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A-STABILITY AND STOCHASTIC MEAN-SQUARE STABILITY*

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Abstract.

This note extends and interprets a result of Saito and Mitsui [SIAM J. Numer. Anal., 33 (1996), pp. 2254–2267] for a method of Milstein. The result concerns mean-square stability on a stochastic differential equation test problem with multiplicative noise. The numerical method reduces to the Theta Method on deterministic problems. Saito and Mitsui showed that the deterministic A-stability property of the Theta Method does not carry through to the mean-square context in general, and gave a condition under which unconditional stability holds. The main purpose of this note is to emphasize that the approach of Saito and Mitsui makes it possible to quantify precisely the point where unconditional stability is lost in terms of the ratio of the drift (deterministic) and diffusion (stochastic) coefficients. This leads to a concept akin to deterministic A(α)-stability that may be useful in the stability analysis of more general methods. It is also shown that mean-square A-stability is recovered if the Theta Method parameter is increased beyond its normal range to the value 3/2.

AMS subject classification: 60H10, 65C20, 65U05, 65L20.

Key words: Mean-square, Milstein method, multiplicative noise, Theta method.

1 Background.

Consider an autonomous scalar Itô stochastic differential equation

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + g(X(t))\mathrm{d}W(t), \quad t > 0, \quad X(0) = X_0, \tag{1.1}$$

driven by the standard Wiener process $W(t)$ [5, 7]. The semi-implicit Milstein scheme [9] for computing approximations $X_n \approx X(t_n)$, with $t_n = n\Delta t$, takes the form

$$\begin{aligned} X_{n+1} = X_n + (1-\theta)\Delta t f(X_n) + \theta \Delta t f(X_{n+1}) + \Delta t^{\frac{1}{2}} g(X_n) V_n \\ + \tfrac{1}{2}\Delta t g'(X_n) g(X_n)(V_n^2 - 1), \end{aligned} \tag{1.2}$$

where each V_n is an independent Normal$(0,1)$ random variable. Here θ is a free parameter and $\Delta t > 0$ is the stepsize. We note that in the deterministic case

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$g \equiv 0$, (1.2) becomes the Theta Method [2, 6]. It is usual to apply the Theta Method with $0 \leq \theta \leq 1$, but we will show that allowing $\theta > 1$ in (1.2) offers benefits in terms of stability.

To study the stability properties of the method (1.2) we introduce the test equation where $f(X(t)) \equiv \lambda X(t)$ and $g(X(t)) \equiv \mu X(t)$ in (1.1), so that

$$\mathrm{d}X(t) = \lambda X(t)\mathrm{d}t + \mu X(t)\mathrm{d}W(t), \quad t > 0, \quad X(0) = X_0. \tag{1.3}$$

Here, $\lambda, \mu \in \mathbb{C}$ are constants and we assume that $X_0 \neq 0$ with probability 1. The zero solution to (1.3) is said to be *mean-square stable* if $\lim_{t\to\infty} \mathsf{E}(|X(t)|^2) = 0$ [1, 7], where $\mathsf{E}(\cdot)$ denotes the expected value. It is known [4, 9] that mean-square stability for (1.3) is equivalent to

$$\Re\{\lambda\} + \tfrac{1}{2}|\mu|^2 < 0. \tag{1.4}$$

Applying the method (1.2) to the test problem (1.3) produces the recurrence

$$X_{n+1} = (p + qV_n + rV_n^2)X_n, \tag{1.5}$$

where

$$p := \frac{1 + (1-\theta)\Delta t\lambda - \frac{1}{2}\Delta t\mu^2}{1 - \theta\Delta t\lambda}, \quad q := \frac{\Delta t^{\frac{1}{2}}\mu}{1 - \theta\Delta t\lambda}, \quad r := \frac{\frac{1}{2}\Delta t\mu^2}{1 - \theta\Delta t\lambda} \tag{1.6}$$

and we have assumed that $1 - \theta\Delta t\lambda \neq 0$. Mimicking the definition for the continuous problem (1.3), we say that the method is *mean-square stable* for a particular λ, μ and Δt if $\lim_{n\to\infty} \mathsf{E}(|X_n|^2) = 0$. We may now follow standard numerical analysis practice and compare the stability properties of the test problem and the numerical method.

Note that setting $\mu = 0$ reduces (1.3) to the classical deterministic test problem. In this case, the stability condition (1.4) becomes $\Re\{\lambda\} < 0$. We recall that a numerical method is then said to be *A-stable* [2, 6] if

$$\text{problem stable} \quad \Rightarrow \quad \text{method stable } \forall \Delta t. \tag{1.7}$$

The Theta Method is known to be A-stable for $\theta \geq \frac{1}{2}$ [2, 6]. To accommodate some useful deterministic methods that have good stability properties but are not A-stable, the concept of A(α)-stability is also used. A method is said to be *A(α)-stable* [2, 6] if, for some $0 < \alpha < \pi/2$,

$$\text{problem stable and} \quad \left|\frac{\Im\{\lambda\}}{\Re\{\lambda\}}\right| < \tan(\alpha) \quad \Rightarrow \quad \text{method stable } \forall \Delta t. \tag{1.8}$$

Our aim here is to investigate the mean-square stability of the method (1.2) on (1.3). The work is inspired by [9], which studies mean-square stability for a number of methods. In [9], Saito and Mitsui derive a condition that characterizes mean-square stability of (1.2) and they give a constraint on λ, μ and θ under which unconditional stability holds. Our contributions are (a) to analyze fully the general case $\lambda, \mu \in \mathbb{C}$, (b) to point out a natural A(α)-stability style interpretation of the stability properties, (c) to show that the mean-square extension of A-stability is achieved for $\theta \geq \frac{3}{2}$, and (d) to show that with an appropriate choice of variables the connection with A(α)-stability is particularly striking when $\lambda, \mu \in \mathbb{R}$. We hope that this interpretation of stability properties will prove useful in the analysis of more general methods.

We mention that removing the last term in (1.2) lowers the strong order of the method and produces the semi-implicit Euler scheme, or Stochastic Theta Method [4, 9]. This method was analyzed in [4], and was shown to inherit the mean-square extension of deterministic A-stability for all $\frac{1}{2} \leq \theta \leq 1$.

We also note that some authors have studied numerical methods applied to the linear test equation with *additive* noise; that is, where the term $\mu X(t)\mathrm{d}W(t)$ in (1.3) is replaced by $\mu \mathrm{d}W(t)$, see, for example, [1, 3, 5]. In this case, the stochastic term has less influence than for the multiplicative noise problem (1.3), and the stability properties of the underlying deterministic method tend to dominate. See [8] for a comprehensive list of references on stochastic stability and a discussion of the relevance of the multiplicative noise problem.

2 Mean square stability analysis.

Taking the expected value of the modulus squared in (1.5), using $\mathsf{E}(V_n) = 0$, $\mathsf{E}(V_n^2) = 1$, $\mathsf{E}(V_n^3) = 0$ and $\mathsf{E}(V_n^4) = 3$, we find that

$$\mathsf{E}(|X_{n+1}|^2) = \left(|p+q|^2 + |q|^2 + 2|r|^2\right) \mathsf{E}(|X_n^2|).$$

We deduce immediately that mean-square stability is equivalent to

$$|p+q|^2 + |q|^2 + 2|r|^2 < 1, \tag{2.1}$$

which agrees with the condition obtained in [9, see (21) and (30)]. Using (1.6) we may re-write (2.1) in terms of the problem parameters λ and μ, the method parameter θ, and the stepsize Δt. After some manipulation this leads to

$$\Re\{\lambda\} + \tfrac{1}{2}|\mu|^2 + \tfrac{1}{2}\Delta t\{(1-2\theta)|\lambda|^2 + \tfrac{1}{2}|\mu|^4\} < 0. \tag{2.2}$$

We note from (1.4) that the first two terms on the left-hand side of (2.2) govern the mean-square stability of the test problem. Hence, the size and sign of the third term, $\frac{1}{2}\Delta t\{(1-2\theta)|\lambda|^2 + \frac{1}{2}|\mu|^4\}$, determines whether the stability of the method matches that of the test problem. The following result can be obtained directly from (1.4) and (2.2).

THEOREM 2.1. *Consider the method* (1.2) *applied to the test problem* (1.3) *and let*

$$\Delta t_{\mathrm{S}} := 2\left|\frac{\Re\{\lambda\} + \frac{1}{2}|\mu|^2}{\frac{1}{2}|\mu|^4 + (1-2\theta)|\lambda|^2}\right|. \tag{2.3}$$

For $0 \leq \theta \leq \frac{1}{2}$,

$$\textit{problem unstable} \Rightarrow \textit{method unstable } \forall \Delta t, \tag{2.4}$$

$$\textit{problem stable} \Rightarrow \textit{method stable for } \Delta t < \Delta t_{\mathrm{S}}. \tag{2.5}$$

For $\theta > \frac{1}{2}$, *if*

$$|\lambda|^2 < \frac{|\mu|^4}{2(2\theta - 1)}$$

(diffusion term dominates) then implications (2.4) *and* (2.5) *are valid, and if*

$$|\lambda|^2 \geq \frac{|\mu|^4}{2(2\theta - 1)}$$

(drift term dominates) then

$$\text{problem unstable} \Rightarrow \text{method unstable for } \Delta t < \Delta t_{\mathrm{S}}, \tag{2.6}$$

$$\text{problem stable} \Rightarrow \text{method stable } \forall \Delta t. \tag{2.7}$$

In the cases where the denominator vanishes in (2.3)*, we interpret the condition* $\Delta t < \Delta t_{\mathrm{S}}$ *as meaning* $\forall \Delta t$.

We remark that the implication (2.7) in Theorem 2.1 coincides with Lemma 5.1 (b) of [9] in the case $\lambda, \mu \in \mathbb{R}$.

The implications (2.4) and (2.5) are of the form often encountered in the analysis of explicit methods for deterministic problems—the stability region for the method is strictly contained in that for the problem. Theorem 2.1 shows that this behavior extends to the method (1.2) with $\theta \geq \frac{1}{2}$ when the diffusion term dominates. However, Theorem 2.1 also shows that if the drift term dominates then the unconditional stability condition (2.7) holds. This is analogous to deterministic A(α)-stability, with $|\Im\{\lambda\}/\Re\{\lambda\}| < \tan(\alpha)$ replaced by

$$|\mu|^2/|\lambda| \leq \sqrt{2(2\theta - 1)}.$$

The following corollary shows that for $\theta \geq \frac{3}{2}$, if the problem is stable then the drift term automatically dominates, giving unconditional stability. Note that the price to be paid for this property is the loss of positivity in the Theta Method weights.

Corollary 2.2. *Consider the method* (1.2) *applied to the test problem* (1.3)*. For* $\theta \geq \frac{3}{2}$,

$$\text{problem stable} \Rightarrow \text{method stable } \forall \Delta t.$$

Proof. The condition (1.4) implies that

$$|\lambda|^2 > \frac{|\mu|^4}{4} \geq \frac{|\mu|^4}{2(2\theta - 1)}$$

for $\theta \geq 3/2$. Hence, (2.7) in Theorem 2.1 applies. □

3 Real coefficients.

When $\lambda, \mu \in \mathbb{R}$ it is possible to plot regions of stability. Saito and Mitsui [9] gave plots in the x-y plane with $x := -\mu^2/\lambda$ and $y := \Delta t\lambda$. Following [4], we find it natural to use $x := \Delta t\lambda$ and $y := \Delta t\mu^2$. In this case, the conditions (1.4) and (2.2) for mean-square stability of the problem and method become, respectively,

$$x + \tfrac{1}{2}y < 0 \tag{3.1}$$

and

$$2(x + \tfrac{1}{2}y) + \tfrac{1}{2}y^2 + (1 - 2\theta)x^2 < 0. \tag{3.2}$$

We let $R_{\mathrm{MS}}(\theta) := \{x, y \in \mathbb{R} : y \geq 0 \text{ and (3.2) holds}\}$ denote the mean-square stability region of the method (1.2). The region $R_{\mathrm{MS}}(\theta)$ is enclosed in the upper part of an ellipse for $0 \leq \theta < \frac{1}{2}$. At $\theta = \frac{1}{2}$, $R_{\mathrm{MS}}(\theta)$ lies inside the parabola $x = -\frac{1}{4}y^2 - \frac{1}{2}y$. For $\frac{1}{2} < \theta < \frac{3}{2}$ the upper boundary of $R_{\mathrm{MS}}(\theta)$ is formed by two separate branches of a hyperbola. At $\theta = \frac{3}{2}$, $R_{\mathrm{MS}}(\theta)$ degenerates to the wedge (3.1) in the left half plane and the wedge $x - \frac{1}{2}y > 1$ in the right half plane. For $\theta > \frac{3}{2}$, $R_{\mathrm{MS}}(\theta)$ is enclosed by a single hyperbolic branch.

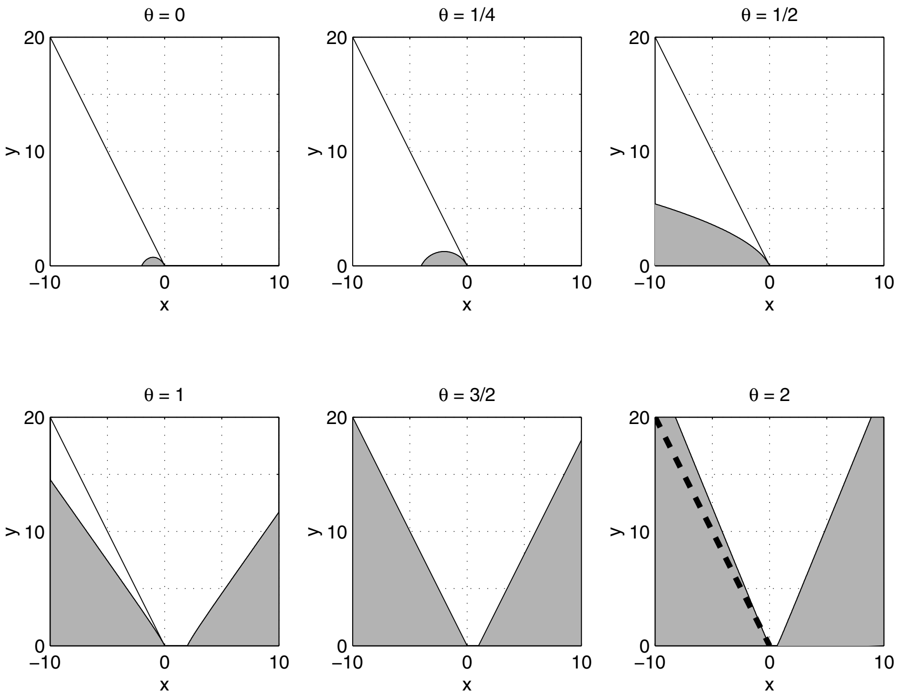

Figure 3.1: Real mean-square stability region for test problem (light) and method (dark). Here, $x := \Delta t\lambda$ and $y := \Delta t\mu^2$.

Figure 3.1 illustrates how $R_{\rm MS}(\theta)$ varies with θ. The light shading marks the region (3.1) where the test problem is stable and the dark shading superimposes $R_{\rm MS}(\theta)$ for $\theta = 0, \frac{1}{4}, \frac{1}{2}, 1, \frac{3}{2}, 2$. (In the case $\theta = 2$, the upper boundary of (3.1) is marked with a dashed line.) Given parameters λ and μ, the test problem is stable if (λ, μ^2) lies in the light region, and the method is stable if $(\Delta t\lambda, \Delta t\mu^2)$ lies in the dark region. The set inclusions implied by Theorem 2.1 and Corollary 2.2 can be clearly seen.

We finish by pointing out two connections between the real mean-square stability region $R_{\rm MS}(\theta)$, where $x = \Delta t\lambda \in \mathbb{R}, y = \Delta t\mu^2 \in \mathbb{R}$, and the corresponding complex deterministic stability region, where $\lambda \in \mathbb{C}$ and $x = \Re\{\Delta t\lambda\}, y = \Im\{\Delta t\lambda\}$ [2, 6]. First, we note that the intersection of $R_{\rm MS}(\theta)$ with the x-axis corresponds to the deterministic real stability interval (since $\mu = 0$ there). Hence, this intersection gives a finite portion of the negative x-axis for $0 \le \theta < \frac{1}{2}$ and contains the negative x-axis for $\theta \ge \frac{1}{2}$. Second, for $\theta > 1/2$, we see from Theorem 2.1, or from [9, Lemma 5.1], that the method is unconditionally stable for all points (x, y) in the left half plane inside the wedge

$$|y/x| < \tan(\alpha), \qquad \alpha = \arctan\sqrt{2(2\theta - 1)}.$$

This is analogous to deterministic A(α)-stability (1.8) with the x and y axes representing the real drift and diffusion coefficients, rather than the real and imaginary parts of the drift. For $\theta \ge \frac{3}{2}$, the wedge $|y/x| < \tan(\alpha)$ in the left half plane contains the wedge $|y/x| < 2$, confirming the mean-square A-stability property established in Corollary 2.2.

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