1. (a) Find the general integer solution (x, y, z) to the equation

$$5x + 7y + 9z = 11.$$

[6]

(b) Define what is meant by an *arithmetic* function, and also what is meant by a *multiplicative* function.

A certain multiplicative function has $f(1) = 1$ and for $k \geq 1$ takes the value k at all prime powers p^k.

What is $f(144)$?

For which values of n is $f(n)$ prime? [7]

(c) For the function f in part (b), show that the Dirichlet series $D_f(s)$ has Euler product

$$\prod_p \frac{1 - p^{-s} + p^{-2s}}{(1 - p^{-s})^2}.$$

[You may find the formal identity $\sum_{k=1}^{\infty} kx^n = \frac{x}{(1-x)^2}$ useful.] [6]

(d) Given any real number x and positive integer n, prove that $\lfloor \frac{\lfloor x \rfloor}{n} \rfloor = \lfloor \frac{x}{n} \rfloor$.

Deduce that for any real number y and positive integers n and m, one has

$$\lfloor \frac{\lfloor \frac{y}{m} \rfloor}{n} \rfloor = \lfloor \frac{\lfloor \frac{y}{n} \rfloor}{m} \rfloor.$$

[6]

2. Let p be a fixed odd prime.

(a) What does it mean for g to be a *primitive root* $\pmod p$?

Taking g to be a fixed primitive root $\pmod p$, describe all the primitive roots $\pmod p$ in terms of g. Justify your answer.

How many primitive roots $\pmod p$ are there? [9]

(b) Taking g as in part (a), and $n \in \{0, 1, \ldots, p-2\}$, write down a necessary and sufficient condition for $x = g^n$ to be a root of $x^5 \equiv 1 \pmod p$. (This condition should depend on n and p only, not on g.)

How many such roots x of this equation are there? [The answer may depend on p.] [9]

(c) Prove that every prime factor q of $2^p - 1$ satisfies the congruence

$$2^{\gcd(p,q-1)} \equiv 1 \pmod q.$$

Deduce that q must be of the form $q = 2kp + 1$. [7]

[Continued overleaf...]

3. (a) Given an odd prime p and an integer $a \in \{1, 2, \ldots, p-1\}$
   - Define what it means for a to be a *quadratic residue* $\pmod p$;
   - Define the Legendre symbol $\left(\frac{a}{p}\right)$. [2]

   (b) State without proof the Law of Quadratic Reciprocity.
   Use it to evaluate $\left(\frac{43}{97}\right)$. [8]
   [Standard properties of the Legendre symbol may be used without proof, provided that they are clearly stated.]

   (c) Apply the Law of Quadratic Reciprocity to prove that for a prime $p > 3$
   $$\left(\frac{3}{p}\right) = \varepsilon\varepsilon',$$
   where $\varepsilon, \varepsilon' \in \{-1, 1\}$ with $p \equiv \varepsilon \pmod 3$ and $p \equiv \varepsilon' \pmod 4$. [8]

   (d) Apply the Law of Quadratic Reciprocity to prove that for a prime $p > 5$
   $$\left(\frac{5}{p}\right) = |2(p \mod 5) - 5| - 2,$$
   where $p \mod 5 \in \{1, 2, 3, 4\}$. [7]

4. (a) State the defining properties of a nonarchimedean valuation $|\cdot|$ on a field F.
   Use these properties to prove that $|1| = 1$ and that $|-1| = 1$. [7]

   (b) Prove that for any positive integer m the number $(m+1)^m - 1$ is divisible by m^2. Deduce that $16 \mid (5^4 - 1)$. [4]

   (c) Calculate the expansions of $-1/16$ and of $15/16$ as 5-adic numbers in standard form. [6]

   (d) For a fixed odd prime p having g as a primitive root, prove that for any integer $k \geq 0$ one has $g^{p^{k+1}} \equiv g^{p^k} \pmod{p^{k+1}}$.
   Deduce that $g^{p^{k'}} \equiv g^{p^k} \pmod{p^k}$ for all $k' \geq k$, and that the sequence $\{g^{p^k}\}_{k\in\mathbb{N}}$ tends to a limit, ℓ say, in $\mathbb{Q}_p$.
   Prove that $\ell^{p-1} = 1$ in $\mathbb{Q}_p$. [8]

[End of Paper]