Mathematics 4: Number Theory Problem Sheet 1

The integer part (= floor) function

The aim of these problems is to gain some facility in obtaining nontrivial results about this simple function.

Workshop

(1) Let $x, y \in \mathbb{R}$. Show that $\lfloor x \rfloor < \lfloor y \rfloor$ iff there is an integer in the half-open interval $(x, y]$, (or, equivalently, $\mathbb{Z} \cap (x, y] \neq \emptyset$).
(In particular, note that $\mathbb{Z} \cap (\lfloor y \rfloor, y] = \emptyset$.)

If $\lfloor x \rfloor < \lfloor y \rfloor$ then $x < \lfloor y \rfloor$ (as otherwise we'd have $\lfloor x \rfloor \geq$ the integer $\lfloor y \rfloor$). So $x < \lfloor y \rfloor \leq y$, and we can take $k = \lfloor y \rfloor$.
Conversely, if $x < k \leq y$ then $\lfloor x \rfloor < k$ and $\lfloor y \rfloor \geq k$ so $\lfloor x \rfloor < \lfloor y \rfloor$.

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(2) For $x \in \mathbb{R}$ and $k \in \mathbb{N}$ show that $\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor$.

As
$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ we have $\lfloor x \rfloor / k \leq x/k < (\lfloor x \rfloor + 1)/k$.
Thus there is no integer in the interval $(\lfloor x \rfloor / k, x/k]$ so, by the previous question, they have the same integer part.

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(3) For $x \in \mathbb{R}$ show that $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor = \lfloor \sqrt{x} \rfloor$ for all $x \geq 0$.

As the integer-part and square root functions are both monotonic non-decreasing functions, we have $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor \leq \lfloor \sqrt{x} \rfloor$. If $\left\lfloor \sqrt{\lfloor x \rfloor} \right\rfloor < \lfloor \sqrt{x} \rfloor$, then we have $\sqrt{\lfloor x \rfloor} < n \leq \sqrt{x}$ for some integer n. Squaring, we have $\lfloor x \rfloor < n^2 \leq x$, contradicting the definition of $\lfloor x \rfloor$.

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(4) (a) Prove that, for $n \in \mathbb{N}$ and all $a_k \in \mathbb{R}$,

$$\sum_{k=1}^{n} a_k = n a_n - \sum_{k=1}^{n-1} k(a_{k+1} - a_k).$$

(b) Deduce that

$$\sum_{k=1}^{n} \lfloor \log_2 k \rfloor = (n+1) \lfloor \log_2 n \rfloor - 2^{\lfloor \log_2 n \rfloor + 1} + 2.$$

(a) Check that each a_k occurs exactly once on the RHS, as it does on the LHS.

(This is a special case of 'Summation by Parts', the discrete version of Integration by Parts.)

(b) Put $a_k = \lfloor \log_2 k \rfloor$. Then, by (a),

$$\sum_{k=1}^{n} \lfloor \log_2 k \rfloor = n \lfloor \log_2 n \rfloor - \sum_{k=1}^{n-1} k(\lfloor \log_2 (k+1) \rfloor - \lfloor \log_2 k \rfloor).$$

Now $\lfloor \log_2 (k+1) \rfloor - \lfloor \log_2 k \rfloor = 0$ unless there is an integer between the two terms, i.e. unless

$$\log_2 k < \ell \leq \log_2 (k+1)$$

or

$$k < 2^\ell \leq k+1.$$

So $k = 2^\ell - 1 \leq n-1$, giving $2^\ell \leq n$, so that $\ell = 1, \ldots, \lfloor \log_2 n \rfloor$. Hence the total is

$$n \lfloor \log_2 n \rfloor - (2^1 - 1 + 2^2 - 1 + \ldots 2^{\lfloor \log_2 n \rfloor} - 1) = n \lfloor \log_2 n \rfloor + \lfloor \log_2 n \rfloor - \sum_{\ell=1}^{\lfloor \log_2 n \rfloor} 2^\ell$$
$$= (n+1) \lfloor \log_2 n \rfloor - (2^{\lfloor \log_2 n \rfloor + 1} - 2),$$

giving the answer.

Handin: due Friday, week 3, 5 Oct, before 12.10 lecture.

This handin only: please hand in to MTO, room 5211

You are expected to write clearly and legibly, giving thought to the presentation of your answer as a document written in mathematical English.

(5) (a) Prove that, for $m \in \mathbb{N}$,

$$\sum_{k=1}^{m} k^2 = \frac{m(m+1)(2m+1)}{6}.$$

(b) Evaluate $\sum_{k=1}^{n} \left\lfloor \sqrt{k} \right\rfloor$. You may find it convenient to express your answer in terms of $N = \lfloor \sqrt{n} \rfloor$.

(c) For which values of N is the sum in (b) divisible by N?

[2 marks] (a) A standard induction, for instance.

(b) Put $a_k = \left\lfloor \sqrt{k} \right\rfloor$, so that (a) gives

$$\sum_{k=1}^{n} \lfloor \sqrt{k} \rfloor = n \lfloor \sqrt{n} \rfloor - \sum_{k=1}^{n-1} k(\left\lfloor \sqrt{k+1} \right\rfloor - \left\lfloor \sqrt{k} \right\rfloor).$$

Now $\lfloor\sqrt{k+1}\rfloor - \lfloor\sqrt{k}\rfloor = 0$ unless there is an integer between the two terms, i.e. unless

$$\sqrt{k} < \ell \leq \sqrt{k+1}$$

or

$$k < \ell^2 \leq k+1.$$

So $k = \ell^2 - 1 \leq n-1$, giving $\ell^2 \leq n$, so that $\ell = 2, \ldots, \lfloor\sqrt{n}\rfloor$. Hence the total is

$$\begin{aligned} n\lfloor\sqrt{n}\rfloor - \sum_{\ell=2}^{\lfloor\sqrt{n}\rfloor}(\ell^2-1) &= n\lfloor\sqrt{n}\rfloor - \sum_{\ell=1}^{\lfloor\sqrt{n}\rfloor}(\ell^2-1) \\ &= (n+1)\lfloor\sqrt{n}\rfloor - \sum_{\ell=1}^{\lfloor\sqrt{n}\rfloor}\ell^2 \\ &= (n+1)N - N(N+1)(2N+1)/6, \end{aligned}$$

using (a). [6 marks]

(c) We need $(N+1)(2N+1)/6$ to be an integer. As $2N+1$ is odd, $N+1$ must be even, so N odd. If N is divisible by 3, then neither of $N+1$ or $2N+1$ are. On the other hand, if $N \equiv \pm 1 \pmod 3$, then $3 \mid (N+1)(2N+1)$.

Hence the given sum is divisible by N iff $N \equiv \pm 1 \pmod 6$. [2 marks]

Further problems

(6) For $m, n \in \mathbb{Z}$, evaluate $\lfloor\frac{n+m}{2}\rfloor + \lfloor\frac{n-m+1}{2}\rfloor$.

If $n+m = 2k$ then $\lfloor\frac{n+m}{2}\rfloor = k$ and $\lfloor\frac{n-m+1}{2}\rfloor = \lfloor\frac{2k+1-2m}{2}\rfloor = k-m$ so that $\lfloor\frac{n+m}{2}\rfloor + \lfloor\frac{n-m+1}{2}\rfloor = 2k-m = n$.

If $n+m = 2k+1$ then $\lfloor\frac{n+m}{2}\rfloor = k$ and $\lfloor\frac{n-m+1}{2}\rfloor = \lfloor\frac{2k+2-2m}{2}\rfloor = k-m+1$ so that $\lfloor\frac{n+m}{2}\rfloor + \lfloor\frac{n-m+1}{2}\rfloor = 2k+1-m = n$ also.

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(7) Show that, for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor = \left\lfloor\frac{n^2}{4}\right\rfloor.$$

(8) (a) Let $n \in \mathbb{N}$ and p be a prime number. Show that the highest power of p dividing $n!$ is p^N where

$$N = \left\lfloor\frac{n}{p}\right\rfloor + \left\lfloor\frac{n}{p^2}\right\rfloor + \left\lfloor\frac{n}{p^3}\right\rfloor + \cdots.$$

(b) Show that this sum can be efficiently computed as

$$\sum_{k=1}^{\lfloor \frac{\log n}{\log p} \rfloor} n_k,$$

where $n_1 = \lfloor n/p \rfloor$ and, for $k > 1$, $n_k = \lfloor n_{k-1}/p \rfloor$.

(a) The term $\left\lfloor \frac{n}{p^j} \right\rfloor$ counts the number of integers up to n that are divisible by p^j. Thus if a particular number $\le n$ is divisible by p^j but not by any higher power of p it will be counted once in each of $\left\lfloor \frac{n}{p} \right\rfloor, \left\lfloor \frac{n}{p^2} \right\rfloor, \left\lfloor \frac{n}{p^3} \right\rfloor, \cdots, \left\lfloor \frac{n}{p^j} \right\rfloor$, a total of j times. Thus e gives the total number of p's dividing all of $1, 2, 3, \ldots, n$, i.e. of $n!$.

(b) Use the earlier result $\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{\lfloor x \rfloor}{k} \right\rfloor$ with $x = n/p^j$ and $k = p$.

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(9) Let $p, q \in \mathbb{N}$ with $\gcd(p, q) = 1$. Show that

$$\sum_{k=1}^{p-1} \left\lfloor \frac{kq}{p} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

The integers $q, 2q, \ldots, kq, \ldots, (p-1)q$ all have different remainders $\bmod p$, giving all possible nonzero remainders $1, 2, \ldots, (p-1)$. Hence, as $\left\lfloor \frac{kq}{p} \right\rfloor = \frac{kq-r}{p}$ for one of these remainders r,

$$\begin{aligned} \sum_{k=1}^{p-1} \left\lfloor \frac{kq}{p} \right\rfloor &= \sum_{k=1}^{p-1} \frac{kq}{p} - \sum_{r=1}^{p-1} \frac{r}{p} \\ &= \frac{(p-1)(q-1)}{2} \end{aligned}$$

on summing these arithmetic progressions.

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(10) Prove Hermite's formula

$$\sum_{k=1}^{\lfloor a \rfloor} \left\lfloor \frac{a}{k} \right\rfloor = 2 \sum_{k=1}^{\lfloor \sqrt{a} \rfloor} \left\lfloor \frac{a}{k} \right\rfloor - \lfloor \sqrt{a} \rfloor^2,$$

for any $a \ge 0$.

Consider the hyperbola $xy = a$ for $x, y \ge 0$. Then then left sum S_1 is the total number of integer points (x, y) with $x, y \ge 1$ below or on the hyperbola.

Note that there are no such integer points with either of x or $y > \lfloor a \rfloor$, so we're working entirely inside the square $0 < x, y < \lfloor a \rfloor$. Now divide the part of the square that's below the hyperbola (ie the part we want to count the integer points in) into 3 pieces by partitioning off the square $0 < x, y < \sqrt{a}$. Draw a picture! Then apart from the square, which has $\lfloor\sqrt{a}\rfloor^2$ integer points, there is the region above the square, with say n_1 integer points, and the region to the right of the square, with say n_2 integer points. The required left-hand sum S_1 is $\lfloor\sqrt{a}\rfloor^2 + n_1 + n_2$. But the total number of integer points with $x \le \lfloor\sqrt{a}\rfloor$ is $\sum_{k=1}^{\lfloor\sqrt{a}\rfloor} \left\lfloor \frac{a}{k} \right\rfloor$, which is also $= \lfloor\sqrt{a}\rfloor^2 + n_1$. Also, by symmetry (interchanging x and y), $n_1 = n_2$. Hence $S_1 = \lfloor\sqrt{a}\rfloor^2 + n_1 + n_2 = 2(\lfloor\sqrt{a}\rfloor^2 + n_1) - \lfloor\sqrt{a}\rfloor^2$, giving the result.

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(11) Let $m \in \mathbb{N}$, and $b > 0$ such that none of the numbers $kb \quad (k = 1, \dots, m)$ are integers. Put $n = \lfloor mb \rfloor$. Prove that

$$\sum_{k=1}^{m} \lfloor kb \rfloor + \sum_{k=1}^{n} \left\lfloor \frac{k}{b} \right\rfloor = mn.$$

On the right upper quadrant of the x-y plane, draw the line $y = bx$, and consider the rectangle with vertices $(0,0)$, $(m,0)$, $(0,bm)$, (m,bm). It has the line as a diagonal. The number of lattice points (points (i,j) with $i,j \in \mathbb{N}$) in this rectangle, but not on either axis, is clearly mn.

Because none of the numbers $kb \quad (k = 1, \dots, m)$ are integers, there are no lattice points on the diagonal.

We now count these lattice points in a different way. The number of lattice points with $x = k$ and below the diagonal is clearly $\lfloor kb \rfloor$ (since $\lfloor x \rfloor$ is the number of integers in $[1,x]$). Summing this for $k = 1, 2, \dots, m$ gives the total number of lattice points below the diagonal as $\sum_{k=1}^{m} \lfloor kb \rfloor$.

To obtain the number of lattice points in the rectangle that are above the diagonal, we reverse the rôles of x and y. The diagonal is $x = y/b$, and the number of lattice points with y-coordinate $= k$ is $\lfloor k/b \rfloor$. Summing this for $k = 1, 2, \dots, n$ gives the total number of lattice points above the diagonal as $\sum_{k=1}^{n} \left\lfloor \frac{k}{b} \right\rfloor$.

Combining these results, we have that the total number of lattice points in the rectangle is equal both to mn and to $\sum_{k=1}^{m} \lfloor kb \rfloor + \sum_{k=1}^{n} \left\lfloor \frac{k}{b} \right\rfloor$.

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(12) A positive irrational number r generates the so-called *Beatty sequence*

$$\mathcal{B}_r = \lfloor r \rfloor, \lfloor 2r \rfloor, \lfloor 3r \rfloor, ... = (\lfloor nr \rfloor)_{n \ge 1}.$$

If $r > 1$, then $s = r/(r-1)$ is also a positive irrational number, and, as is trivial to check, $\frac{1}{r} + \frac{1}{s} = 1$. Prove that

$$\mathcal{B}_r = (\lfloor nr \rfloor)_{n \ge 1}$$

and

$$\mathcal{B}_s = (\lfloor ns \rfloor)_{n \geq 1}$$

form a pair of *complementary* Beatty sequences: that is, they have empty intersection and their union is $\mathbb{N}$.

Hint: One approach to a proof is to first show that collisions are impossible, and then to show that gaps are impossible.

No collisions: Suppose that there are integers $j > 0$ and k and m such that

$$j = \lfloor k \cdot r \rfloor = \lfloor m \cdot s \rfloor .$$

(So j is in both sequences.)

This is equivalent to the inequalities

$$j \leq k \cdot r < j+1 \text{ and } j \leq m \cdot s < j+1.$$

For non-zero j, the irrationality of r and s is incompatible with equality, so

$$j < k \cdot r < j+1 \text{ and } j < m \cdot s < j+1$$

which lead to

$$\frac{j}{r} < k < \frac{j+1}{r} \text{ and } \frac{j}{s} < m < \frac{j+1}{s}.$$

Adding these together and using the hypothesis, we get $j < k + m < j + 1$, a contradiction.

No gaps: Suppose there are integers $j > 0$ and k and m such that

$$k \cdot r < j \text{ and } j+1 \leq (k+1) \cdot r \text{ and } m \cdot s < j \text{ and } j+1 \leq (m+1) \cdot s .$$

(So j is a 'gap'.)

Since $j+1$ is non-zero and r and s are irrational, we can exclude equality, so

$$k \cdot r < j \text{ and } j+1 < (k+1) \cdot r \text{ and } m \cdot s < j \text{ and } j+1 < (m+1) \cdot s.$$

Then we get

$$k < \frac{j}{r} \text{ and } \frac{j+1}{r} < k+1 \text{ and } m < \frac{j}{s} \text{ and } \frac{j+1}{s} < m+1$$

Adding corresponding inequalities, we get

$$k + m < j \text{ and } j+1 < k+m+2$$

so that $k + m < j < k + m + 1$, a contradiction.
