

THE MAHLER MEASURE OF ALGEBRAIC NUMBERS: A SURVEY

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ABSTRACT. A survey of results for Mahler measure of algebraic numbers, and one-variable polynomials with integer coefficients is presented. Related results on the maximum modulus of the conjugates ('house') of an algebraic integer are also discussed. Some generalisations are given too, though not to Mahler measure of polynomials in more than one variable.

1. INTRODUCTION

Let $P(x) = a_0z^d + \cdots + a_d = a_0 \prod_{i=1}^d (z - \alpha_i)$ be a nonconstant polynomial with (at first) complex coefficients. Then, following Mahler [101] its *Mahler measure* is defined to be

$$M(P) := \exp \left(\int_0^1 \log |P(e^{2\pi it})| dt \right), \quad (1)$$

the geometric mean of $|P(z)|$ for z on the unit circle. However $M(P)$ had appeared earlier in a paper of Lehmer [94], in an alternative form

$$M(P) = |a_0| \prod_{|\alpha_i| \geq 1} |\alpha_i|. \quad (2)$$

The equivalence of the two definitions follows immediately from Jensen's formula [88]

$$\int_0^1 \log |e^{2\pi it} - \alpha| dt = \log_+ |\alpha|.$$

Here $\log_+ x$ denotes $\max(0, \log x)$. If $|a_0| \geq 1$, then clearly $M(P) \geq 1$. This is the case when P has integer coefficients; we assume henceforth that P is of this form. Then, from a result of Kronecker [90], $M(P) = 1$ occurs only if $\pm P$ is a power of z times a cyclotomic polynomial.

In [101] Mahler called $M(P)$ the measure of the polynomial P , apparently to distinguish it from its (naïve) height. This was first referred to as Mahler's measure by Waldschmidt [165, p.21] in 1979 ('mesure de Mahler'), and soon afterwards by Boyd [33] and Durand [75], in the sense of "the function that Mahler called 'measure'", rather than as a name. But it soon *became* a name. In 1983 Louboutin [98] used the term to apply to an algebraic number. We shall follow this convention too — $M(\alpha)$ for an algebraic number α will mean

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the Mahler measure of the minimal polynomial P_α of α , with d the degree of α , having conjugates $\alpha = \alpha_1, \alpha_2, \dots, \alpha_d$. The Mahler measure is actually a height function on polynomials with integer coefficients, as there are only a finite number of such polynomials of bounded degree and bounded Mahler measure. Indeed, in the MR review of [98], it is called the Mahler height; but ‘Mahler measure’ has stuck.

For the Mahler measure in the form $M(\alpha)$, there is a third representation to add to (1) and (2). We consider a complete set of inequivalent valuations $|\cdot|_\nu$ of the field $\mathbb{Q}(\alpha)$, normalised so that, for $\nu|p$, $|\cdot|_\nu = |\cdot|_p$ on \mathbb{Q}_p . Here \mathbb{Q}_p is the field of p -adic numbers, with the usual valuation $|\cdot|_p$. Then for a_0 as in (2),

$$|a_0| = \prod_{p < \infty} |a_0|_p^{-1} = \prod_{p < \infty} \prod_{\nu|p} \max(1, |\alpha|_\nu^{d_\nu}), \quad (3)$$

coming from the product formula, and from considering the Newton polygons of the irreducible factors (of degree d_ν) of P_α over \mathbb{Q}_p (see e.g. [170, p. 73]).

Then [169, pp. 74–79], [23] from (2) and (3)

$$M(\alpha) = \prod_{\text{all } \nu} \max(1, |\alpha|_\nu^{d_\nu}), \quad (4)$$

and so also

$$h(\alpha) := \frac{\log M}{d} = \sum_{\text{all } \nu} \log_+ |\alpha|_\nu^{d_\nu/d}. \quad (5)$$

Here $h(\alpha)$ is called the *Weil*, or *absolute* height of α .

2. LEHMER’S PROBLEM

While Mahler presumably had applications of his measure to transcendence in mind, Lehmer’s interest was in finding large primes. He sought them amongst the Pierce numbers $\prod_{i=1}^d (1 \pm \alpha_i^m)$, where the α_i are the roots of a monic polynomial P having integer coefficients. Lehmer showed that for P with no roots on the unit circle these numbers grew with m like $M(P)^m$. Pierce [120] had earlier considered the factorization of these numbers. Lehmer posed the problem of whether, among those monic integer polynomials with $M(P) > 1$, polynomials could be chosen with $M(P)$ arbitrarily close to 1. This has become known as ‘Lehmer’s problem’, or ‘Lehmer’s conjecture’, the ‘conjecture’ being that they could not, although Lehmer did not in fact make this conjecture.¹ The smallest value of $M(P) > 1$ he could find was

$$M(L) = 1.176280818\dots,$$

¹‘Lehmer’s conjecture’ is also used to refer to a conjecture on the non-vanishing of Ramanujan’s τ -function. But I do not know that Lehmer actually made that conjecture either: in [95, p. 429] he wrote “... and it is natural to ask whether $\tau(n) = 0$ for any $n > 0$.”

where $L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ is now called ‘Lehmer’s polynomial’. To this day no-one has found a smaller value of $M(P) > 1$ for $P(z) \in \mathbb{Z}[z]$.

Lehmer’s problem is central to this survey. We concentrate on results for $M(P)$ with P having integer coefficients. We do not attempt to survey results for $M(P)$ for P a polynomial in several variables. For this we refer the reader to [18], [33], [163], [39], [43], [40], [79, Chapter 3]. However, the one-variable case should not really be separated from the general case, because of the fact that for every P with integer coefficients, irreducible and in genuinely more than one variable (i.e., its Newton polytope is not one-dimensional) $M(P)$ is known [33, Theorem 1] to be the limit of $\{M(P_n)\}$ for some sequence $\{P_n\}$ of one-variable integer polynomials. This is part of a far-reaching conjecture of Boyd [33] to the effect that the set of all $M(P)$ for P an integer polynomial in any number of variables is a closed subset of the real line.

Our survey of results related to Lehmer’s problem falls into three categories. We report lower bounds, or sometimes exact infima, for $M(P)$ as P ranges over certain sets of integer polynomials. Depending on this set, such lower bounds can either tend to 1 as the degree d of P tends to infinity (Section 4), be constant and greater than 1 (Section 5), or increase exponentially with d (Section 6). We also report on computational work on the problem (Section 8).

In Sections 3 and 7 we discuss the closely-related function $|\overline{\alpha}|$ and the Schinzel-Zassenhaus conjecture. In Section 9 connections between Mahler measure and the discriminant are covered. In Section 10 the known properties of $M(\alpha)$ as an algebraic number are outlined. Section 11 is concerned with counting integer polynomials of given Mahler measure, while in Section 12 a dynamical systems version of Lehmer’s problem is presented. In Section 13 variants of Mahler measure are discussed, and finally in Section 14 some applications of Mahler measure are given.

3. THE HOUSE $|\overline{\alpha}|$ OF α AND THE CONJECTURE OF SCHINZEL AND ZASSENHAUS

Related to the Mahler measure of an algebraic integer α is $|\overline{\alpha}|$, the *house* of α , defined as the maximum modulus of its conjugates (including α itself). For α with $r > 0$ roots of modulus greater than 1 we have the obvious inequality

$$M(\alpha)^{1/d} \leq M(\alpha)^{1/r} \leq |\overline{\alpha}| \leq M(\alpha) \quad (6)$$

(see e.g. [34]). If α is in fact a unit (which is certainly the case if $M(\alpha) < 2$) then $M(\alpha) = M(\alpha^{-1})$ so that

$$M(\alpha) \leq (\max(|\overline{\alpha}|, |\overline{1/\alpha}|))^{d/2}.$$

In 1965 Schinzel and Zassenhaus [144] proved that if $\alpha \neq 0$ an algebraic integer that is not a root of unity and if $2s$ of its conjugates are nonreal, then

$$|\overline{\alpha}| > 1 + 4^{-s-2}. \quad (7)$$

This was the first unconditional result towards solving Lehmer's problem, since by (6) it implies the same lower bound for $M(\alpha)$ for such α . They conjectured, however, that a much stronger bound should hold: that under these conditions in fact

$$|\overline{\alpha}| \geq 1 + c/d \quad (8)$$

for some absolute constant $c > 0$. Its truth is implied by a positive answer to Lehmer's 'conjecture'. Indeed, because $|\overline{\alpha}| \geq M(\alpha)^{1/d}$ where $d = \deg \alpha$, we have

$$|\overline{\alpha}| \geq 1 + \frac{\log M(\alpha)}{d} = 1 + h(\alpha), \quad (9)$$

so that if $M(\alpha) \geq c_0 > 1$ then $|\overline{\alpha}| > 1 + \frac{\log(c_0)}{d}$.

Likewise, from this inequality any results in the direction of solving Lehmer's problem will have a corresponding 'Schinzel-Zassenhaus conjecture' version. In particular, this applies to the results of Section 5.1 below, including that of Breusch. His inequality appears to be the first, albeit conditional, result in the direction of the Schinzel-Zassenhaus conjecture or the Lehmer problem.

4. UNCONDITIONAL LOWER BOUNDS FOR $M(\alpha)$ THAT TEND TO 1 AS $d \rightarrow \infty$

4.1. The bounds of Blanksby and Montgomery, and Stewart. The lower bound for $M(\alpha)$ coming from (7) was dramatically improved in 1971 by Blanksby and Montgomery [22], who showed, again for α of degree $d > 1$ and not a root of unity, that

$$M(\alpha) > 1 + \frac{1}{52d \log(6d)}.$$

Their methods were based on Fourier series in several variables, making use of the nonnegativity of Fejér's kernel

$$\frac{1}{2} + \sum_{k=1}^K \left(1 - \frac{k}{K+1}\right) \cos(kx) = \frac{1}{2(K+1)} \left(\sum_{j=0}^K e^{ix(\frac{K}{2}-j)} \right)^2.$$

They also employed a neat geometric lemma for bounding the modulus of complex numbers near the unit circle: if $0 < \rho \leq 1$ and $\rho \leq |z| \leq \rho^{-1}$ then

$$|z - 1| \leq \rho^{-1} \left| \rho \frac{z}{|z|} - 1 \right|. \quad (10)$$

In 1978 Stewart [158] caused some surprise by obtaining a lower bound of the same strength $1 + \frac{C}{d \log d}$ by the use of a completely different argument. He based his proof on the construction of an auxiliary function of the type used in transcendence proofs.

In such arguments it is of course necessary to make use of some arithmetic information, because of the fact that the polynomials one is dealing with, here the minimal polynomials of algebraic integers, are monic, have integer coefficients, and no root is a root of unity. In the three proofs of the results given above, this is done by making use of the fact that, for α not a root of unity, the Pierce numbers $\prod_{i=1}^d (1 - \alpha_i^m)$ are then nonzero integers for all $m \in \mathbb{N}$. Hence they are at least 1 in modulus.

4.2. Dobrowolski's lower bound. In 1979 a breakthrough was achieved by Dobrowolski, who, like Stewart, used an argument based on an auxiliary function to get a lower bound for $M(\alpha)$. However, he also employed more powerful arithmetic information: the fact that for any prime p the resultant of the minimal polynomials of α and of α^p is an integer multiple of p^d . Since this can be shown to be nonzero for α not a root of unity, it is at least p^d in modulus. Dobrowolski [54] was able to apply this fact to obtain for $d \geq 2$ the much improved lower bound

$$M(\alpha) > 1 + \frac{1}{1200} \left(\frac{\log \log d}{\log d} \right)^3. \quad (11)$$

He also has an asymptotic version of his result, where the constant $1/1200$ can be increased to $1 - \varepsilon$ for α of degree $d \geq d_0(\varepsilon)$. Improvements in the constant in Dobrowolski's Theorem have been made since that time. Cantor and Straus [49] proved the asymptotic version of his result with the larger constant $2 - \varepsilon$, by a different method: the auxiliary function was replaced by the use of generalised Vandermonde determinants. See also [125] for a similar argument (plus some early references to these determinants). As with Dobrowolski's argument, the large size of the resultant of α and α^p was an essential ingredient. Louboutin [98] improved the constant further, to $9/4 - \varepsilon$, using the Cantor-Straus method. A different proof of Louboutin's result was given by Meyer [108]. Later Voutier [164], by a very careful argument based on Cantor-Straus, has obtained the constant $1/4$ valid for all α of degree $d \geq 2$. However, no-one has been able to improve the dependence on the degree d in (11), so that Lehmer's problem remains unsolved!

4.3. Generalisations of Dobrowolski's Theorem. Amoroso and David [3, 4] have generalised Dobrowolski's result in the following way. Let $\alpha_1, \dots, \alpha_n$ be n multiplicatively independent algebraic numbers in a number field of degree d . Then for some constant $c(n)$ depending only on n

$$h(\alpha_1) \dots h(\alpha_n) \geq \frac{1}{d \log(3d)^{c(n)}}. \quad (12)$$

Matveev [107] also has a result of this type, but using instead the modified Weil height $h_*(\alpha) := \max(h(\alpha), d^{-1} |\log \alpha|)$.

Amoroso and Zannier [11] have given a version of Dobrowolski's result for α , not 0 or a root of unity, of degree D over an finite abelian extension of a

number field. Then

$$h(\alpha) \geq \frac{c}{D} \left(\frac{\log \log 5D}{\log 2D} \right)^{13}, \quad (13)$$

where the constant c depends only on the number field, not on its abelian extension. Amoroso and Delsinne [9] have recently improved this result, for instance essentially reducing the exponent 13 to 4.

Analogues of Dobrowolski's Theorem have been proved for elliptic curves by Anderson and Masser [12], Hindry and Silverman [84], Laurent [93] and Masser [104]. In particular Masser proved that for an elliptic curve E defined over a number field K and a nontorsion point P defined over a degree $\leq d$ extension F of K that the canonical height $\hat{h}(P)$ satisfies

$$\hat{h}(P) \geq \frac{C}{d^3(\log d)^2}.$$

Here C depends only on E and K . When E has non-integral j -invariant Hindry and Silverman improved this bound to $\hat{h}(P) \geq \frac{C}{d^2(\log d)^2}$. In the case where E has complex multiplication, however, Laurent obtained the stronger bound

$$\hat{h}(P) \geq \frac{C}{d} (\log \log d / \log d)^3.$$

This is completely analogous to the formulation of Dobrowolski's result (11) in terms of the Weil height $h(\alpha) = \log M(\alpha)/d$.

5. RESTRICTED RESULTS OF LEHMER STRENGTH: $M(\alpha) > c > 1$.

5.1. Results for nonreciprocal algebraic numbers and polynomials.

Recall that a polynomial $P(z)$ of degree d is said to be *reciprocal* if it satisfies $z^d P(1/z) = \pm P(z)$. (With the negative sign, clearly $P(z)$ is divisible by $z-1$.) Furthermore an algebraic number α is *reciprocal* if it is conjugate to α^{-1} (as then P_α is a reciprocal polynomial). One might at first think that it should be possible to prove stronger results on Lehmer's problem if we restrict our attention to reciprocal polynomials. However, this is far from being the case: reciprocal polynomials seem to be the most difficult to work with, perhaps because cyclotomic polynomials are reciprocal; we can prove stronger results on Lehmer's problem if we restrict our attention to nonreciprocal polynomials!

The first result in this direction was due to Breusch [44]. Strangely, this paper was unknown to number theorists until it was recently unearthed by Narkiewicz. Breusch proved that for α a nonreciprocal algebraic integer

$$M(\alpha) \geq M(z^3 - z^2 - \tfrac{1}{4}) = 1.1796 \dots \quad (14)$$

Breusch's argument is based on the study of the resultant of α and α^{-1} , for α a root of P . On the one hand, this resultant must be at least 1 in modulus. But, on the other hand, this is not possible if $M(P)$ is too close to 1, because then all the distances $|\alpha_i - \overline{\alpha_i^{-1}}|$ are too small. (Note that $\alpha_i = \overline{\alpha_i^{-1}}$ implies that P is reciprocal.)

In 1971 Smyth [155] independently improved the constant in (14), showing for α a nonreciprocal algebraic integer

$$M(\alpha) \geq M(z^3 - z - 1) = \theta_0 = 1.3247\dots, \quad (15)$$

the real root of $z^3 - z - 1 = 0$. This constant is best possible here, $z^3 - z - 1$ being nonreciprocal. Equality $M(\alpha) = \theta_0$ occurs only for α conjugate to $(\pm\theta_0)^{\pm 1/k}$ for k some positive integer.² Otherwise in fact $M(\alpha) > \theta_0 + 10^{-4}$ ([156]), so that θ_0 is an isolated point in the spectrum of Mahler measures of nonreciprocal algebraic integers. The lower bound 10^{-4} for this gap in the spectrum was increased to $0.000260\dots$ by Dixon and Dubickas [52, Th. 15]. It would be interesting to know more about this spectrum. All of its known small points come from trinomials, or their irreducible factors:

$$\begin{aligned} 1.324717959\dots &= M(z^3 - z - 1) = M\left(\frac{z^5 - z^4 - 1}{z^2 - z + 1}\right); \\ 1.349716105\dots &= M(z^5 - z^4 + z^2 - z + 1) = M\left(\frac{z^7 + z^2 + 1}{z^2 + z + 1}\right); \\ 1.359914149\dots &= M(z^6 - z^5 + z^3 - z^2 + 1) = M\left(\frac{z^8 + z + 1}{z^2 + z + 1}\right); \\ 1.364199545\dots &= M(z^5 - z^2 + 1); \\ 1.367854634\dots &= M(z^9 - z^8 + z^6 - z^5 + z^3 - z + 1) = M\left(\frac{z^{11} + z^4 + 1}{z^2 + z + 1}\right). \end{aligned}$$

The smallest known limit point of nonreciprocal measures is

$$\lim_{n \rightarrow \infty} M(z^n + z + 1) = 1.38135\dots$$

([31]). The spectrum clearly contains the set of all Pisot numbers, except perhaps the reciprocal ones. But in fact it does contain those too, a result due to Boyd [36, Proposition 2]. There are however smaller limit points of reciprocal measures (see [33], [42]).

The method of proof of (15) was based on the Maclaurin expansion of the rational function $F(z) = P(0)P(z)/z^d P(1/z)$, which has integer coefficients and is nonconstant for P nonreciprocal. This idea had been used in 1944 by Salem [133] in his proof that the set of Pisot numbers is closed, and in the same year by Siegel [147] in his proof that θ_0 is the smallest Pisot number. One can write $F(z)$ as a quotient $f(z)/g(z)$ where f and g are both holomorphic and bounded above by 1 in modulus in the disc $|z| < 1$. Furthermore, $f(0) = g(0) = M(P)^{-1}$. These functions were first studied by Schur [146], who completely specified the conditions on the coefficients of a power series $\sum_{n=0}^{\infty} c_n z^n$ for it to belong to this class. Then study of functions of this type, combined with the fact that the series of their quotient has integer coefficients, enables one to get the required lower bound for $M(P)$. To prove that θ_0 is an isolated point of the nonreciprocal spectrum, it was necessary to consider the quotient $F(z)/F_1(z)$, where $F_1(z) = P_1(0)P_1(z)/z^d P_1(1/z)$. Here P_1 is chosen as the minimal polynomial of some $(\pm\theta_0)^{\pm 1/k}$ so that, if $F(z) = 1 + a_k z^k + \dots$, where $a_k \neq 0$ then also $F_1(z) \equiv 1 + a_k z^k \pmod{z^{k+1}}$.

²As Boyd [36] pointed out, however, this does not preclude the possibility of equality for some *reciprocal* α . But it was proved by Dixon and Dubickas [52, Cor. 14] that this could not happen.

Thus this quotient, assumed nonconstant, had a first nonzero term of higher order, enabling one to show that $M(P) > \theta_0 + 10^{-4}$.

5.2. Nonreciprocal case: generalizations. Soon afterwards Schinzel [137] and then Bazylewicz [15] generalised Smyth's result to polynomials over Kroneckerian fields. (These are fields that are either totally real extensions of the rationals, or totally nonreal quadratic extensions of such fields.) For a further generalisation to polynomials in several variables see [142, Theorem 70]. In these generalisations the optimal constant is obtained. If the field does not contain a primitive cube root of unity ω_3 then the best constant is again θ_0 , while if it does contain ω_3 then the best constant is the maximum modulus of the roots θ of $\theta^2 - \omega_3\theta - 1 = 0$.

Generalisations to algebraic numbers were proved by Notari [116] and Lloyd-Smith [97]. See also Skoruppa's Heights notes [154] and Schinzel [142].

5.3. The case where $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois. In 1999 Amoroso and David [4], as a Corollary of a far more general result concerning heights of points on subvarieties of \mathbb{G}_m^n , solved Lehmer's problem for $\mathbb{Q}(\alpha)/\mathbb{Q}$ a Galois extension: they proved that there is a constant $c > 1$ such that if α is not zero or a root of unity and $\mathbb{Q}(\alpha)$ is Galois of degree d then $M(\alpha) \geq c$.

5.4. Other restricted results of Lehmer strength. Mignotte [109, Cor. 2] proved that if α is an algebraic number of degree d such that there is a prime less than $d \log d$ that is unramified in the field $\mathbb{Q}(\alpha)$ then $M(\alpha) \geq 1.2$.

Mignotte [109, Prop. 5] gave a very short proof, based on an idea of Dobrowolski, of the fact that for an irreducible noncyclotomic polynomial P of length $L = \|P\|_1$ that $M(P) \geq 2^{1/2L}$. For a similar result (where $2^{1/2L}$ is replaced by $1 + 1/(6L)$), see Stewart [159].

In 2004 P. Borwein, Mossinghoff and Hare [28] generalised the argument in [155] to nonreciprocal polynomials P all of whose coefficients are odd, proving that in this case

$$M(P) \geq M(z^2 - z - 1) = \phi.$$

Here $\phi = (1 + \sqrt{5})/2$. This lower bound is clearly best possible. Recently Borwein, Dobrowolski and Mossinghoff have been able to drop the requirement of nonreciprocity: they proved in [27] that for a noncyclotomic irreducible polynomial with all odd coefficients then

$$M(P) \geq 5^{1/4} = 1.495348 \dots \quad (16)$$

In the other direction, in a search [28] of polynomials up to degree 72 with coefficients ± 1 and no cyclotomic factor the smallest Mahler measure found was $M(z^6 + z^5 - z^4 - z^3 - z^2 + z + 1) = 1.556030 \dots$.

Dobrowolski, Lawton and Schinzel [59] first gave a bound for the Mahler measure of an noncyclotomic integer polynomial P in terms of the number k

of its nonzero coefficients:

$$M(P) \geq 1 + \frac{1}{\exp_{k+1} 2k^2}. \quad (17)$$

Here \exp_{k+1} is the $(k+1)$ -fold exponential. This was later improved by Dobrowolski [56] to $1 + \frac{1}{13911} \exp(-2.27k^k)$, and lately [57] to

$$M(P) \geq 1 + \frac{1}{\exp(a3^{\lfloor (k-2)/4 \rfloor} k^2 \log k)}, \quad (18)$$

where $a < 0.785$. Furthermore, in the same paper he proves that if P has no cyclotomic factors then

$$M(P) \geq 1 + \frac{0.31}{k!}. \quad (19)$$

With the additional restriction that P is irreducible, Dobrowolski [55] gave the lower bound

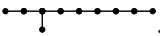
$$M(P) \geq 1 + \frac{\log(2e)}{2e} (k+1)^{-k}. \quad (20)$$

In [57] he strengthened this to

$$M(P) \geq 1 + \frac{0.17}{2^m m!}, \quad (21)$$

where $m = \lceil k/2 \rceil$.

Recently Dobrowolski [58] has proved that for an integer symmetric $n \times n$ matrix A with characteristic polynomial $\chi_A(x)$, the reciprocal polynomial $z^n \chi_A(z + 1/z)$ is either cyclotomic or has Mahler measure at least 1.043. The Mahler measure of A can then be defined to be the Mahler measure of this polynomial. McKee and Smyth [100] have just improved the lower bound in Dobrowolski's result to the best possible value $\tau_0 = 1.176 \dots$ coming from Lehmer's polynomial. The adjacency matrix of the graph below is an example of a matrix where this value is attained.

The Mahler measure of a graph, defined as the Mahler measure of its adjacency matrix, has been studied by McKee and Smyth [99]. They showed that its Mahler measure was either 1 or at least τ_0 , the Mahler measure of the graph . They further found all numbers in the interval $[1, \phi]$ that were Mahler measures of graphs. All but one of these numbers is a Salem number.

6. RESTRICTED RESULTS WHERE $M(\alpha) > C^d$.

6.1. Totally real α . Suppose that α is a totally real algebraic integer of degree d , $\alpha \neq 0$ or ± 1 . Then Schinzel [137] proved that

$$M(\alpha) \geq \phi^{d/2}. \quad (22)$$

A one-page proof of this result was later provided by Höhn and Skoruppa [86]. The result also holds for any nonzero algebraic number α in a Kroneckerian

field, provided $|\alpha| \neq 1$. Amoroso and Dvornicich [10, p. 261] gave the interesting example of $\alpha = \frac{1}{2}\sqrt{3 + \sqrt{-7}}$, not an algebraic integer, where $|\alpha| = 1$, $\mathbb{Q}(\alpha)$ is Kroneckerian, but $M(\alpha) = 2 < \phi^2$.

Smyth [157] studied the spectrum of values $M(\alpha)^{1/d}$ in $(1, \infty)$. He showed that this spectrum was discrete at first, and found its smallest four points. The method used is semi-infinite linear programming (continuous real variables and a finite number of constraints), combined with resultant information. One takes a list of judiciously chosen polynomials $P_i(x)$, and then finds the largest c such that for some $c_i \geq 0$

$$\log_+ |x| \geq c - \sum_i c_i \log |P_i(x)| \quad (23)$$

for all real x . Then, averaging this inequality over the conjugates of α , one gets that $M(\alpha) \geq e^c$, unless some $P_i(\alpha) = 0$.

Two further isolated points were later found by Flammang [80], giving the six points comprising the whole of the spectrum in $(1, 1.3117)$. On the other hand Smyth also showed that this spectrum was dense in (ℓ, ∞) , where $\ell = 1.31427\dots$. The number ℓ is $\lim_{n \rightarrow \infty} M(\alpha_n)$, where $\beta_0 = 1$ and β_n , of degree 2^n , is defined by $\beta_n - \beta_n^{-1} = \beta_{n-1}$ ($n \geq 1$). The limiting distribution of the conjugates of β_n was studied in detail by Davie and Smyth [51]. It is highly irregular: indeed, the Hausdorff dimension of the associated probability measure is $0.800611138269168784\dots$. It is the invariant measure of the map $\mathbb{C} \rightarrow \mathbb{C}$ taking $t \mapsto t - 1/t$, whose Julia set (and thus the support of the measure) is \mathbb{R} .

Bertin [17] pointed out that from a result of Matveev (22) could be strengthened when α was a nonunit.

6.2. Langevin's Theorem. In 1988 Langevin [92] proved the following general result, which included Schinzel's result (22) as a special case (though not with the explicit and best constant given by Schinzel). Suppose that V is an open subset of \mathbb{C} that has nonempty intersection with the unit circle $|z| = 1$, and is stable under complex conjugation. Then there is a constant $C(V) > 1$ such that for every irreducible monic integer polynomial P of degree d having all its roots outside V one has $M(P) > C(V)^d$. The proof is based on the beautiful result of Kakeya to the effect that, for a compact subset of \mathbb{C} stable under complex conjugation and of transfinite diameter less than 1 there is a nonzero polynomial with integer coefficients whose maximum modulus on this set is less than 1. (Kakeya's result is applied to the unit disc with V removed.) For Schinzel's result take $V = \mathbb{C} \setminus \mathbb{R}$, $C(\mathbb{R}) = \phi^{1/2}$, where the value of $C(\mathbb{R})$ given here is best possible. It is of course of interest to find such best possible constants for other sets V .

Stimulated by Langevin's Theorem, Rhin and Smyth [129] studied the case where the subset of \mathbb{C} was the sector $V_\theta = \{z \in \mathbb{C} : |\arg z| > \theta\}$. They found a value $C(V_\theta) > 1$ for $0 \leq \theta \leq 2\pi/3$, including 9 subintervals of this range for

which the constants found were best possible. In particular, the best constant $C(V_{\pi/2})$ was evaluated. This implied that for $P(z)$ irreducible, of degree d , having all its roots with positive real part and not equal to $z - 1$ or $z^2 - z + 1$ we have

$$M(P)^{1/d} \geq M(z^6 - 2z^5 + 4z^4 - 5z^3 + 4z^2 - 2z + 1)^{1/6} = 1.12933793\dots, \quad (24)$$

all roots of $z^6 - 2z^5 + 4z^4 - 5z^3 + 4z^2 - 2z + 1$ having positive real part. Curiously, for some root α of this polynomial, $\alpha + 1/\alpha = \theta_0^2$, where as above θ_0 is the smallest Pisot number.

Recently Rhin and Wu [131] extended these results, so that there are now 13 known subintervals of $[0, \pi]$ where the best constant $C(V_\theta)$ is known. It is of interest to see what happens as θ tends to π ; maybe one could obtain a bound connected to Lehmer's original problem. Mignotte [112] has looked at this, and has shown that for $\theta = \pi - \varepsilon$ the smallest limit point of the set $M(P)^{1/d}$ for P having all its roots outside V_θ is at least $1 + c\varepsilon^3$ for some positive constant c .

Dubickas and Smyth [73] applied Langevin's Theorem to the annulus

$$V(R^{-\gamma}, R) = \{z \in \mathbb{C} \mid R^{-\gamma} < |z| < R\},$$

where $R > 1$ and $\gamma > 0$, proving that the best constant $C(V(R^{-\gamma}, R))$ is $R^{\gamma/(1+\gamma)}$.

6.3. Abelian number fields. In 2000 Amoroso and Dvornicich [10] showed that when α is a nonzero algebraic number, not a root of unity, and $\mathbb{Q}(\alpha)$ is an abelian extension of \mathbb{Q} then $M(\alpha) \geq 5^{d/12}$. They also give an example with $M(\alpha) = 7^{d/12}$. It would be interesting to find the best constant $c > 1$ such that $M(\alpha) \geq c^d$ for these numbers. Baker and Silverman [13], [151], [14] generalised this lower bound first to elliptic curves, and then to abelian varieties of arbitrary dimension.

6.4. Totally p -adic fields. Bombieri and Zannier [25] proved an analogue of Schinzel's result (22) for 'totally p -adic' numbers: that is, for algebraic numbers α of degree d all of whose conjugates lie in \mathbb{Q}_p . They showed that then $M(\alpha) \geq c_p^d$, for some constant $c_p > 1$.

6.5. The heights of Zagier and Zhang and generalisations. Zagier [171] gave a result that can be formulated as proving that the Mahler measure of any irreducible nonconstant polynomial in $\mathbb{Z}[(x-1)]$ has Mahler measure at least $\phi^{d/2}$, apart from $\pm(x(x-1)+1)$. Doche [60, 61] studied the spectrum resulting from the measures of such polynomials, giving a gap to the right of the smallest point $\phi^{1/2}$, and finding a short interval where the smallest limit point lies. He used the semi-infinite linear programming method outlined above. For this problem, however, finding the second point of the spectrum seems to be difficult. Zagier's work was motivated by a far-reaching result of Zhang [173] (see also [169, p. 103]) for curves on a linear torus. He proved

that for all such curves, apart from those of the type $x^i y^j = \omega$, where $i, j \in \mathbb{Z}$ and ω is a root of unity, there is a constant $c > 0$ such that the curve has only finitely many algebraic points (x, y) with $h(x) + h(y) \leq c$. Zagier's result was for the curve $x + y = 1$.

Following on from Zhang, there have been recent deep and diverse generalisations in the area of small points on subvarieties of \mathbb{G}_m^n . In particular see Bombieri and Zannier [24], Schmidt [145] and Amoroso and David [5, 6, 7, 8].

Rhin and Smyth [130] generalised Zagier's result by replacing polynomials in $\mathbb{Z}(x(x-1))$ by polynomials in $\mathbb{Z}[Q(x)]$, where $Q(x) \in \mathbb{Z}[x]$ is not \pm a power of x . Their proof used a very general result of Beukers and Zagier [21] on heights of points on projective hypersurfaces. Noticing that Zagier's result has the same lower bound as Schinzel's result above for totally real α , Samuels [135] has recently shown that the same lower bound holds for a more general height function. His result includes those of both Zagier and Schinzel. The proof is also based on [21].

7. LOWER BOUNDS FOR $|\overline{\alpha}|$

7.1. General lower bounds. We know that any lower bound for $M(\alpha)$ immediately gives a corresponding lower bound for $|\overline{\alpha}|$, using (9). For instance, from [164] it follows that for α of degree $d > 2$ and not a root of unity

$$|\overline{\alpha}| \geq 1 + \frac{1}{4d} \left(\frac{\log \log d}{\log d} \right)^3. \quad (25)$$

Some lower bounds, though asymptotically weaker, are better for small degrees. For example Matveev [105] has shown that for such α

$$|\overline{\alpha}| \geq \exp \frac{\log(d+0.5)}{d^2}, \quad (26)$$

which is better than (25) for $d \leq 1434$ (see [132]). Recently Rhin and Wu have improved (26) for $d \geq 13$ to

$$|\overline{\alpha}| \geq \exp \frac{3 \log(d/2)}{d^2}, \quad (27)$$

which is better than (25) for $d \leq 6380$. See also the paper of Rhin and Wu in this volume.

Matveev [105] also proves that if α is a reciprocal (conjugate to α^{-1}) algebraic integer, not a root of unity, then $|\overline{\alpha}| \geq (p-1)^{1/(pm)}$, where p is the least prime greater than $m = n/2 \geq 3$.

Indeed, Dobrowolski's first result in this area [53] was for $|\overline{\alpha}|$ rather than $M(\alpha)$: he proved that

$$|\overline{\alpha}| > 1 + \frac{\log d}{6d^2}.$$

His argument is a beautifully simple one, based on the use of the power sums $s_k = \sum_{i=1}^d \alpha_i^k$, the Newton identities, and the arithmetic fact that, for any prime p , $s_{kp} \equiv s_k \pmod{p}$.

The strongest asymptotic result to date in the direction of the Schinzel-Zassenhaus conjecture is due to Dubickas [62]: that given $\varepsilon > 0$ there is a constant $d(\varepsilon)$ such that any nonzero algebraic integer α of degree $d > d(\varepsilon)$ not a root of unity satisfies

$$|\bar{\alpha}| > 1 + \left(\frac{64}{\pi^2} - \varepsilon \right) \left(\frac{\log \log d}{\log d} \right)^3 \frac{1}{d}. \quad (28)$$

Cassels [46] proved that if an algebraic number α of degree d has the property $|\bar{\alpha}| \leq 1 + \frac{1}{10d^2}$ then at least one of the conjugates of α has modulus 1. Although this result has been superseded by Dobrowolski's work, Dubickas [66] applied the inequality

$$\prod_{k < j} |z_k \bar{z}_j - 1| \leq n^{n/2} \left(\prod_{m=1}^n \max(1, |z_m|) \right)^{n-1} \quad (29)$$

for complex numbers z_1, \dots, z_n , a variant of one in [46], to prove that

$$M(\alpha)^2 \left| \prod \log |\alpha_i| \right|^{1/d} \geq 1/(2d)$$

for a nonreciprocal algebraic number α of degree d with conjugates α_i .

7.2. The house $|\bar{\alpha}|$ for α nonreciprocal. The Schinzel-Zassenhaus conjecture (8) restricted to nonreciprocal polynomials follows from Breusch's result above, with $c = \log 1.1796 \dots = 0.165 \dots$, using (9). Independently Cassels [46] obtained this result with $c = 0.1$, improved by Schinzel to 0.2 ([136]), and by Smyth [155] to $\log \theta_0 = 0.2811 \dots$. He also showed that c could not exceed $\frac{3}{2} \log \theta_0 = 0.4217 \dots$. In 1985 Lind and Boyd (see [34]), as a result of extensive computation (see Section 8), conjectured that, for degree d , the extremal α are nonreciprocal and have $\sim \frac{2}{3}d$ roots outside the unit circle. What a contrast with Mahler measure, where all small $M(\alpha)$ are reciprocal! This would imply that the best constant c is $\frac{3}{2} \log \theta_0$. In 1997 Dubickas [64] proved that $c > 0.3096$ in this nonreciprocal case.

7.3. The house of totally real α . Suppose that α is a totally real algebraic integer. If $|\bar{\alpha}| \leq 2$ then by [90, Theorem 2] α is of the form $\omega + 1/\omega$, where ω is a root of unity. If for some $\delta > 0$ we have $2 < |\bar{\alpha}| \leq 2 + \delta^2/(1 + \delta)$, then, on defining γ by $\gamma + 1/\gamma = \alpha$, we see that γ and its conjugates are either real or lie on the unit circle, and $1 < |\bar{\gamma}| \leq 1 + \delta$. This fact readily enables us to deduce a lower bound greater than 2 for $|\bar{\alpha}|$ whenever we have a lower bound greater than 1 for $|\bar{\gamma}|$. Thus from (7) [144] it follows that for α not of the form $2 \cos \pi r$ for any $r \in \mathbb{Q}$

$$|\bar{\alpha}| \geq 2 + 4^{-2d-3} \quad (30)$$

[144]. In a similar way (28) above implies that for such α , and $d > d(\varepsilon)$ that

$$|\overline{\alpha}| > 2 + \left(\frac{4096}{\pi^4} - \varepsilon \right) \left(\frac{\log \log d}{\log d} \right)^6 \frac{1}{d^2} \quad (31)$$

[62]. However Dubickas [63] managed to improve this lower bound to

$$|\overline{\alpha}| > 2 + 3.8 \frac{(\log \log d)^3}{d(\log d)^4}. \quad (32)$$

He improved the constant 3.8 to 4.6 in [64].

7.4. The Kronecker constant. Callahan, Newman and Sheingorn [48] define the *Kronecker constant* of a number field K to be the least $\varepsilon > 0$ such that $|\overline{\alpha}| \geq 1 + \varepsilon$ for every algebraic integer $\alpha \in K$. The truth of the Schinzel-Zassenhaus conjecture (8) would imply that the Kronecker constant of K is at least $c/[K : \mathbb{Q}]$. They give [48, Theorem 2] a sufficient condition on K for this to be the case. They also point out, from considering $\alpha\overline{\alpha} - 1$, that if α is a nonzero algebraic integer not a root of unity in a Kroneckerian field then $|\overline{\alpha}| \geq \sqrt{2}$ (See also [111]), so that the Kronecker constant of a Kroneckerian field is at least $\sqrt{2} - 1$.

8. SMALL VALUES OF $M(\alpha)$ AND $|\overline{\alpha}|$

8.1. Small values of $M(\alpha)$. The first recorded computations on Mahler measure were performed by Lehmer in his 1933 paper [94]. He found the smallest values of $M(\alpha)$ for α of degrees 2, 3 and 4, and the smallest $M(\alpha)$ for α reciprocal of degrees 2, 4, 6 and 8. Lehmer records the fact that Poulet (?unpublished) "...has made a similar investigation of symmetric polynomials with practically the same results". Boyd has done extensive computations, searching for 'small' algebraic integers of various kinds. His first major published table was of Salem numbers less than 1.3 [29], with four more found in [30]. Recall that these are positive reciprocal algebraic integers of degree at least 4 having only one conjugate (the number itself) outside the unit circle. These numbers give many examples of small Mahler measures, most notably (from (2)) $M(L) = 1.176\dots$ from the Lehmer polynomial itself, which is the minimal polynomial of a Salem number. In later computations [32], [38], he finds all reciprocal α with $M(\alpha) \leq 1.3$ and degree up to 20, and those with $M(\alpha) \leq 1.3$ and degree up to 32 having coefficients in $\{-1, 0, 1\}$ ('height 1').

Mossinghoff [114] extended Boyd's tables from degree 20 to degree 24 for $M(\alpha) < 1.3$, and to degree 40 for height 1 polynomials, finding four more Salem numbers less than 1.3. He also has a website [115] where up-to-date tables of small Salem numbers and Mahler measures are conveniently displayed (though unfortunately without their provenance). Flammang, Grandcolas and Rhin [82] proved that Boyd's table, with the additions by Mossinghoff, of the 47 known Salem numbers less than 1.3 is complete up to degree 40. Recently Flammang, Rhin and Sac-Épée [83] have extended these tables, finding

all $M(\alpha) < \theta_0$ for α of degree up to 36, and all $M(\alpha) < 1.31$ for α of degree up to 40. This latter computation showed that the earlier tables of Boyd and Mossinghoff for α of degree up to 40 with $M(\alpha) < 1.3$ are complete.

8.2. Small values of $|\overline{\alpha}|$. Concerning $|\overline{\alpha}|$, Boyd [34] gives tables of the smallest values of $|\overline{\alpha}|$ for α of degree d up to 12, and for α reciprocal of degree up to 16. Further computation has recently been done on this problem by Rhin and Wu [132]. They computed the smallest house of algebraic numbers of degree up to 28. All are nonreciprocal, as predicted by Boyd's conjecture (see Section 7.2). Their data led the authors to conjecture that, for a given degree, an algebraic number of that degree with minimal house was a root of a polynomial consisting of at most four monomials.

9. MAHLER MEASURE AND THE DISCRIMINANT

9.1. Mahler [103] showed that for a complex polynomial

$$P(z) = a_0 z^d + \cdots + a_d = a_0(z - \alpha_1) \cdots (z - \alpha_d)$$

its discriminant $\text{disc}(P) = a_0^{2d-2} \prod_{i < j} (\alpha_i - \alpha_j)^2$ satisfies

$$|\text{disc}(P)| \leq d^d M(P)^{2d-2}. \quad (33)$$

From this it follows immediately that if there is an absolute constant $c > 1$ such that $|\text{disc}(P)| \geq (cd)^d$ for all irreducible $P(z) \in \mathbb{Z}[z]$, then $M(P) \geq c^{d/(2d-2)}$, which would solve Lehmer's problem. This consequence of Mahler's inequality has been noticed in various variants by several people, including Mignotte [109] and Bertrand [16].

In 1996 Matveev [106] showed that in Dobrowolski's inequality, the degree $d \geq 2$ of α could be replaced by a much smaller (for large d) quantity

$$\delta = \max(d / \text{disc}(\alpha)^{1/d}, \delta_0(\varepsilon))$$

for those α for which α^p had degree d for all primes p . (Such α do not include any roots of unity.) Specifically, he obtained for given $\varepsilon > 0$

$$M(\alpha) \geq \exp \left((2 - \varepsilon) \left(\frac{\log \log \delta}{\log \delta} \right)^3 \right) \quad (34)$$

for these α .

Mahler [103] also gives the lower bound

$$\delta(P) > \sqrt{3} |\text{disc}(P)|^{1/2} d^{-(d+2)/2} M(P)^{-(m-1)} \quad (35)$$

for the minimum distance $\delta(P) = \min_{i < j} |\alpha_i - \alpha_j|$ between the roots of P .

9.2. Generalisation involving the discriminant of Schinzel's lower bound. Rhin [127] generalised Schinzel's result (22) by proving, for α a totally positive algebraic integer of degree d at least 2 that

$$M(\alpha) \geq \left(\frac{\delta_1 + \sqrt{\delta_1^2 + 4}}{2} \right)^{d/2}. \quad (36)$$

Here $\delta_1 = |\text{disc}(\alpha)|^{1/d(d-1)}$. This result apparently also follows from an earlier result of Zaïmi [172] concerning a lower bound for a weighted product of the moduli of the conjugates of an algebraic integer — see the Math Review of Rhin's paper.

10. PROPERTIES OF $M(\alpha)$ AS AN ALGEBRAIC NUMBER

A *Perron number* is an algebraic integer with exactly one conjugate of maximum modulus. It is clear from (2) that $M(\alpha)$ is a Perron number for any algebraic integer α ; this seems to have been first observed by Adler and Marcus [1] (see [36]). In the other direction: is the Perron number $1 + \sqrt{17}$ a Mahler measure? See Schinzel [143], Dubickas [71]. Dubickas [70] proves that for any Perron number β some integer multiple of β is a Mahler measure. (These papers also contains other interesting properties of the set of Mahler measures.) Boyd [35] proves that if $\beta = M(\alpha)$ for some algebraic integer α , then all conjugates of β other than β itself either lie in the annulus $\beta^{-1} < |z| < \beta$ or are equal to $\pm\beta^{-1}$.

If α were reciprocal, it might be expected that $M(\alpha)$ would be reciprocal too, while if α were nonreciprocal, then $M(\alpha)$ would be nonreciprocal. However neither of these need be the case: in [36, Proposition 6] Boyd exhibits a family of degree 4 Pisot numbers that are the Mahler measures of reciprocal algebraic integers of degree 6, and in [36, Proposition 2] he notes that for $q \geq 3$ a root α_q of the irreducible nonreciprocal polynomial $z^4 - qz^3 + (q+1)z^2 - 2z + 1$ then $M(\alpha_q) = \frac{1}{2}(q + \sqrt{q^2 - 4})$ is reciprocal. In fact, since $M(\frac{1}{2}(q + \sqrt{q^2 - 4})) = \frac{1}{2}(q + \sqrt{q^2 - 4})$, this also shows that a number can be both a reciprocal and a nonreciprocal measure. See also [37]. Dixon and Dubickas [52] prove that the set of all $M(\alpha)$ does not form a semigroup, as for instance $\sqrt{2} + 1$ and $\sqrt{3} + 2$ are Mahler measures, while their product is not. (In terms of polynomials, this set is of course equal to the set of all $M(P)$ for P irreducible. If instead we take the set of all (reducible and irreducible) polynomials, then, because of $M(PQ) = M(P)M(Q)$ this larger set *does* form a semigroup.)

In [69] Dubickas proves that the additive group generated by all Mahler measures is the group of all real algebraic numbers, while the multiplicative group generated by all Mahler measures is the group of all positive real algebraic numbers.

We know that $M(P(z)) = M(P(\pm z^k))$ for either choice of sign, and any $k \in \mathbb{N}$. Is this the only way that Mahler measures of irreducible polynomials can be equal? Boyd [32] gives some illuminating examples to show that there can be other reasons that make this happen. The examples were discovered during his computation of reciprocal polynomials of small Mahler measure (see Section 8). For example, for $P_6 = z^6 + 2z^5 + 2z^4 + z^3 + 2z^2 + 2z + 1$ and $P_8 = z^8 + z^7 - z^6 - z^5 + z^4 - z^3 - z^2 + z + 1$ we have

$$M(P_6) = M(P_8) = 1.746793\dots = M,$$

say, where both polynomials are irreducible. Boyd explains how such examples arise. If $\alpha_i (i = 1, \dots, 8)$ are the roots of P_8 , then for different i $M(\alpha_1 \alpha_i)$ can equal M , M^2 or M^3 . The roots of P_6 are the three $\alpha_1 \alpha_i$ with $M(\alpha_1 \alpha_i) = M$ and their reciprocals. Clearly $M(\alpha_1^2) = M^2$, while for three other α_i the product $\alpha_1 \alpha_i$ is of degree 12 and has $M(\alpha_1 \alpha_i) = M^3$. (P_8 has the special property that it has roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\alpha_1 \alpha_2 = \alpha_3 \alpha_4 \neq 1$.)

Dubickas [67] gives a lower bound for the distance of an algebraic number γ of degree n and leading coefficient c , not a Mahler measure, from a Mahler measure $M(\alpha)$ of degree D :

$$|M(\alpha) - \gamma| > c^{-D} (2|\overline{\gamma}|)^{-nD}. \quad (37)$$

11. COUNTING POLYNOMIALS WITH GIVEN MAHLER MEASURE

Let $\#(d, T)$ denote the number of integer polynomials of degree d and Mahler measure at most T . This function has been studied by several authors. Boyd and Montgomery [41] give the asymptotic formula

$$c(\log d)^{-1/2} d^{-1} \exp\left(\frac{1}{\pi} \sqrt{105\zeta(3)d}\right) (1 + o(1)), \quad (38)$$

where $c = \frac{1}{4\pi^2} \sqrt{105\zeta(3)e^{-\gamma}}$, for the number $\#(d, 1)$ of cyclotomic polynomials of degree d , as $d \rightarrow \infty$.

Dubickas and Konyagin [72] obtain by simple arguments the lower bound $\#(d, T) > \frac{1}{2} T^{d+1} (d+1)^{-(d+1)/2}$, and upper bound $\#(d, T) < T^{d+1} \exp(d^2/2)$, the latter being valid for d sufficiently large. For $T \geq \theta_0$ they derived the upper bound $\#(d, T) < T^{d(1+16 \log \log d / \log d)}$. Chern and Vaaler [50] obtained the asymptotic formula $V_{d+1} T^{d+1} + O_d(T^d)$ for $\#(d, T)$ for fixed d , as $T \rightarrow \infty$. Here V_{d+1} is an explicit constant (the volume of a certain star body). Recently Sinclair [153] has produced corresponding estimates for counting functions of reciprocal polynomials.

12. A DYNAMICAL LEHMER'S PROBLEM

Given a rational map $f(\alpha)$ of degree $d \geq 2$ defined over a number field K , one can define for α in some extension field of K a canonical height

$$h_f(\alpha) = \lim_{n \rightarrow \infty} d^{-n} h(f^n(\alpha)),$$

where f^n is the n th iterate of f , and h is, as before, the Weil height of α . Then $h_f(\alpha) = 0$ if and only if the iterates $f^n(\alpha)$ form a finite set, and an analogue of Lehmer's problem would be to decide whether or not

$$h_f(\alpha) \geq \frac{C}{\deg(\alpha)}$$

for some constant C depending only on f and K . Taking $f(\alpha) = \alpha^d$ we retrieve the Weil height and the original Lehmer problem. There seem to be no good estimates, not even of polynomial decay, for any f not associated to an endomorphism of an algebraic group. See [152, Section 3.4] for more details.

13. VARIANTS OF MAHLER MEASURE

Everest and ní Fhlathúin [77] and Everest and Pinner [78] (see also [79, Chapter 6]) have defined the *elliptic Mahler measure*, based on a given elliptic curve $E = \mathbb{C}/L$ over \mathbb{C} , where $L = \langle \omega_1, \omega_2 \rangle \subset \mathbb{C}$ is a lattice, with \wp_L its associated Weierstrass \wp -function. Then for $F \in \mathbb{C}[z]$ the (logarithmic) elliptic Mahler measure $m_E(F)$ is defined as

$$\int_0^1 \int_0^1 \log |F(\wp_L(t_1\omega_1 + t_2\omega_2))| dt_1 dt_2. \quad (39)$$

If E is in fact defined over \mathbb{Q} and has a rational point Q with x -coordinate M/N then often $m_E(Nz - M) = 2\hat{h}(Q)$, showing that m_E is connected with the canonical height on E .

Kurokawa [91] and Oyanagi [118] have defined a q -analogue of Mahler measure, for a real parameter q . As $q \rightarrow 1$ the classical Mahler measure is recovered.

Dubickas and Smyth [74] defined the *metric Mahler measure* $\mathcal{M}(\alpha)$ as the infimum of $\prod_i M(\beta_i)$, where $\prod_i \beta_i = \alpha$. They used this to define a metric on the group of nonzero algebraic numbers modulo torsion points, the metric giving the discrete topology on this group if and only if Lehmer's 'conjecture' is true (i.e., $\inf_{\alpha: M(\alpha) > 1} M(\alpha) > 1$).

Very recently Pritsker [122, 123] has studied an areal analogue of Mahler measure, defined by replacing the normalised arclength measure on the unit circle by the normalised area measure on the unit disc.

14. APPLICATIONS

14.1. Polynomial factorization. I first met Andrzej Schinzel at the ICM in Nice in 1970. There he mentioned to me an application of Mahler measure to irreducibility of polynomials. (After this we had some correspondence about the work leading to [155], which was very helpful to me.) If a class of irreducible polynomials had Mahler measure at least B , then any polynomial of Mahler measure less than B^2 can have at most one factor from that class. For instance, a trinomial $z^d \pm z^m \pm 1$ has, by Vicente Gonçalves' inequality

[162], [117], [124, Th. 9.1.1] $M(P)^2 + M(P)^{-2} \leq \|P\|_2^2$, Mahler measure at most ϕ . Since $\phi < \theta_0^2$, by (15) such trinomials can have at most one irreducible noncyclotomic factor. Here $\|P\|_2$ is the 2-norm of P (the square root of the sum of the squares of its coefficients).

More generally Schinzel (see [54]) pointed out the following consequence of (11): that for any fixed $\varepsilon > 0$ and polynomial P of degree d with integer coefficients, the number of its noncyclotomic irreducible factors counted with multiplicities is $O(d^\varepsilon \|P\|_2^{1-\varepsilon})$. See also [138], [140], [121].

14.2. Ergodic theory. One-variable Mahler measures have applications in ergodic theory. Consider an automorphism of the torus $\mathbb{R}^d/\mathbb{Z}^d$ defined by a $d \times d$ integer matrix of determinant ± 1 , with characteristic polynomial $P(z)$. Then the topological entropy of this map is $\log M(P)$ (Lind [96] — see also [33], [79, Theorem 2.6]).

14.3. Transcendence and diophantine approximation. Mahler measure, or rather the Weil height $h(\alpha) = \log M(\alpha)/d$, plays an important technical rôle in modern transcendence theory, in particular for bounding the coefficients of a linear form in logarithms known to be dependent.

As remarked by Waldschmidt [169, p65], the fact that this height has three equivalent representations, coming from (1), (2) and (4) makes it a very versatile height function for these applications.

If $\alpha_1, \dots, \alpha_n$ are algebraic numbers such that their logarithms are \mathbb{Q} -linearly dependent, then it is of importance in Baker's transcendence method to get small upper estimates for the size of integers m_1, \dots, m_n needed so that $m_1 \log \alpha_1 + \dots + m_n \log \alpha_n = 0$. Such estimates can be given using Weil heights of the α_i . See [169, Lemma 7.19] and the remark after it.

Chapter 3 ('Heights of Algebraic Numbers') of [169] contains a wealth of interesting material on the Weil height and other height functions, connections between them, and applications. For instance, for a polynomial $f \in \mathbb{Z}[z]$ of degree at most N for which the algebraic number α is not a root one has

$$|f(\alpha)| \geq \frac{1}{M(\alpha)^N \|f\|_1^{d-1}},$$

where $\|f\|_1$ is the length of f , the sum of the absolute values of its coefficients, and $d = \deg \alpha$ ([169, p83]).

In particular, for a rational number $p/q \neq \alpha$ with $q > 0$, and $f(x) = qx - p$ we obtain

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{1}{M(\alpha)q(\max(|p| + q))^{d-1}}. \quad (40)$$

14.4. Distance of α from 1. From (40) we immediately get for $\alpha \neq 1$

$$|\alpha - 1| \geq \frac{1}{2^{d-1}M(\alpha)}. \quad (41)$$

Better lower bounds for $|\alpha - 1|$ in terms of its Mahler measure have been given by Mignotte [110], Mignotte and Waldschmidt [113], Bugeaud, Mignotte and Normandin [45], Amoroso [2], Dubickas [63], and [65]. For instance Mignotte and Waldschmidt prove that

$$|\alpha - 1| > \exp\{-(1 + \varepsilon)(d(\log d)(\log M(\alpha)))^{1/2}\} \quad (42)$$

for $\varepsilon > 0$ and α of degree $d \geq d(\varepsilon)$. Dubickas [63] improves the constant 1 in this result to $\pi/4$, and in the other direction [65] proves that for given $\varepsilon > 0$ there is an infinite sequences of degrees d for which an α of degree d satisfies

$$|\alpha - 1| < \exp \left\{ -(c - \varepsilon) \left(\frac{d \log M(\alpha)}{\log d} \right)^{1/2} \right\}. \quad (43)$$

Here Dubickas uses the following simple result: if $F \in \mathbb{C}[z]$ has degree t and $F'(1) \neq 0$ then there is a root a of F such that $|a - 1| \leq t|F(1)/F'(1)|$.

14.5. Shortest unit lattice vector. Let K be a number field with unit lattice of rank r , and $M = \min M(\alpha)$, the minimum being taken over all units $\alpha \in K$, α not a root of unity. Kessler [89] showed that then the shortest vector λ in the unit lattice has length $\|\lambda\|_2$ at least $\sqrt{\frac{2}{r+1}} \log M$.

14.6. Knot theory. Mahler measure of one-variable polynomials arises in knot theory in connection with Alexander polynomials of knots and reduced Alexander polynomials of links — see Silver and Williams [148]. Indeed, in Reidemeister's classic book on the subject [126], the polynomial $L(-z)$ appears as the Alexander polynomial of the $(-2, 3, 7)$ -pretzel knot. Hironaka [85] has shown that among a wide class of Alexander polynomials of pretzel links, this one has the smallest Mahler measure. Champanerkar and Kofman [47] study a sequence of Mahler measures of Jones polynomials of hyperbolic links L_m obtained using $(-1/m)$ -Dehn surgery, starting with a fixed link. They show that it converges to the Mahler measure of a 2-variable polynomial. (The many more applications of Mahler measures of several-variable polynomials to geometry and topology are outside the scope of this survey.)

15. FINAL REMARKS

15.1. Other sources on Mahler measure. Books covering various aspects of Mahler measure include the following: Bertin and Pathiaux-Delefosse [19], Bertin *et al* [20], Bombieri and Gubler [23], Borwein [26], Schinzel [139], Schinzel [142], Waldschmidt [169].

Survey articles and lecture notes on Mahler measure include: Boyd [31], Boyd [33], Everest [76], Hunter [87], Schinzel [141], Skoruppa [154], Stewart [159], Vaaler [160], Waldschmidt [167].

15.2. Memories of Mahler. As one of a small group of undergraduates in ANU, Canberra in the mid-1960s, we were encouraged to attend graduate courses at the university's Institute of Advanced Studies, where Mahler had a research chair. I well remember his lectures on transcendence with his blackboard copperplate handwriting, all the technical details being carefully spelt out.

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