting substitute for $\mathbb{R}$-nondegeneracy is discussed in [6].
- If $\beta > 1$ and $\beta\mathbf{1}$ lies in the interior of the compact edge E_j, the constant c_1 in part i) of Theorem 1.1 is equal to

$$\chi(0,0) \int\int e^{i\phi_{E_j}(s,t)}\, ds dt;$$

the existence of this oscillatory integral is discussed in section 4. The precise values for the constants c_1 and c_2 in all cases can be determined from the proofs given below.

It is interesting to compare Varčenko's result with the bilinear form $I_\lambda(K)$ on $L^2(\mathbb{R})$ where $K(s,t) = f(s)g(t)$ is the product of two arbitrary L^2 functions. This effectively fixes the coordinate axes (s,t) and a result of Phong and Stein [9] states that the sharp decay estimate for the L^2 norm of this bilinear form is $O(\lambda^{-1/2\beta})$ for *any* real-analytic ϕ (here β is the Newton distance to the Newton diagram associated to $\partial^2_{s,t}\phi$). Such results arise from the study of certain degenerate Fourier integral operators associated to generalised Radon transforms along curves in the plane which is a topic studied by many authors. The C^∞ case has been successfully treated by Seeger [13] and Rychkov [11] (see also [5]).

Another instance where one has sharp results for *any* real-analytic phase ϕ occurs when $K(s,t) = 1/st$ is the double Hilbert transform singular kernel. In fact we have

Theorem 1.3. *Let ϕ be any real-valued phase function which is real-analytic at $(0,0)$ and $K(s,t) = 1/st$. Then for $\chi \in C_c^\infty(\mathbb{R}^2)$ supported in a sufficiently small neighbourhood of the origin and identically equal to 1 near $(0,0)$,*

$$I_\lambda(K) \;=\; C_\phi \log\lambda \;+\; O(1)$$

where C_ϕ is an explicit constant which may or may not vanish, depending on ϕ.

Remarks 1.4.

- A similar result for polynomial phases was established in [8].
- Consider the translation-invariant singular integral operator $Tf = f * S$, where S is the principal-valued distribution defined on a test function ψ by
$$\langle S, \psi\rangle \;=\; \int\!\!\int \psi(s,t,\phi(s,t))\chi(s,t)\; ds/s\, dt/t.$$
The multiplier $m = \widehat{S}$ for this operator is related to $I_\lambda(K)$ in Theorem 1.3 by $m(0,0,\lambda) = I_\lambda(K)$. The proof of Theorem 1.3 can be modified to show that T is bounded on all $L^p(\mathbb{R}^3), 1 < p < \infty$ if and only if every vertex $V_j, 1 \le j \le N$, of the Newton diagram of ϕ has at least one even component. This extends the result in [2] from polynomial to real-analytic surfaces and we will indicate the required modifications in section 5 (see also [10] for a further extension). Interestingly this result for T is false in the C^∞ category, even if ϕ has some nonvanishing derivative; that is, even if ϕ is of finite-type in some sense. An example is produced in section 5.
- Recently certain variants of Theorem 1.3 have been used in the study of real-analytic mappings $\phi : \mathbb{T}^2 \to \mathbb{T}^k$ between tori which have the property that the change of variable $f \to f\circ\phi$ linear transformation maps absolutely convergent Fourier series to uniformly convergent (with respect to rectangular summation) Fourier series. See [4].

In each of the three cases, $K \equiv 1$, $K(s,t) = f(s)g(t)$, or $K(s,t) = 1/st$, the nature of K dictates the decomposition of $I_\lambda(K)$ needed to understand its behaviour for large λ. When $K(s,t) = f(s)g(t)$ is the product of two arbitrary $L^2(\mathbb{R})$ functions, a subtle decomposition away from the zero set of $\partial^2_{st}\phi$ is used by Phong and Stein [9] to estimate the norm of the form $I_\lambda(fg)$. We will use a more elementary decomposition, one with respect to the edges $\{E_j\}$ of the Newton diagram Π of ϕ in the proof of Theorem 1.1, and one with respect to the vertices $\{V_j\}$ of Π in the proof of Theorem 1.3. In both cases the two decompositions are similar as well as the method used to analyse $I_\lambda(1)$ and $I_\lambda(1/st)$.

To illustrate the method in a simple setting we prove the following proposition in the next section.

Proposition 1.5. *For any real-valued ϕ of a single variable which is real-analytic at* 0*,*

$$I_\lambda \;=\; \int_{|s|\le 1} e^{i\lambda\phi(s)}\, ds/s \;=\; O(1). \tag{1}$$

Remark 1.6. Proposition 1.5 is well-known; in fact, higher dimensional versions, where $1/s$ is replaced by a general homogeneous Calderón-Zygmund kernel $K(x) = \Omega(x)/|x|^d$ with $\Omega \in L\log L(\mathbb{S}^{d-1})$ having mean value zero, also hold. These are special instances of the theory of generalised singular Radon transforms; see for example, [14].

In the next section we will sketch the proof of Proposition 1.5, highlighting an idea which will be used in the proofs of Theorems 1.1 and 1.3. In section 3 we describe the basic decomposition of $I_\lambda(K)$ for both $K \equiv 1$ and $K(s,t) = 1/st$ and prove some basic estimates. In section 4 we complete the proof of Theorem 1.1. The final section is devoted to the proof of Theorem 1.3 as well describing how to extend the main result in [2] regarding the singular integral operator T (defined in the remarks after the statement of Theorem 1.3) from polynomial to real-analytic surfaces.

2. Proof of Proposition 1.5

We may assume that $\phi(0) = 0$. The Newton diagram of ϕ simply picks out the first nonvanishing $b_k \ne 0$ Taylor coefficient of $\phi(s) = \sum_{n\ge k} b_n s^n$. In particular this tells us that $\phi(s) \sim b_k s^k$ for s small (note that we may restrict the integration of I_λ in (1) to an arbitrarily small interval $|s| \le \epsilon$ - independent of λ - which creates an $O(1)$ error). Thus for small s the monomial $b_k s^k$ dominates the other terms in the expansion of ϕ and we will see that for sufficiently small $\epsilon > 0$,

$$\int_{|s|\le\epsilon} e^{i\lambda\phi(s)}\, ds/s \;=\; \int_{|s|\le\epsilon} e^{i\lambda b_k s^k}\, ds/s \;+\; O(\lambda^{-\delta/k}) \tag{2}$$

for some $\delta > 0$. The second integral in (2) is zero if k is even whereas when k is odd, it is equal to $\pi\mathrm{sgn}(b_k)/k + O(1/\lambda)$ which gives us an asymptotic description of I_λ and in particular proves (1).

We decompose the first integral in (2) dyadically in s (in higher dimensions it is natural to decompose into dyadic annuli since $\Omega \in L\log L(\mathbb{S}^{d-1})$ possesses some regularity which should be compared to the homogeneous example $K(s,t) = 1/st$ of Theorem 1.3),

$$\sum_{p>p_0} \int_{2^{-p}\le|s|\le 2^{-p+1}} e^{i\lambda\phi(s)} ds/s := \sum_{p>p_0} I_p(\lambda)$$

where we write

$$I_p(\lambda) = \int_{1\le|s|\le 2} e^{i\lambda 2^{-pk}\phi_p(s)} ds/s \quad \text{with} \quad \phi_p(s) = b_k s^k + \sum_{n>k} 2^{-(n-k)p} b_n s^n.$$

Here ϕ_p is a normalised phase adapted to the dyadic interval $2^{-p} \le |s| \le 2^{-p+1}$ indexed by p and on which ϕ has size 2^{-pk}. Similarly we decompose the second integral in (2)

$$\int_{|s|\le\epsilon} e^{i\lambda b_k s^k}\, ds/s := \sum_{p>p_0} II_p(\lambda)$$

where $II_p(\lambda) = \int_{1\le|s|\le 2} e^{i\lambda 2^{-pk} b_k s^k} ds/s$. We examine the difference $I_p(\lambda) - II_p(\lambda)$ for each p.

The idea is very simple. For small $\lambda 2^{-pk}$ we gain in the difference since $\phi_p(s) - b_k s^k = O(2^{-p})$ for large p and so

$$|I_p(\lambda) - II_p(\lambda)| \le C 2^{-p}\ \lambda 2^{-pk}.$$

For large $\lambda 2^{-pk}$ we treat I_p and II_p separately, integrating by parts to obtain

$$|I_p(\lambda) - II_p(\lambda)| \le C\,[\lambda 2^{-pk}]^{-N}$$

for any $N > 0$. Putting these estimates together shows that $|I_p(\lambda) - II_p(\lambda)| \le C\, 2^{-p\delta} \min(\lambda 2^{-pk}, [\lambda 2^{-pk}]^{-\delta})$ for some $\delta > 0$. Summing in p establishes (2).

The basic idea for the proofs of Theorems 1.1 and 1.3 is the same; however a single monomial of $\phi(s,t) = \sum_\alpha b_\alpha s^{\alpha_1} t^{\alpha_2}$ no longer dominates all the other monomials. For $I_\lambda(1)$ we will decompose the integration into various regions corresponding to each edge $E_j, 0 \le j \le N$ of the Newton diagram Π. In the region corresponding to E_k, say, the monomials along E_k (that is, the monomials appearing in ϕ_{E_k}) will dominate in a certain sense. For $I_\lambda(1/st)$ we will decompose the integration into various regions corresponding to each vertex $V_j, 1 \le j \le N$ of Π. In the region corresponding to V_k, say, the monomial of ϕ corresponding to V_k will dominate in a certain sense. In both cases we will compare matters to the corresponding integral where the phase ϕ is replaced by ϕ_{E_k} or the monomial corresponding to the vertex V_k, creating an allowable error.

3. Basic decompositions

In this section we fix a real-valued, real-analytic phase function $\phi(s,t) = \sum_\alpha b_\alpha s^{\alpha_1} t^{\alpha_2}$ with Newton diagram Π consisting of vertices $\{V_j\}_{j=1}^N$ and edges $\{E_j\}_{j=0}^N$.

Let n_j denote an inward normal vector to the edge $E_j, 0 \le j \le N$, as indicated in Figure 2. The components of n_j can be chosen to be rational and for notational convenience, we will normalise the normals $n_j, 0 \le j \le N$, so that all components

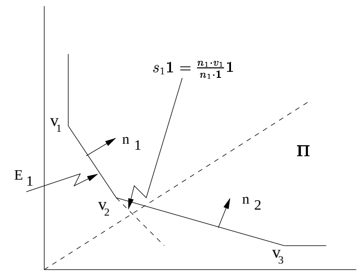

FIGURE 2.

have a common denominator. To each compact edge $E_j = \overline{V_j V_{j+1}}, 1 \leq j \leq N-1$, we associate the positive parameter $s_j = (n_j \cdot V_j)/(n_j \cdot \mathbf{1})$ which will serve to measure the decay rate of the part of $I_\lambda(1)$ corresponding to E_j. Similarly, if the end vertices V_0 and V_N do not lie along the coordinate axes, we set $s_0 = (n_0 \cdot V_1)/(n_0 \cdot \mathbf{1})$ and $s_N = (n_N \cdot V_N)/(n_N \cdot \mathbf{1})$ for the noncompact edges E_0 and E_N. If either V_0 or V_N lie along one of the coordinate axes, we set $s_0 = (n_1 \cdot V_0)/(n_0 + n_1) \cdot \mathbf{1}$ or $s_N = (n_{N-1} \cdot V_N)/(n_{N-1} + n_N) \cdot \mathbf{1}$, respectively. Geometrically s_j is the parameter such that $s_j\mathbf{1}$ lies on the line extension of E_j. Hence if the ray $\{s\mathbf{1}\}_{s \geq 0}$ intersects the edge E_j, then $s_j = \beta$ is the Newton distance of Π. The situation is depicted in Figure 2 with E_1 and s_1.

We begin the analysis of

$$I_\lambda(K) = \int\int e^{i\lambda\phi(s,t)} K(s,t)\chi(s,t)\, dsdt$$

where $\chi \in C_c^\infty(\mathbb{R}^2)$ is supported in a small neighbourhood of $(0,0)$ and $K \equiv 1$ or $K(s,t) = 1/st$. Fix a nonnegative, even $\psi \in C_c^\infty$ supported in $\{s : 1/2 \leq |s| \leq 2\}$ such that $\sum_{p \in \mathbb{Z}} \psi(2^p s) = 1$ for $s \neq 0$. Then

$$I_\lambda(K) = \sum_{P=(p,q)} \int\int e^{i\lambda\phi(s,t)} K(s,t)\chi(s,t)\psi(2^p s)\psi(2^q t)\, dsdt \tag{3}$$

and the integral in the sum is supported in the dyadic rectangle $\{(s,t) : |s| \sim 2^{-p}, |t| \sim 2^{-q}\}$, indexed by the integer lattice point $P = (p,q)$ where both p,q are large and positive due to the small support of χ.

The basic decomposition of $I_\lambda(K)$ will be expressed as a decomposition of $L = \{P = (p,q) \in \mathbb{N} \times \mathbb{N}\}$. We begin with $K(s,t) = 1/st$ and define, for each vertex $V_j, 1 \leq j \leq N$, of Π, the cone $C(V_j) = \{P = \sigma n_{j-1} + \rho n_j \in L : \sigma, \rho \geq 0\}$ in L. See Figure 3.

It is clear that $L = \cup_{j=1}^N C(V_j)$ gives an essentially disjoint decomposition of L. By our convention that all rational components of the normals $\{n_j\}$ have a common denominator, $P = \sigma n_{j-1} + \rho n_j \in C(V_j)$ implies that $\sigma = k/d_j$ and $\rho = \ell/d_j$ for some fixed positive integer d_j and integers $k, \ell \geq 0$. Hence the points of $C(V_j)$ are parameterised by a certain subcollection $\mathcal{A}_j \subset \{(k,\ell) \in \mathbb{N} \times \mathbb{N}\}$ of positive integer lattice points. Furthermore for any $\alpha \in \Pi$, $P \cdot (\alpha - V_j) \geq 0$ or $2^{-P\cdot\alpha} \leq 2^{-P\cdot V_j}$ for all

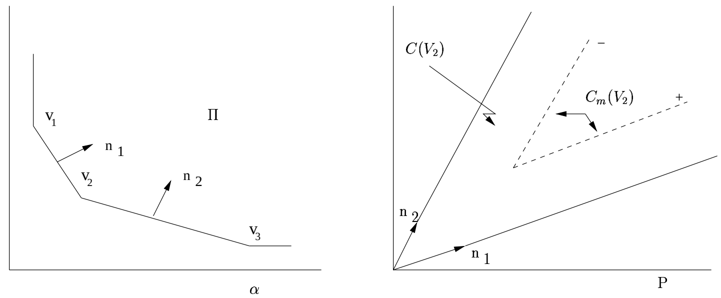

FIGURE 3.

$P \in C(V_j)$ and hence the monomial $b_{V_j} s^{V_{j,1}} t^{V_{j,2}}$ of ϕ corresponding to the vertex V_j dominates all the other monomials $b_\alpha s^{\alpha_1} t^{\alpha_2}$ of ϕ on those dyadic rectangles indexed by $P \in C(V_j)$. This gives us the basic decomposition of

$$I_\lambda(1/st) = \sum_{j=1}^{N} S_{\lambda,j}(1/st) := \sum_{j=1}^{N} \sum_{P \in C(V_j)} I_{j,P}(1/st)$$

where $I_{j,P}(K)$ ($K(s,t) = 1/st$ in this instance) is the $P = (p,q)$ integral in (3). We will compare this to $M_{\lambda,j}(1/st) = \sum_{P \in C(V_j)} II_{j,P}(1/st)$ where

$$II_{j,P}(1/st) \; = \; \int\!\!\int e^{i\lambda b_{V_j} s^{V_{j,1}} t^{V_{j,2}}} \chi(s,t) \psi(2^p s) \psi(2^q t) \; ds/s \, dt/t.$$

In fact, we will show that

$$S_{\lambda,j}(1/st) - M_{\lambda,j}(1/st) = O(1) \tag{4}$$

for each $1 \le j \le N$ and the behaviour of each $M_{\lambda,j}(1/st)$ is easy to understand.

We shall need a further decomposition of $C(V_j) = \cup_{m \ge 0} C_m(V_j)$ where $C_m(V_j) =$

$$\{P = \frac{m+k}{d_j} n_{j-1} + \frac{m}{d_j} n_j \in L : \, k \in \mathbb{N}\} \; \cup \{P = \frac{m}{d_j} n_{j-1} + \frac{m+\ell}{d_j} n_j \in L : \, \ell \in \mathbb{N}\}$$

$$:= \;\; C_m^+(V_j) \;\cup\; C_m^-(V_j).$$

See Figure 3. In particular this divides each cone $C(V_j)$ into two parts, $C^-(V_j) = \cup_{m \ge 0} C_m^-(V_j)$ and $C^+(V_j) = \cup_{m \ge 0} C_m^+(V_j)$. This leads us to the cones $C(E_j) = C^-(V_j) \cup C^+(V_{j+1})$ in L associated to each compact edge $E_j = \overline{V_j V_{j+1}}, 1 \le j \le N-1$. To the noncompact edges E_0 and E_N we associate $C(E_0) = C^+(V_1)$ and $C(E_N) = C^-(V_N)$ respectively. This gives us another decomposition of $L = \cup_{j=0}^{N} C(E_j)$ but now with respect to the edges $\{E_j\}$ of the Newton diagram Π of ϕ; each cone $C(E_j) = \cup_{m \ge 0} C_m(E_j)$ decomposes further where $C_m(E_j) = C_m^-(V_j) \cup C_m^+(V_{j+1})$. We will use this decomposition to analyse $I_\lambda(1)$. In fact we decompose

$$I_\lambda(1) \;=\; \sum_{j=0}^{N} S_{\lambda,j}(1) := \sum_{j=0}^{N} \sum_{P \in C(E_j)} I_{j,P}(1)$$

and then compare each $S_{\lambda,j}(1)$ to $M_{\lambda,j}(1) = \sum_{P\in C(E_j)} II_{j,P}(1)$ where

$$II_{j,P}(1) \;=\; \int\!\!\int e^{i\lambda\phi_{E_j}(s,t)}\chi(s,t)\psi(2^p s)\psi(2^q t)\,dsdt.$$

We will show that

$$S_{\lambda,j}(1) \;-\; M_{\lambda,j}(1) \;=\; O(\lambda^{-(1/s_j+\delta_j)}) \tag{5}$$

for some $\delta_j > 0$; recall that $s_j = (n_j\cdot V_j)/(n_j\cdot\mathbf{1}) \le \beta$ where β is the Newton distance of Π. This shows that in some sense, the monomials appearing in ϕ_{E_j} dominate the other monomials of ϕ on those dyadic rectangles indexed by $P \in C(E_j)$.

In either case $K \equiv 1$ or $K(s,t) = 1/st$, if $P \in C(V_j)$, we write

$$I_{j,P}(K) = 2^{-P\cdot\mathbf{1}}\int\!\!\int e^{i\lambda 2^{-P\cdot V_j}\phi_{j,P}(s,t)}\chi(2^{-p}s,2^{-q}t)K(2^{-p}s,2^{-q}t)\psi(s)\psi(t)\;dsdt$$

where

$$\phi_{j,P}(s,t) = 2^{P\cdot V_j}\phi(2^{-p}s,2^{-q}t) = b_{V_j}s^{V_{j,1}}t^{V_{j,2}} + \sum_{\alpha\in[\Pi\setminus V_j]\cap\Lambda} 2^{-P\cdot(\alpha-V_j)}b_\alpha s^{\alpha_1}t^{\alpha_2}$$

is a normalised phase with respect to $P \in C(V_j)$. We will compare each $I_{j,P}(K)$, for $P \in C(V_j)$, to $II_{j,P}(K)$ defined above which can be written as

$$II_{j,P}(K) = 2^{-P\cdot\mathbf{1}}\int\!\!\int e^{i\lambda 2^{-P\cdot V_j}\phi_{K,j,P}(s,t)}\chi(2^{-p}s,2^{-q}t)K(2^{-p}s,2^{-q}t)\psi(s)\psi(t)\;dsdt$$

where $\phi_{1/st,j,P}(s,t) = b_{V_j}s^{V_{j,1}}t^{V_{j,2}}$ if $P\in C(V_j)$ and $\phi_{1,j,P}(s,t) = 2^{P\cdot V_j}\phi_{E_j}(2^{-p}s,2^{-q}t)$ if $P \in C^-(V_j)$ whereas $\phi_{1,j,P}(s,t) = 2^{P\cdot V_j}\phi_{E_{j-1}}(2^{-p}s,2^{-q}t)$ if $P\in C^+(V_j)$. Recall that $C(V_j) = C^-(V_j)\cup C^+(V_j)$ and $C(E_j) = C^-(V_j)\cup C^+(V_{j+1})$.

As in Proposition 1.5 we split the analysis of the difference $I_{j,P}(K) - II_{j,P}(K)$ for $P \in C(V_j)$ into the cases when $\lambda 2^{-P\cdot V_j}$ is small and large. Again we will gain in the difference. To understand this when $K(s,t) = 1/st$ and $P\in C(V_j)$, we need to estimate the difference

$$\phi_{j,P}(s,t) - \phi_{1/st,j,P}(s,t) = \sum_{\alpha\in[\Pi\setminus V_j]\cap\Lambda} b_\alpha 2^{-(\alpha-V_j)\cdot P}s^{\alpha_1}t^{\alpha_2}$$

for $|s|,|t|\sim 1$. We observe that $\delta_{j,1} > 0$ and $\delta_{j,2} > 0$ where

$$\delta_{j,1} := \inf_{\alpha\in[\Pi\setminus E_j]\cap\Lambda}(\alpha - V_j)\cdot n_j \quad\text{and}\quad \delta_{j,2} := \inf_{\alpha\in[\Pi\setminus E_{j-1}]\cap\Lambda}(\alpha - V_j)\cdot n_{j-1}.$$

Hence for $P \in C_m(V_j)$,

$$(\alpha - V_j)\cdot P \ge m/d_j(\alpha - V_j)\cdot(n_{j-1}+n_j) \ge \delta_j m$$

for some $\delta_j > 0$, uniformly for $\alpha\in\Pi\setminus V_j$. This implies that $\phi_{j,P}(s,t) - \phi_{1/st,j,P}(s,t) = O(2^{-\delta_j m})$ and thus

$$I_{\lambda,P}(1/st) - II_{\lambda,P}(1/st) = O(2^{-\delta_j m}[\lambda 2^{-P\cdot V_j}]), \tag{6}$$

uniformly for $P\in C_m(V_j)$.

In order to understand the difference $I_{j,P}(K) - II_{j,P}(K)$ when $K \equiv 1$ and $P \in C(E_j) = C^-(V_j) \cup C^+(V_{j+1})$, we need to estimate the difference

$$\phi_{j,P}(s,t) - \phi_{1,j,P}(s,t) = \sum_{\alpha \in [\Pi \setminus E_j] \cap \Lambda} b_\alpha 2^{-(\alpha - V_j)\cdot P} s^{\alpha_1} t^{\alpha_2}$$

for $|s|, |t| \sim 1$ if $P \in C^-(V_j)$, and the difference

$$\phi_{j+1,P}(s,t) - \phi_{1,j+1,P}(s,t) = \sum_{\alpha \in [\Pi \setminus E_j] \cap \Lambda} b_\alpha 2^{-(\alpha - V_{j+1})\cdot P} s^{\alpha_1} t^{\alpha_2}$$

for $|s|, |t| \sim 1$ if $P \in C^+(V_{j+1})$. In the first case for $P \in C^-_m(V_j)$, we have

$$(\alpha - V_j) \cdot P \;\geq\; \frac{m+k}{d_j}\,(\alpha - V_j)\cdot n_j \;\geq\; \frac{\delta_{j,1}}{d_j}\,[m+k],$$

and in the second case, for $P \in C^+_m(V_{j+1})$,

$$(\alpha - V_{j+1}) \cdot P \;\geq\; \frac{m+k}{d_{j+1}}\,(\alpha - V_{j+1})\cdot n_j \;\geq\; \frac{\delta_{j+1,2}}{d_{j+1}}\,[m+k];$$

in both instances, these hold uniformly for $\alpha \in \Pi \setminus E_j$. Thus for some $\epsilon_j > 0$,

$$I_{j,P}(1) - II_{j,P}(1) = O(2^{-\epsilon_j(m+k)} 2^{-P\cdot \mathbf{1}}[\lambda 2^{-P\cdot V_r}]), \tag{7}$$

uniformly for $P \in C_m(E_j) = C^-_m(V_j) \cup C^+_m(V_{j+1})$ where $r = j$ or $j+1$ depending on whether $P \in C^-_m(V_j)$ or $P \in C^+_m(V_{j+1})$, respectively. Estimates (6) and (7) are good when $\lambda 2^{-P\cdot V_j}$ is small.

Complementary estimates when $\lambda 2^{-P\cdot V_j}$ is large are easily obtained for $II_{j,P}(K)$ in both cases $K \equiv 1$ and $K(s,t) = 1/st$. When $K(s,t) = 1/st$, integration by parts shows that for $P \in C(V_j)$, $II_{\lambda,P}(1/st) =$

$$\int\!\!\int e^{i\lambda 2^{-P\cdot V_j} b_{V_j} s^{V_{j,1}} t^{V_{j,2}}} \chi(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\; ds/s\, dt/t = O([\lambda 2^{-P\cdot V_j}]^{-N}) \tag{8}$$

for any $N > 0$.

On the other hand, when $K \equiv 1$, we have

$$|\nabla \phi_{1,j,P}(s,t)| = |\nabla[2^{P\cdot V_j}\phi_{E_j}(2^{-p}s, 2^{-q}t)]| \geq \delta_j > 0 \tag{9}$$

on the support of $\psi(s)\psi(t)$, uniformly for $P \in C^-(V_j) \subset C(E_j)$, say, whenever E_j is a compact edge (similarly for $P \in C^+(V_{j+1}) \subset C(E_j)$). This follows from the $\mathbb{R}$-nondegeneracy hypothesis that $\nabla \phi_{E_j}$ never vanishes away from the coordinate axes. In fact, more generally, for $P = \sigma n_0 + \tau n_j$ with $\sigma, \tau > 0$, $2^{P\cdot V_j}\phi_{E_j}(2^{-p}s, 2^{-q}t) =$

$$b_{V_j} s^{V_{j,1}} t^{V_{j,2}} + \sum_{\alpha \in [E_j \setminus V_j] \cap \Lambda} \delta^{(\alpha - V_j)\cdot n_0} b_\alpha s^{\alpha_1} t^{\alpha_2}$$

where $\delta = 2^{-\sigma}$ and $(\alpha - V_j)\cdot n_0 > 0$ whenever $\alpha \in E_j \setminus V_j$. The $\mathbb{R}$-nondegeneracy hypothesis implies that the gradient of $A\phi_{E_j}(Bs, Ct)$ does not vanish whenever $st \neq 0$ and A, B and C positive fixed constants; therefore, we see that the gradient of the above expression, denoted by $\mathbf{F}(s,t,\delta)$ say, is nonzero for (s,t) in the support of $\psi(s)\psi(t)$ and $\delta > 0$. But the above expression also shows that $\mathbf{F}(s,t,0) \neq 0$ and since $\mathbf{F}$ is clearly continuous on the compact product $supp(\psi(s)\psi(t)) \times [0,1]$ we see that $\mathbf{F}$ is uniformly bounded below on this product, establishing (9). A similar argument gives a bound from below of the gradient of $2^{P\cdot V_{j+1}}\phi_{E_j}(2^{-p}s, 2^{-q}t)$, uniformly for $P = \sigma n_j + \tau n_N$ with $\sigma, \tau > 0$.

Even for the noncompact edges E_0 and E_N, (9) continues to hold whether or not ϕ is $\mathbb{R}$-nondegenerate, as long as the components of $P = (p, q)$ are large and positive which is the situation when the support of χ is sufficiently small. For $P = \frac{m+k}{d_1}n_0 + \frac{m}{d_1}n_1 \in C_m(E_0) = C_m^+(V_1)$, say, $\phi_{1,0,P}(s,t) =$

$$\sum_{\alpha \in E_0 \cap \Lambda} 2^{-P\cdot(\alpha - V_1)} b_\alpha s^{\alpha_1} t^{\alpha_2} = s^{V_{1,1}}[b_{V_1} t^{V_{1,2}} + \sum_{\substack{\alpha \in E_0 \cap \Lambda \\ \alpha_2 > V_{1,2}}} 2^{-\frac{m}{d_1}(\alpha - V_1)\cdot n_1} b_\alpha t^{\alpha_2}].$$

However $m = c\,q$ since n_0 is proportional to $(1,0)$ and from this, it is easily seen that (9) also holds for the noncompact edges as well since q can be chosen to be large if the support of χ is small.

Hence, for $P \in C^-(V_j) \subset C(E_j)$ say, since any C^k norm of $\phi_{1,j,P}$ is bounded above, an integration by parts argument shows that $II_{j,P}(1) =$
(10)

$$2^{-P\cdot\mathbf{1}} \int\int e^{i\lambda 2^{-P\cdot V_j}\phi_{1,j,P}(s,t)} \chi(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\, dsdt = O(2^{-P\cdot\mathbf{1}}[\lambda 2^{-P\cdot V_j}]^{-N})$$

for any $N > 0$. Similarly for $P \in C^+(V_{j+1}) \subset C(E_j)$.

To prove similar estimates for $I_{j,P}(K)$, we need similar derivative bounds for the normalised phases $\phi_{j,P}(s,t) = 2^{P\cdot V_j}\phi(2^{-p}s, 2^{-q}t)$ which we establish in the following lemma.

Lemma 3.1. *For every $M > 0$ and $1 \le j \le N$, there exists constants $\delta_j, C_{M,j} > 0$ such that for $(s,t) \in supp(\psi(s)\psi(t))$ and $P \in C(V_j)$ large in the sense that both p and q in $P = (p,q)$ are large,*

i) $\|\phi_{j,P}\|_{C^M} \le C_{M,j}$;

ii) if $j = 1$ and $P \in C^+(V_1)$ or if $j = N$ and $P \in C^-(V_N)$,

$$|\nabla\phi_{j,P}(s,t)| \;\ge\; \delta_j;$$

iii) there is some derivative ∂^α such that

$$|\partial^\alpha\phi_{j,P}(s.t)| \;\ge\; \delta_j;$$

iv) if in addition, ϕ is $\mathbb{R}$-nondegenerate,

$$|\nabla\phi_{j,P}(s,t)| \;\ge\; \delta_j$$

holds for any $1 \le j \le N$.

Proof. Since

$$\phi_{j,P}(s,t) = 2^{P\cdot V_j}\phi(2^{-p}s, 2^{-q}t) = \sum_\alpha 2^{-P\cdot(\alpha - V_j)} b_\alpha s^{\alpha_1} t^{\alpha_2}$$

and $2^{-P\cdot(\alpha - V_j)} \le 1$ for $P \in C(V_j)$ and $\alpha \in \Pi$, we see that $i)$ holds. The proof of part $ii)$ is similar to the proof given above that the gradient of $\phi_{1,0,j}$ is bounded below. We leave the details to the reader.

For parts $iii)$ and $iv)$, suppose that $P \in C^-(V_j)$ (the proof when $P \in C^+(V_j)$ is similar). Furthermore, we may suppose that $1 \le j \le N-1$ so that $P \in C(E_j)$ and E_j is a compact edge; otherwise we are in the situation of part $ii)$. For part $iii)$, we write

$$\phi_{j,P}(s,t) = b_{V_j} s^{V_{j,1}} t^{V_{j,2}} + \sum_{\alpha \in \Pi \setminus V_j} 2^{-P\cdot(\alpha - V_j)} b_\alpha s^{\alpha_1} t^{\alpha_2}$$

and consider the ∂^{V_j} derivative of $\phi_{j,P}$:

$$\partial^{V_j} \phi_{j,P}(s,t) \;=\; c_j \;+ \sum_{\substack{\alpha \in \Pi \setminus V_j : \\ \alpha_1 \ge V_{j,1},\ \alpha_2 \ge V_{j,2}}} 2^{-P\cdot(\alpha - V_j)} c_\alpha s^{\alpha_1 - V_{j,1}} t^{\alpha_2 - V_{j,2}}$$

where c_j is nonzero. Since $P \in C^-(V_j)$ and $1 \le j \le N-1$, we have that $\alpha \in \Pi \setminus V_j$ such that $\alpha_1 \ge V_{j,1}, \alpha_2 \ge V_{j,2}$ implies that $\alpha \in \Pi \setminus E_j$. Hence, for $P = \frac{m}{d_j} n_{j-1} + \frac{m+k}{d_j} n_j \in C_m^-(V_j)$ and $\alpha \in [\Pi \setminus E_j] \cap \Lambda$,

$$(\alpha - V_j)\cdot P \ge \frac{m+k}{d_j}(\alpha - V_j)\cdot n_j \ge \frac{\delta_{j,1}}{d_j}[m+k]$$

and in this case, $m+k \sim \max(p,q)$ which we are taking to be large. This shows that $|\partial^{V_j}\phi_{j,P}(s,t)| \ge |c_j|/2$ if p and q are large, completing the proof of part $iii)$.

For part $iv)$, we write

$$\phi_{j,P}(s,t) = 2^{P\cdot V_j} \phi_{E_j}(2^{-p}s, 2^{-q}t) + \sum_{\alpha \in \Pi \setminus E_j} 2^{-P\cdot(\alpha - V_j)} b_\alpha s^{\alpha_1} t^{\alpha_2}$$

and use (9) to uniformly bound from below the gradient of the first term, $\phi_{1,j,P}$. It suffices to show that the gradient of the second term can be made as small as we like by taking $P = (p,q)$ large enough. This follows by the same argument in part $iii)$ to show that $2^{-P\cdot(\alpha - V_j)}$ is uniformly small if the $\max(p,q)$ is large. This completes the proof of Lemma 3.1. □

As a consequence of Lemma 3.1 we obtain the complementary estimates for $I_{j,P}(K)$, $P \in C(V_j)$, when $\lambda 2^{-P\cdot V_j}$ is large. For instance, when $K(s,t) = 1/st$, parts $i)$ and $iii)$ of Lemma 3.1, together with an integration by parts argument (using a version of van der Corput's lemma in higher dimensions; see for example, [14]) shows that for $P \in C(V_j)$, $I_{j,P}(1/st) =$

$$\int\!\!\int e^{i\lambda 2^{-P\cdot V_j}\phi_{j,P}(s,t)} \chi(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\; ds/s\, dt/t \;=\; O([\lambda 2^{-P\cdot V_j}]^{-\delta}) \tag{11}$$

for some $\delta > 0$. On the other hand, when $K \equiv 1$, parts $i)$, $ii)$ and $iv)$ of Lemma 3.1, together with an integration by parts argument, imply that for $P \in C^-(V_j) \subset C(E_j)$, say, $I_{j,P}(1) =$

$$2^{-P\cdot\mathbf{1}} \int\!\!\int e^{i\lambda 2^{-P\cdot V_j}\phi_{j,P}(s,t)} \chi(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\, dsdt \;=\; O(2^{-P\cdot\mathbf{1}}[\lambda 2^{-P\cdot V_j}]^{-N}) \tag{12}$$

for any $N > 0$. A similar estimate holds for $I_{j,P}(1)$ when $P \in C^+(V_j) \subset C(E_{j-1})$.

4. Proof of Theorem 1.1

Recall that we are trying to understand the oscillatory integrals

$$I_\lambda(K) \;=\; \int\int e^{i\lambda\phi(s,t)} K(s,t)\chi(s,t)\, dsdt$$

where ϕ is a real-valued, real-analytic phase at $(0,0)$, $\chi \in C_c^\infty(\mathbb{R}^2)$ is supported in a sufficiently small neighbourhood of $(0,0)$, and either $K \equiv 1$ or $K(s,t) = 1/st$. In both cases $I_\lambda(K) = \sum_j S_{\lambda,j}(K)$ where for $K \equiv 1$, $S_{\lambda,j}(1) = \sum_{P\in C(E_j)} I_{j,P}(1)$ and $0 \le j \le N$, and for $K(s,t) = 1/st$, $S_{\lambda,j}(1/st) = \sum_{P\in C(V_j)} I_{j,P}(1/st)$ and $1 \le j \le N$. Here, if $P \in C(V_j)$,

$$I_{j,P}(K) = 2^{-P\cdot\mathbf{1}} \int\int e^{i\lambda 2^{-P\cdot V_j}\phi_{j,P}(s,t)} \chi(2^{-p}s, 2^{-q}t) K(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\, dsdt$$

where $\phi_{j,P}(s,t) = 2^{P\cdot V_j}\phi(2^{-p}s, 2^{-q}t)$.

In this section we complete the proof of Theorem 1.1 which concerns the case $K \equiv 1$ under the additional hypothesis that ϕ is $\mathbb{R}$-nondegenerate. As described in the previous section we compare $S_{\lambda,j}(1)$ with $M_{\lambda,j}(1) = \sum_{P\in C(E_j)} II_{j,P}(1)$. From (7), (10) and (12), we see that for $P \in C_m(E_j) = C_m^-(V_j) \cup C_m^+(V_{j+1})$ (that is, $P = \frac{m}{d_j} n_{j-1} + \frac{m+k}{d_j} n_j$ or $P = \frac{m+k}{d_{j+1}} n_j + \frac{m}{d_{j+1}} n_{j+1}$),

$$|I_{j,P}(1) - II_{j,P}(1)| \le C_{N,j} 2^{-\epsilon_j(m+k)} 2^{-P\cdot\mathbf{1}} \min(1, [\lambda 2^{-P\cdot V_r}]^{-N}) \tag{13}$$

for some $\epsilon_j > 0$ and any $N > 0$. Here $r = j$ or $r = j+1$ depending on whether $P \in C_m^-(V_j)$ or $P \in C_m^+(V_{j+1})$ respectively. By choosing N large enough and summing over all $m, k \ge 0$, we obtain

$$S_{\lambda,j}(1) \;-\; M_{\lambda,j}(1) \;=\; O(\lambda^{-(1/s_j + \delta_j)})$$

for some $\delta_j > 0$, establishing (5) and reducing the analysis of $I_\lambda(1)$ to $\sum_j M_{\lambda,j}(1)$ (it is convenient to sum first in k and then m if V_r does not lie on one of the coordinate axes; otherwise sum in the opposite order).

To bound $M_{\lambda,j}(1) = \sum_{P\in C(E_j)} II_{j,P}(1)$, we use (10) to see that for $P \in C(E_j)$,

$$|II_{j,P}(1)| \le C_{N,j} 2^{-P\cdot\mathbf{1}} \min(1, [\lambda 2^{-P\cdot V_r}]^{-N})$$

for any $N > 0$ and this leads to the estimate $M_{\lambda,j}(1) = O(\lambda^{-1/s_j})$, for each $0 \le j \le N$ as long as the vertex V_r does not lie along the line $\{s\mathbf{1}\}_{s>0}$. When V_r lies along this line, summing the above estimates (say, in the case $r = j$ so that we are summing over $P \in C^-(V_j)$) adds an extra factor of $\log\lambda$ due to the fact that $s_{j-1} = s_j$ in this case (after summing in k, we are left with $O(\log\lambda)$ terms of order 1 in the m sum).

This gives us the correct estimate for $I_\lambda(1)$ when the Newton distance β is strictly larger than 1. To get the asymptotic refinement we first consider the case when $\beta\mathbf{1} \notin \{V_1, \dots, V_N\}$. Let E_{j_0} denote the edge whose interior contains $\beta\mathbf{1}$. For $j \ne j_0$, the bounds $M_{\lambda,j}(1) = O(\lambda^{-1/s_j})$ mentioned above contribute to the error estimate.

Next we observe that

$$(14)\qquad \int\int e^{i\lambda\phi_{E_{j_0}}(s,t)}\chi(s,t)\,dsdt \;-\; M_{\lambda,j_0}(1) \;=\; O(\lambda^{-(1/\beta+\epsilon)})$$

for some $\epsilon > 0$. In fact the above difference is equal to

$$\sum_{P\notin C(E_{j_0})} \int\int e^{i\lambda\phi_{E_{j_0}}(s,t)}\chi(s,t)\psi(2^p s)\psi(2^q t)\,dsdt$$

$$=: \sum_{P\notin C(E_{j_0})} III_{\lambda,P}(1).$$

If $P \notin C(E_{j_0})$ then there exist $\sigma > 0$ and positive numbers a, b, c and d such that either $P = kan_0 + \ell bn_{j_0}$ for certain positive integers k, ℓ satisfying $k \geq \sigma\ell$, or $P = kcn_{j_0} + \ell dn_N$ for certain positive integers k, ℓ satisfying $\ell \geq \sigma k$. Concentrating on those $P \notin C(E_{j_0})$ which are linear combinations of n_0 and n_{j_0}, we write

$$III_{\lambda,P}(1) = 2^{-P\cdot\mathbf{1}} \int\int e^{i\lambda 2^{-P\cdot V_{j_0}}\widetilde{\phi_P}(s,t)}\chi(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\,dsdt$$

where $\widetilde{\phi_P}(s,t) = 2^{P\cdot V_{j_0}}\phi_{E_{j_0}}(2^{-p}s, 2^{-q}t)$; the general argument establishing (9) shows that the gradient of this normalised phase is also uniformly bounded below. Hence integration by parts shows

$$|III_{\lambda,P}(1)| \;\leq\; C2^{-P\cdot\mathbf{1}}\min(1, [\lambda 2^{-P\cdot V_{j_0}}]^{-N})$$

for any $N > 0$. Summing over all such $P = kan_0 + \ell bn_{j_0}$, choosing N large enough, establishes (14).

This leaves us with developing the asymptotic behaviour of

$$I(\lambda) = \int\int e^{i\lambda\phi_{E_{j_0}}(s,t)}\chi(s,t)\;dsdt$$

as λ tends to infinity. This is fairly straightforward and so we will be brief. Let m denote the absolute value of the slope of the edge E_{j_0} and assume that m is positive and finite (that is, E_{j_0} is a compact edge); the other cases are easier to handle. Finally we may assume that $\mathbf{1} \notin E_{j_0}$; otherwise both vertices $(2,0)$ and $(0,2)$ lie on E_{j_0} and the $\mathbb{R}$-nondegeneracy hypothesis implies that $\phi_{E_{j_0}}$ has a nondegenerate critical point at $(0,0)$ and so stationary phase asymptotics can be invoked.

Let (A, B) denote the strictly positive components of the vector $n_{j_0}/(V_{j_0}\cdot n_{j_0})$ and note that $\alpha\cdot(A,B) = 1$ for all $\alpha \in E_{j_0}$ since for such α, $(\alpha - V_{j_0})\cdot n_{j_0} = 0$. Making the change of variables $s \to \lambda^{-A}s$ and $t \to \lambda^{-B}t$ gives us

$$I(\lambda) = \lambda^{-1/\beta}\int\int e^{i\phi_{E_{j_0}}(s,t)}\chi(\lambda^{-A}s, \lambda^{-B}t)\;dsdt.$$

We split the above integral by writing $\chi(\lambda^{-A}s, \lambda^{-B}t) = [\chi(\lambda^{-A}s, \lambda^{-B}t) - \chi(\lambda^{-A}s, 0)]$ $+[\chi(\lambda^{-A}s, 0) - \chi(0,0)] + \chi(0,0)$. We denote the first difference by $\chi_1(s,t)$ and the second difference as $\chi_2(s)$. Here we are implicitly assuming the existence of the oscillatory integral $\int\int e^{i\phi_{E_{j_0}}(s,t)}dsdt$ for the case we are considering; however the argument sketched below also shows that this integral does indeed exist. We concentrate on showing

$$(15)\qquad S_2(\lambda) := \int\int e^{i\phi_{E_{j_0}}(s,t)}\chi_2(s)\,dsdt = O(\lambda^{-\epsilon_0})$$

for some $\epsilon_0 > 0$. It is slightly easier to show that $S_1(\lambda) = O(\lambda^{-\delta_0})$ for some $\delta_0 > 0$ and this, together with (15), gives the desired result. We split the region of integration defining $S_2(\lambda)$ into three parts; $|s| \geq C|t|^m$, $|s| \leq C^{-1}|t|^m$ and $C^{-1}|t|^m \leq |s| \leq C|t|^m$. The first and second regions correspond to where the monomials associated to the endpoint vertices V_{j_0} and V_{j_0+1}, respectively, are pointwise larger than the other monomials in $\phi_{E_{j_0}}$. In either case, the size of any derivative of the phase $\phi_{E_{j_0}}$ is understood (being determined by the endpoint vertices) and straightforward integration by parts arguments show the decay estimates $O(\lambda^{-\epsilon})$ for some $\epsilon > 0$ in these cases.

We shall concentrate on estimating the part of the integral defining $S_2(\lambda)$ over the third region where all the monomials in $\phi_{E_{j_0}}$ have the same size. We make the change of variable $t \to s^{1/m}t$ (treating the positive and negative s integrals separately), reducing the analysis of $S_2(\lambda)$ to

$$\int\int_{1/C\leq|t|\leq C} e^{is^{\alpha_1+\alpha_2/m}\phi_{E_{j_0}}(1,t)} s^{1/m}\chi_2(s)dsdt.$$

Here the exponent $\alpha_1 + \alpha_2/m = \alpha \cdot (1, 1/m)$ is constant as α varies over $E_{j_0} \cap \Lambda$ and the basic observation is that the constant

$$\eta \ := \ (\alpha - \mathbf{1}) \cdot (1, 1/m)$$

is strictly positive since we are assuming that $\mathbf{1} \notin E_{j_o}$. Consider first the part of the integral where $s > \lambda^\delta$ for *any* $\delta > 0$; that is

$$S_{2,\delta} \equiv \int_{s>\lambda^\delta} s^{1/m} \int_{\frac{1}{C}\leq|t|\leq C} e^{is^r Q(t)} dt\, ds$$

where $Q(t) \equiv \phi_{E_{j_0}}(1,t)$ and $r = 1 + \frac{1}{m} + \eta$.

We split the t integral in $S_{2,\delta}$ around the critical points of Q. Away from the critical points of Q (where $|Q'(t)| \gtrsim 1$) an integration by parts argument shows that the t integral is $O(1/s^{1+\eta})$ which allows us to estimate that part of $S_{2,\delta}$ successfully. In a small neighbourhood of a critical point of Q, say $|t-\alpha| < \epsilon$ for small $\epsilon > 0$ where $Q'(\alpha) = 0, 1/C \leq |\alpha| \leq C$, we make the change of variable $t \to t - \alpha$ to write this part of $S_{2,\delta}$ as

$$S_{2,\delta,\alpha} \equiv \int_{s>\lambda^\delta} e^{iQ(\alpha)s^r} s^{1/m} \int_{|t|<\epsilon} e^{is^r P(t)} dt\, ds$$

where $P(t) \equiv Q(t+\alpha) - Q(\alpha)$ is a polynomial satisfying $|P(t)| \lesssim |t|^{k_0}$, $|P'(t)| \gtrsim |t|^{k_0-1}$ on the interval $|t| < \epsilon$ for some $k_0 \geq 2$. Since ϕ is $\mathbb{R}$-nondegenerate, we see that $Q(\alpha) \neq 0$. An integration by parts argument (in s) shows that

$$S_{2,\delta,\alpha} = C \int_{s>\lambda^\delta} e^{iQ(\alpha)s^r} s^{1/m} \int_{|t|<\epsilon} e^{is^r P(t)} P(t) dt\, ds \ + \ O(\lambda^{-\varepsilon})$$

for some constnat C and $\varepsilon > 0$. Now integrating by parts in the t integral shows that $S_{2,\delta,\alpha} = O(\lambda^{-\varepsilon})$ for every nonzero critical point α of Q and *any* $\delta > 0$.

For the part where $s \leq \lambda^\delta$, we write $\chi_2(s) = s\lambda^{-A} \int_0^1 \partial\chi/\partial s(\lambda^{-A}s\sigma, 0)d\sigma$ and trivially estimate

$$\int_0^1 \int_{|t|\sim 1} \int_{s\leq\lambda^\delta} e^{is^{\alpha\cdot(1,1/m)}\phi_{E_{j_0}}(1,t)} \frac{s}{\lambda^A} \frac{\partial\chi}{\partial s}(\lambda^{-A}s\sigma, 0) ds dt\, d\sigma = O(\lambda^{-(A-2\delta)}).$$

Taking $\delta < A/2$ establishes (15), completing the proof that

$$I(\lambda) = \lambda^{-1/\beta}\chi(0,0) \int\int e^{i\phi_{E_{j_0}}(s,t)} ds dt + O(\lambda^{-(1/\beta+\epsilon)}).$$

For the case $\beta\mathbf{1} \in \{V_1, \ldots, V_N\}$, say $\beta\mathbf{1} = V_{j_0}$, we consider only the situation when $\beta > 1$ since otherwise stationary phase methods apply. From the above analysis we have

$$I_\lambda(1) = \sum_{P\in C(E_{j_0-1})\cup C(E_{j_0})} \int\int e^{i\lambda\phi(s,t)}\chi(s,t)\psi(2^p s)\psi(2^q t) ds dt + O(\lambda^{-(1/\beta+\epsilon)})$$

for some $\epsilon > 0$. Furthermore, similar arguments already used show that the above sum is equal to

$$\sum_{P\in C(V_{j_0})} \int\int e^{i\lambda b_{V_{j_0}}(st)^\beta}\chi(s,t)\psi(2^p s)\psi(2^q t) ds dt + O(\lambda^{-1/\beta})$$

and the sum is easily seen to be equal to $c\lambda^{-1}\log\lambda + O(1/\lambda)$ for some $c \neq 0$ since β is a positive integer larger than 1. We omit the details. This completes the proof of Theorem 1.1.

5. Analysis of $I_\lambda(1/st)$ and T

In this section we complete the proof of Theorem 1.3. Recall that we are trying to understand the oscillatory integral

$$I_\lambda(1/st) \;=\; \int\int e^{i\lambda\phi(s,t)}\chi(s,t)\; ds/s\, dt/t$$

where ϕ is a real-valued, real-analytic phase at $(0,0)$ and $\chi \in C_c^\infty(\mathbb{R}^2)$ is supported in a sufficiently small neighbourhood of $(0,0)$. Furthermore $I_\lambda(1/st) = \sum_{1\leq j\leq N} S_{\lambda,j}(1/st)$ where $S_{\lambda,j}(1/st) = \sum_{P\in C(V_j)} I_{j,P}(1/st)$ and for $P \in C(V_j)$,

$$I_{j,P}(1/st) = \int\int e^{i\lambda 2^{-P\cdot V_j}\phi_{j,P}(s,t)}\chi(2^{-p}s, 2^{-q}t)\psi(s)\psi(t)\; ds/s\, dt/t$$

where $\phi_{j,P}(s,t) = 2^{P\cdot V_j}\phi(2^{-p}s, 2^{-q}t)$.

As described in section 3 we compare $S_{\lambda,j}(1/st)$ with $M_{\lambda,j}(1/st) = \sum_{P\in C(V_j)} II_{j,P}(1/st)$. From (6), (8) and (11), we see that for $P \in C_m(V_j)$,

(16) $$|I_{j,P}(1/st) - II_{j,P}(1/st)| \leq C_j 2^{-\epsilon_j m}\min(\lambda 2^{-P\cdot V_j}, [\lambda 2^{-P\cdot V_j}]^{-\epsilon_j})$$

for some $\epsilon_j > 0$. If the endpoint vertices V_0 and V_N do not lie along the coordinate axes, then we can sum over $P \in C_m(V_j)$ to obtain

(17) $$\sum_{P\in C_m(V_j)} |I_{j,P}(1/st) - II_{j,P}(1/st)| \;\leq\; C2^{-\delta_j m}$$

for some $\delta_j > 0$. Summing in m establishes (4).

With regard to the singular integral operator $Tf = f * S$ mentioned in the remarks after the statement of Theorem 1.3, the operator corresponding to $I_{j,P}(1/st)$ is the convolution operator $T_{j,P}f = f * S_{j,P}$ where for $P \in C(V_j)$, $S_{j,P}$ is the distribution defined on a test function ρ by

$$\langle S_{j,P}, \rho \rangle = \int\int \rho(s,t,\phi(s,t))\chi(s,t)\psi(2^p s)\psi(2^q t) ds/s dt/t.$$

Similarly the operator $M_{j,P}f = f * U_{j,P}$ corresponding to $II_{j,P}$ is defined exactly in the same way except ϕ is replaced by the monomial $b_{V_j} s^{V_{j,1}} t^{V_{j,2}}$. The above bounds translate in this setting to the fact that the difference operators $\{T_{j,P} - M_{j,P}\}_{P \in C_m(V_j)}$ are almost orthogonal whose sum has an L^2 operator norm bound of $O(2^{-\delta_j m})$. Using appropriate Littlewood-Paley theory these L^2 estimates can be converted into $L^p, 1 < p < \infty$ estimates; see [2].

Thus, if the vertices V_0 and V_N do not lie along the coordinate axes, summing over $m \geq 0$ reduces the analysis of $I_\lambda(1/st)$ and T to $\sum_j M_{\lambda,j}(1/st)$ and $\sum_j M_j f = \sum_j \sum_{P \in C(V_j)} M_{j,P} f$, respectively. As in [2], if each vertex V_j has at least one even component, the operator $\sum_j M_j$ is bounded on all $L^p, 1 < p < \infty$ (if one of the components of V_j is even, then clearly $M_{\lambda,j}(1/st) \equiv 0$). If there exists a vertex V_j whose components are both odd, then one can argue exactly as in [2] to show that T is not bounded on L^2. Finally, it is not difficult to show that $\sum_j M_{\lambda,j}(1/st) = C_\phi \log \lambda + O(1)$ for an explicit C_ϕ depending on the signs of the coefficients b_{V_j} for those vertices V_j which have both components odd. This is carried out in [8] where one can find a formula for C_ϕ.

If either V_0 or V_N lies along the coordinate axes, the sum (17) collapses. In this case (at least for those $P \in C^+(V_1)$ or $P \in C^-(V_N)$), we need to replace $II_{1,P}$, say, with

$$II_{1,P} = \int\int e^{i\lambda\phi(0,t)}\chi(s,t)\psi(2^p s)\psi(2^q t)\ ds/s\, dt/t.$$

Similarly we need appropriate replacements for $II_{N,P}$ as well as for the operators $M_{1,P}$ and $M_{N,P}$. With these substitutions, the sum estimate (17) now holds as well as the fact that the difference operators $\{T_{1,P} - M_{1,P}\}_{P \in C_m^+(V_1)}$, say, are almost orthogonal whose sum has an L^2 operator norm bound of $O(2^{-\delta m})$ for some $\delta > 0$. This case was overlooked in [2].

We shall now show that the result determining the L^p boundedness for the singular integral operator T does not extend to $\phi \in C^\infty$, even in the finite-type category. For any $\epsilon > 0$, we consider the operator

$$(18) \qquad T_\epsilon f(x,y,z) \ = \ p.v. \int_{|s|,} \int_{|t| \leq \epsilon} f(x-s, y-t, z-\phi(s,t))\ ds/s\, dt/t$$

where $\phi(s,t) = s^2 t + \psi(s)$ and ψ is an appropriate smooth function near $s = 0$ such that $\psi^{(k)}(0) = 0$ for all $k \geq 0$. In this case there is only one vertex, $(2,1)$, for the Newton polygon Π of ϕ. We will show that when ψ is convex and odd, a necessary and sufficient condition for (18) to be unbounded on L^2 for all $\epsilon > 0$ is that there exists a sequence $s_j \searrow 0$ such that for

(19) $\sigma_j < s_j$ satisfying $\psi'(\sigma_j) = \psi(s_j)/s_j$, then we have $s_j/\sigma_j \to \infty$.

This is just the contrapositive to the (local) h doubling condition used in [7] to analyse Hilbert transforms along convex curves in the plane. In fact we will show that for every $\epsilon > 0$,

$$m_\epsilon(\xi,\eta,\gamma) \;=\; \int\!\!\int_{|s|,\,|t|\le\epsilon} e^{i[\xi s+\eta t+\gamma\phi(s,t)]}\,ds/s\,dt/t$$

is an unbounded function. We take $\eta = 0$ and perform the t integral first; $m_\epsilon(\xi,0,\gamma) =$

$$\int_{|s|\le\epsilon} e^{i[\xi s+\gamma\psi(s)]}\int_{|t|\le\epsilon} e^{i\gamma s^2 t}\,dt/t\,ds/s \;=\; -2\int_0^\epsilon \sin(\xi s+\gamma\psi(s))I(s^2)\,ds/s$$

where $I(s^2) = 2\int_0^\epsilon \sin(\gamma s^2 t)dt/t$. Here we are assuming that ψ is odd. Since $I(s^2) = O(\gamma s^2)$ and $I(s^2) = \mathrm{sgn}(\gamma)\pi + O(1/\gamma s^2)$, we see that (for $\gamma < 0$)

$$m_\epsilon(\xi,0,\gamma) \;=\; 2\pi\int_{|\gamma|^{-\frac12}}^{\epsilon} \sin(\xi s+\gamma\psi(s))\,ds/s \;+\; O(1).$$

Now take j so large in (19) that $s_j < \epsilon$ and $\psi''(\sigma_j) < \pi$. For such a j, consider $-\gamma = \pi/[2h(\sigma_j)]$ and $\xi = -\gamma\psi'(\sigma_j)$. Then since $s_j < \epsilon$, we have

$$\int_{|\gamma|^{-\frac12}}^{\epsilon} \sin(\xi s+\gamma\psi(s))\,ds/s \;=\; \int_{|\gamma|^{-\frac12}}^{s_j} \sin(\xi s+\gamma\psi(s))\,ds/s \;+\; O(1)$$

by the convexity of ψ (see [7]). Also $\psi''(\sigma_j) < \pi$ gaurantees that $|\gamma|^{-\frac12} < \sigma_j$ and so (see [7], page 740)

$$\int_{|\gamma|^{-\frac12}}^{s_j} \sin(\xi s+\gamma\psi(s))ds/s \ge \int_{\sigma_j}^{(s_j+\sigma_j)/2} \sin(\xi s+\gamma\psi(s))ds/s > \frac{1}{\sqrt{2}}\log((1+(s_j/\sigma_j))/2)$$

and by (19) this completes the proof that m_ϵ is an unbounded function of ξ, η and γ.

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