Mathematics 4: Number Theory Problem Sheet 1

The integer part (= floor) function

The aim of these problems is to gain some facility in obtaining nontrivial results about this simple function.

Workshop

(1) Let $x, y \in \mathbb{R}$. Show that $\lfloor x \rfloor < \lfloor y \rfloor$ iff there is an integer in the half-open interval $(x, y]$, (or, equivalently, $\mathbb{Z} \cap (x, y] \neq \emptyset$).
(In particular, note that $\mathbb{Z} \cap (\lfloor y \rfloor, y] = \emptyset$.)

(2) For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ show that $\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor$.

(3) For $x \in \mathbb{R}$ show that $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$ for all $x \geq 0$.

(4) (a) Prove that, for $n \in \mathbb{N}$ and all $a_k \in \mathbb{R}$,

$$\sum_{k=1}^{n} a_k = n a_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k).$$

(b) Deduce that

$$\sum_{k=1}^{n} \lfloor \log_2 k \rfloor = (n+1) \lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor + 1} + 2.$$

Handin: due Friday, week 3, 5 Oct, before 12.10 lecture.

This handin only: please hand in to MTO, room 5211

You are expected to write clearly and legibly, giving thought to the presentation of your answer as a document written in mathematical English.

(5) (a) Prove that, for $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}.$$

(b) Evaluate $\sum_{k=1}^{n} \left\lfloor \sqrt{k} \right\rfloor$. You may find it convenient to express your answer in terms of $N = \lfloor \sqrt{n} \rfloor$.

(c) For which values of N is the sum in (b) divisible by N?

Further problems

(6) For $m, n \in \mathbb{Z}$, evaluate $\left\lfloor \frac{n+m}{2} \right\rfloor + \left\lfloor \frac{n-m+1}{2} \right\rfloor$.

(7) Show that, for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n^2}{4} \right\rfloor .$$

(8) (a) Let $n \in \mathbb{N}$ and p be a prime number. Show that the highest power of p dividing $n!$ is p^N where

$$N = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots .$$

(b) Show that this sum can be efficiently computed as

$$\sum_{k=1}^{\left\lfloor \frac{\log n}{\log p} \right\rfloor} n_k,$$

where $n_1 = \lfloor n/p \rfloor$ and, for $k > 1$, $n_k = \lfloor n_{k-1}/p \rfloor$.

(9) Let $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$. Show that

$$\sum_{k=1}^{p-1} \left\lfloor \frac{kq}{p} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

(10) Prove Hermite's formula

$$\sum_{k=1}^{\lfloor a \rfloor} \left\lfloor \frac{a}{k} \right\rfloor = 2 \sum_{k=1}^{\lfloor \sqrt{a} \rfloor} \left\lfloor \frac{a}{k} \right\rfloor - \left\lfloor \sqrt{a} \right\rfloor^2,$$

for any $a \geq 0$.

(11) Let $m \in \mathbb{N}$, and $b > 0$ such that none of the numbers $kb \quad (k = 1, \ldots, m)$ are integers. Put $n = \lfloor mb \rfloor$. Prove that

$$\sum_{k=1}^{m} \lfloor kb \rfloor + \sum_{k=1}^{n} \left\lfloor \frac{k}{b} \right\rfloor = mn.$$

(12) A positive irrational number r generates the so-called *Beatty sequence*

$$\mathcal{B}_r = \lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, ... = (\lfloor nr \rfloor)_{n\geq 1}.$$

If $r > 1$, then $s = r/(r-1)$ is also a positive irrational number, and, as is trivial to check, $\frac{1}{r} + \frac{1}{s} = 1$. Prove that

$$\mathcal{B}_r = (\lfloor nr \rfloor)_{n\geq 1}$$

and

$$\mathcal{B}_s = (\lfloor ns \rfloor)_{n\geq 1}$$

form a pair of *complementary* Beatty sequences: that is, they have empty intersection and their union is $\mathbb{N}$.

Hint: One approach to a proof is to first show that collisions are impossible, and then to show that gaps are impossible.

/home/chris/NTh/wkp1_12.tex