Mathematics 3: Algebra

Workshop 7

Nilpotent matices

Recall that a square matrix is *nilpotent* is some positive power of it is the zero matrix.

Let F be a field.

(1) (a) Suppose that $A \in F^{n\times n}$ has a nonzero eigenvalue λ. Find a vector $\underline{x}$ such that $A^k\underline{x} \neq 0$ for all $k \in \mathbb{N}$. Deduce that A is *not* nilpotent.
(b) Show that all eigenvalues of a nilpotent matrix are 0.
(c) Deduce, by proving the converse to (b), that a matrix $A \in F^{n\times n}$ is nilpotent if and only if all its eigenvalues are 0. (Hint: Cayley-Hamilton)
(d) Deduce that if $A \in F^{n\times n}$ satisfies $A^m = 0$ for some $m \in \mathbb{N}$ then $A^n = 0$.

(a) Take $\underline{x}$ to be an eigenvector of A with eigenvalue λ. Then $A\underline{x} = \lambda\underline{x}$ and so (induction) $A^k\underline{x} = \lambda^k\underline{x}$ for each $k \in \mathbb{N}$. So no power of A can be the zero matrix.

(b) By (a), a nilpotent matrix can have no nonzero eigenvalues, i.e., all its eigenvalues are 0.

(c) Suppose A has all eigenvalues equal to 0. Then the characteristic polynomial of A is $\chi(x) = (x-\lambda_1)\dots(x-\lambda_n) = x^n$ so, by the Cayley-Hamilton Theorem, $\chi(A) = A^n = 0$, making A nilpotent.

Then, using (b), we have (c) in full.

(d) Follows from (c), as nilpotentcy implies $A^n = 0$.

(2) For a nilpotent matrix $A \in F^{n\times n}$, denote by $\nu(A)$ the least exponent m such that $A^m = 0$. From Q1, we know that $\nu(A) \leq n$.

We investigate whether $\nu(A)$ can take all integer values $1, 2, \dots, n$.
(a) Describe the (i, j)th entry of A^m in terms of the entries of $A = (a_{ij})$.
(b) Use your description in (a) to show that the companion matrix – call it C_n – of the polynomial x^n has $C_n^{n-1} \neq 0$. Hence write down $\nu(C_n)$.
(Maybe try some small values of n first.)
(c) For $k = 1, 2, \dots, n$, use $C_k \in F^{k\times k}$ to construct a nilpotent matrix in $F^{n\times n}$ – call it C_{kn} – that has $\nu(C_{kn}) = k$.

(a) The (i, j)th entry of A^m is a sum of all possible nonzero products

$$a_{ik_1}a_{k_1k_2}a_{k_2k_3}\cdots a_{k_{(m-1)}j}.$$

(b) The $(1,n)$th term of C_n^{n-1} is $c_{12}c_{23}c_{34}\cdots c_{(n-1)n} = 1\cdot 1\cdot 1\cdot\cdots\cdot 1 = 1$, this being the only nonzero product, and so giving a nonzero term. Hence $\nu(C_n) = n$.

(c) Construct C_{kn} by padding out C_k with $n-k$ extra rows and columns of zeroes, placed e.g. below and to the right of C_k.

(3) (a) For $A \in \mathbb{C}^{n\times n}$ define $\exp(A)$. Show that if A is nilpotent then $\exp(A) = \sum_{i=0}^{n-1} A^i/i!$.

(b) For $A \in \mathbb{C}^{n\times n}$ and nilpotent, show that

$$(I-A)^{-1} = I + A + A^2 + \cdots + A^{n-1},$$

where (as usual) I is the $n\times n$ identity matrix.

(a) We have $\exp(A) = \sum_{i=0}^{\infty} A^i/i!$, which reduces to $\sum_{i=0}^{n-1} A^i/i!$ as $A^n = 0$.

(b) We have

$$(I-A)(I+A+A^2+\cdots+A^{n-1}) = I+(A-A)+(A^2-A^2)+\cdots+(A^{n-1}-A^{n-1})-A^n = I,$$

as $A^n = 0$. This gives the result.

(4) For A and B in $F^{n\times n}$ with A nilpotent and B nonsingular, show in two different ways that $B^{-1}AB$ is nilpotent:

- by direct multiplication;
- by considering the eigenvalues of $B^{-1}AB$.

We have

$$B^{-1}AB\cdot B^{-1}AB\cdot\ldots B^{-1}AB = B^{-1}A^nB = B^{-1}0B = 0.$$

But also, we know from lectures that $B^{-1}AB$ and A have the same eigenvalues, so by Q1, the eigenvalues of A are all 0, and thus so are all the eigenvalues of $B^{-1}AB$. Hence (again by Q1, $B^{-1}AB$ is nilpotent.