1. (a) Given two real numbers x and y, prove that $\lfloor x \rfloor < \lfloor y \rfloor$ iff there is an integer k such that $x < k \leq y$. [4 marks]

(b) Deduce that for any positive real number x and positive integers ℓ and m one has

$$\left\lfloor \frac{\lfloor \frac{x}{\ell} \rfloor}{m} \right\rfloor = \left\lfloor \frac{x}{\ell m} \right\rfloor.$$

[5]

(c) Let $n \in \mathbb{N}$ and p be a prime number. Show that the highest power of p dividing $n!$ is p^e where

$$e = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

[6]

(d) Show that this sum can be computed as

$$e = \sum_{j=1}^{\lfloor \frac{\log n}{\log p} \rfloor} n_j,$$

where $n_1 = \lfloor n/p \rfloor$ and, for $j > 1$, $n_j = \lfloor n_{j-1}/p \rfloor$. [5]

(e) (Unconnected with earlier parts.) Let $m \in \mathbb{N}$, and suppose that t is a positive irrational number. Put $n = \lfloor mt \rfloor$. Prove that

$$\sum_{k=1}^{m} \lfloor kt \rfloor + \sum_{k=1}^{n} \left\lfloor \frac{k}{t} \right\rfloor = mn.$$

[5]

[Continued overleaf]

2. (a) State without proof the Chinese Remainder Theorem for three moduli m_1, m_2, m_3. Give a formula for the solution x of the congruences of the Theorem. [4]

(b) Deduce that there are integers a, b, c, d such that

$$\begin{aligned} 41b &= a - 1 \\ 61c &= a \\ 101d &= a + 1. \end{aligned}$$

(You do not need to find them.) [2]

(c) Define a *Carmichael number*. [3]

(d) Show that $N = 252601 = 41 \cdot 61 \cdot 101$ is a Carmichael number. [5]

(e) For N as in part (d) and an integer a, describe a property of the prime factorisation of a which determines whether or not a belongs to the set

$$S = \{a : 1 \leq a < N \text{ and } a^{N-1} \not\equiv 1 \pmod{N}\}.$$

[3]

(f) Show that S contains $3! = 6$ sets of three consecutive integers. [4]

(g) Show that in any sequence of 41 consecutive integers $a, a+1, \ldots, a+40$, at most three of them belong to S.
Deduce that S does not contain four consecutive integers. [4]

3. (a) Define the Legendre symbol $\left(\frac{a}{p}\right)$ for an odd prime p and an integer a coprime to p.
State the Law of Quadratic Reciprocity, without proof.
Evaluate $\left(\frac{91}{107}\right)$, stating without proof any properties of the Legendre symbol that you need to use for this evaluation. [8]

(b) State without proof necessary and sufficient conditions on the prime factorisation of a positive integer n for n to be expressible as $n = x^2 + y^2$ for some integers x, y. [3]

(c) Deduce that if a rational n/m is the sum of two squares of rationals then nm is the sum of two squares of integers. [6]

(d) Now let $p > 3$ be a prime that can be written in the form $p = u^2 + 3v^2$ for some integers u, v. Prove that then $\left(\frac{-3}{p}\right) = 1$ and that $p \equiv 1 \pmod{6}$. [8]

[Continued overleaf]

4. (a) State the defining properties of a nonarchimedean valuation $|\cdot|$ on the field $\mathbb{Q}$ of rational numbers. [3]

   (b) For such a valuation, show that $|1| = 1 = |-1|$ and $|n| \leq 1$ for all $n \in \mathbb{Z}$. [6]

   (c) When such a valuation has $|n| < 1$ for at least one $n \in \mathbb{N}$, show that there is a prime p such that $\{n \in \mathbb{N} : |n| < 1\} = \{n \in \mathbb{N} : p \text{ divides } n\}$. [6]

   (d) Calculate the standard 2-adic expansion of 1/5. [5]

   (e) A standard expansion $a_0 + a_1 p + a_2 p^2 + \cdots + a_k p^k + \ldots$ of a p-adic integer is called *purely periodic* if there is some positive integer r such that $a_i = a_{r+i}$ for $i = 0, 1, 2, 3, \ldots$. Show that such an expansion is the p-adic representation of a nonpositive rational number. [5]

[End of Paper]