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NONNORMALITY AND STOCHASTIC DIFFERENTIAL EQUATIONS*

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In memory of Germund Dahlquist (1925–2005).

Abstract.

A highly nonnormal Jacobian may give rise to large transients. This behaviour has been shown to have implications for (a) the relevance of linearising a nonlinear system and (b) the timestep restrictions required to keep a numerical method stable. Here, we show that nonnormality also manifests itself for stochastic differential equations. We give an example of a family of systems that is stable without noise, but can be made exponentially unstable in mean-square by a noise perturbation that shrinks to zero as the nonnormality increases. We then show via finite-time convergence theory that an Euler approximation shares the same property, giving a discrete analogue of the result.

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1 Introduction.

Constant coefficient linear ordinary differential equation (ODE) systems play a key role in our understanding of numerical methods [1, 2, 3]. Properties such as the norm, the logarithmic norm and the spectrum of the Jacobian have been used as the basis for analysis. More recently, pseudospectra have proved useful for characterising a range of features. We refer to [7] for a comprehensive coverage of pseudospectra and nonnormality, and their relevance across a diverse range of topics.

An important message from the pseudospectra/nonnormality viewpoint is that if a linear system is (a) stable in the long term, but (b) highly nonnormal, then the nonnormality may manifest itself through significant transient growth. Such

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behaviour has direct consequences: for example, (i) the neighbourhood inside which a nonlinear system behaves like its linearisation may be tiny, and (ii) when an adaptive time-stepping method based on error control is applied, it may forever choose "small" timesteps in order to guard against the possible large transients.

In this work we give what we believe to be the first study of the impact of *noise* on a highly nonnormal system. We are motivated by the following premise.

> An ODE system that is stable yet highly nonnormal may exhibit large transient growth. In the same way that such a system can become unstable when a nonlinear term is added, it should also be possible to de-stabilise by adding a small amount of noise.

We illustrate this effect analytically with an explicit example.

Section 2 introduces the example and gives the analysis. In Section 3 we show a discrete analogue by drawing upon results from numerical analysis. Section 4 concludes by mentioning possibilities for developing a more general theory of highly nonnormal systems that can be de-stabilised by noise.

2 Nonnormality result.

We begin with the parametrised linear system of two ODEs

$$\frac{dx(t)}{dt} = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^2 \text{ given,} \tag{2.1}$$

where

$$A := \begin{bmatrix} -1 & b \\ 0 & -1 \end{bmatrix}. \tag{2.2}$$

Here $b > 0$ is a parameter that determines the departure from normality of the Jacobian; increasing b makes A more nonnormal. For all b, the matrix A has eigenvalues of -1, and so it follows that all solutions eventually decay like e^{-t}. For later comparison we may state this formally as follows. Given any $b > 0$, there exists a constant $C = C(b)$ such that

$$\| x(t) \|_2^2 \le C(1 + t^2) \| x_0 \|_2^2 \, e^{-2t}, \quad \text{for all } t > 0, \tag{2.3}$$

and hence if $\| x_0 \|_2 \neq 0$ we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \| x(t) \|_2^2 \le -2. \tag{2.4}$$

Figure 2.1 illustrates the vector field for the system (2.1). We have superimposed a particular solution, with initial data $(-1.5, 0.6)$ indicated by a circle. Here we used $b = 10$. There is a single, stable, manifold along the x_1-axis. So all solutions approach the origin in a horizontal direction. However, solution curves

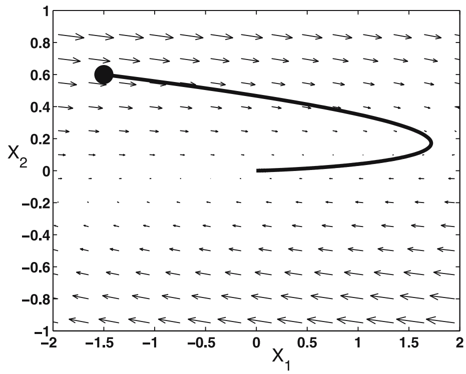

Figure 2.1: Vector field for the ODE (2.1).

can be swept far away in the x_1 direction before they eventually succumb to the decay.

It is trivial to check that perturbing the $(2,1)$ element of A to ϵ changes the eigenvalues to $-1 \pm \sqrt{b\epsilon}$. Hence, a perturbation of size $\epsilon > 1/b$ creates an unstable system.

We now consider a stochastic perturbation of (2.1). More precisely, we add a multiplicative noise term to give an Ito SDE system of the form

$$dx(t) = Ax(t)dt + Gx(t)dw(t), \quad x(0) = x_0 \text{ given}, \ \mathbb{E}\left[\| x_0 \|_2^2\right] < \infty, \tag{2.5}$$

where A is defined in (2.2) and

$$G := \begin{bmatrix} 0 & \sigma \\ -\sigma & 0 \end{bmatrix}, \quad \text{with } \sigma = b^{-\frac{1}{4}}. \tag{2.6}$$

Here, $w(t)$ denotes a scalar Brownian motion [5, 6].

Our choice of G in (2.6) was motivated partly by the fact that this skew-symmetric structure has been observed to de-stabilise other types of SDE system [6, Section 4.5]. Moreover, the noise perturbation can be interpreted as a rotation, and from the vector field diagram in Figure 2.1 it is intuitively reasonable that the system might be sensitive to small, random rotations.

At this stage, we make two remarks. First, our model (2.5) has connections with the simple fluid model proposed by Trefethen et al. in [8, p. 582]. In that case, a nonlinearity was added to a system of two ODEs. Here, we are using a noise perturbation. Second, the literature on the stabilising/de-stabilising effects of noise tends to focus on $O(1)$ or "sufficiently large" noise. Our work differs in that the noise term becomes arbitrarily small in the limit $b \to \infty$ of interest.

To analyse the SDE we will use the Lyapunov function

$$v(x(t)) := x(t)^T Q x(t), \tag{2.7}$$

where

$$Q := \begin{bmatrix} \frac{1}{2}b^{-2} & \frac{1}{4}b^{-1} \\ \frac{1}{4}b^{-1} & \frac{1}{4} + \frac{1}{2}b^{-4} \end{bmatrix}. \tag{2.8}$$

The next lemma confirms that Q is positive definite.

LEMMA 2.1. *For all sufficiently large b, the matrix Q in* (2.8) *is positive definite. Moreover, Q has eigenvalues*

$$\lambda_Q^{\min} = \tfrac{1}{4}b^{-2} + O(b^{-4}) \quad \text{and} \quad \lambda_Q^{\max} = \tfrac{1}{4} + O(b^{-2}), \quad \text{as } b \to \infty. \tag{2.9}$$

PROOF. The result can be derived by explicitly computing the eigenvalues of Q and expanding for large b. ◻

Because Q is positive definite, $v(\cdot)$ is a valid Lyapunov function. From Ito's Lemma, $v(x(t))$ satisfies the SDE

$$dv(t) = x(t)^T M x(t)\, dt + x(t)^T N x(t)\, dw(t), \tag{2.10}$$

where

$$\begin{aligned} M &:= QA + A^T Q + G^T QG, \\ N &:= 2QG. \end{aligned} \tag{2.11}$$

The key feature of this transformation is that M is positive definite, as we now show.

LEMMA 2.2. *For all sufficiently large b, the matrix M in* (2.11) *is positive definite. Moreover, M has eigenvalues*

$$\lambda_M^{\min} = \tfrac{1}{4}b^{-\frac{5}{2}} + O(b^{-4}) \quad \text{and} \quad \lambda_M^{\max} = \tfrac{1}{4}b^{-\frac{1}{2}} + O(b^{-2}), \quad \text{as } b \to \infty. \tag{2.12}$$

PROOF. The result can be derived by explicitly computing the eigenvalues of M and expanding for large b. ◻

Because M is positive definite, we can prove exponential mean-square growth of $v(x(t))$. Then via positive definiteness of Q we can establish the same behaviour for $x(t)$. The following theorem formalises this result.

THEOREM 2.3. *Given sufficiently large b, there exist positive constants $D = D(b)$ and $\delta = \delta(b)$ such that*

$$\mathbb{E}\left[\| x(t) \|_2^2\right] \geq D\,\mathbb{E}\left[\| x_0 \|_2^2\right] e^{\delta t}, \quad \text{for all } t > 0, \tag{2.13}$$

and hence if $\mathbb{E}\left[\| x_0 \|_2^2\right] \neq 0$ we have

$$\liminf_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}\left[\| x(t) \|_2^2\right]\right) \geq \delta. \tag{2.14}$$

PROOF. From (2.10), we may write

$$\frac{\mathbb{E}\left[v(t+h)\right]-\mathbb{E}\left[v(t)\right]}{h}=\mathbb{E}\left[\frac{1}{h}\int_t^{t+h} x(s)^T M x(s)\, ds\right].$$

Taking the limit $h \to 0$ and using the positive definiteness of Q and M gives

$$\begin{aligned}\frac{d}{dt}\mathbb{E}\left[v(t)\right] = \mathbb{E}[x(t)^T M x(t)] &\geq \lambda_M^{\min}\mathbb{E}\left[\| x(t) \|_2^2\right] \\ &\geq \lambda_M^{\min}\frac{\mathbb{E}[x(t)^T Q x(t)]}{\lambda_Q^{\max}} \\ &= \frac{\lambda_M^{\min}}{\lambda_Q^{\max}}\mathbb{E}\left[v(t)\right].\end{aligned}$$

So

$$\mathbb{E}\left[v(t)\right] \geq e^{(\lambda_M^{\min}/\lambda_Q^{\max})t}\mathbb{E}\left[v(0)\right].$$

Hence,

$$\begin{aligned}\mathbb{E}\left[\| x(t) \|_2^2\right] &\geq \frac{1}{\lambda_Q^{\max}}\mathbb{E}\left[v(t)\right] \geq \frac{\mathbb{E}\left[v(0)\right]}{\lambda_Q^{\max}}e^{(\lambda_M^{\min}/\lambda_Q^{\max})t} \\ &\geq \frac{\lambda_Q^{\min}}{\lambda_Q^{\max}}\mathbb{E}\left[\| x_0 \|_2^2\right] e^{(\lambda_M^{\min}/\lambda_Q^{\max})t}.\end{aligned}$$

□

In words, comparing (2.3) and (2.4) with (2.13) and (2.14) we see that as the nonnormality in the problem increases, a vanishingly small noise term can de-stabilise a stable ODE in the mean-square sense.

3 Discrete analogue.

The Euler–Maruyama method [5] applied to the SDE (2.5) produces approximations $y_n \approx x(t_n)$, with $t_n = n\Delta t$, according to the recurrence

$$y_{n+1} = (I + \Delta t A + \Delta w_n G)y_n, \tag{3.1}$$

where $\Delta w_n := w(t_{n+1}) - w(t_n)$ is the increment in the Brownian path, and hence $\{\Delta w_n\}_{n\geq 0}$ are i.i.d. and $N(0,\Delta t)$. We may therefore write

$$y_n = (I + \Delta t A + \Delta w_{n-1}G)(I + \Delta t A + \Delta w_{n-2}G)\cdots(I + \Delta t A + \Delta w_0 G)x_0,$$

which shows that y_n arises from a *random matrix product.* The noncommutativity of matrix multiplication makes it very difficult to prove sharp results in this area in general; however, mean-square stability in the small Δt regime can be studied by appealing to numerical analysis convergence theory. In Theorem 3.1 below we show that the de-stabilising small noise effect in Theorem 2.3 is also present for the discrete analogue. The proof uses a repeated application of a finite-time error bound in order to obtain the infinite-time result. In particular, it exploits the fact that the error bound is linear in the initial data. A related analysis was done in [4], but here we are showing instability rather than stability.

For convenience, we begin by analysing a continuous-time extension of (3.1), defined as

$$y(t) = (I + (t - t_n)A + (w(t) - w(t_n))\, G)\, y_n, \quad \text{for } t \in [t_n, t_{n+1}), \tag{3.2}$$

before giving a corollary for the discrete iteration.

THEOREM 3.1. *Given sufficiently large b, there exist positive constants $\gamma = \gamma(b)$, $T = T(b)$ and $\Delta t^\star = \Delta t^\star(b)$ such that*

$$\mathbb{E}\left[\| y(kT)\|_2^2\right] \geq \mathbb{E}\left[\| x_0 \|_2^2\right] e^{\gamma k T}, \quad \text{for all } 0 < \Delta t \leq \Delta t^\star \text{ and } k \in \mathbb{Z}^+. \tag{3.3}$$

PROOF. The result is trivial when $\mathbb{E}[\| x_0 \|_2^2] = 0$.

Consider now the case where $\mathbb{E}[\| x_0 \|_2^2] \geq 1$. From the identity $\| u \|_2^2 \geq \frac{1}{2}\| v \|_2^2 - \| u - v \|_2^2$, we have

$$\| y(t) \|_2^2 \geq \tfrac{1}{2}\| x(t)^2 \|_2^2 - \| y(t) - x(t) \|_2^2,$$

and hence

$$\mathbb{E}\left[\| y(t) \|_2^2\right] \geq \tfrac{1}{2}\mathbb{E}\left[\| x(t) \|_2^2\right] - \mathbb{E}\left[\| y(t) - x(t) \|_2^2\right]. \tag{3.4}$$

Given b, we know from Theorem 2.3 that (2.13) holds for some D and δ. We may then choose T sufficiently large that

$$\tfrac{D}{4} e^{\delta T} \geq e^{\frac{1}{2}\delta T}. \tag{3.5}$$

Now, from finite-time error analysis of Euler's method, there exist positive constants $\Delta t^\star = \Delta t^\star(b, T)$ and $C = C(b, T) > 0$ such that

$$\mathbb{E}\left[\|x(T) - y(T) \|_2^2\right] \leq C\, \mathbb{E}\left[\| x_0 \|_2^2\right] \Delta t, \tag{3.6}$$

for all $0 < \Delta t \leq \Delta t^\star$. The $O(\Delta t)$ mean-square error behaviour shown in (3.6) is well known; see for example [5, Theorem 9.6.2]. In (3.6), we make it clear that when $\mathbb{E}[\| x_0 \|_2^2] \geq 1$ the mean-square error can be bounded linearly in terms of the second moment of the initial data. This can be established with a slight generalisation of the arguments in [4, Appendix A]. By decreasing $\Delta t^\star$, if necessary, we may ensure that

$$C \Delta t^\star \leq \tfrac{D}{4} e^{\delta T}. \tag{3.7}$$

It follows from (2.13), (3.4), (3.6) and (3.7) that

$$\mathbb{E}\left[\| y(T) \|_2^2\right] \geq \tfrac{D}{4} e^{\delta T} \mathbb{E}\left[\| x_0 \|_2^2\right]. \tag{3.8}$$

Now, by applying the same arguments to $y(t)$ and $\widehat{x}(t)$ over $[T, 2T]$, where $\widehat{x}(t)$ solves the SDE (2.5) with $\widehat{x}(T) = y(T)$, we obtain

$$\begin{aligned}
\mathbb{E}\left[\| y(2T) \|_2^2\right] &\geq \tfrac{1}{2}\mathbb{E}\left[\| \widehat{x}(2T) \|_2^2\right] - \mathbb{E}\left[\| y(2T) - \widehat{x}(2T) \|_2^2\right] \\
&\geq \tfrac{1}{2}De^{\delta T}\mathbb{E}\left[\| y(T) \|_2^2\right] - C\mathbb{E}\left[\| y(T) \|_2^2\right]\Delta t \\
&\geq \tfrac{D}{4}e^{\delta T}\mathbb{E}\left[\| y(T) \|_2^2\right] \\
&\geq \left(\tfrac{D}{4}e^{\delta T}\right)^2 \mathbb{E}\left[\| x_0 \|_2^2\right].
\end{aligned}$$

Continuing with this argument gives

$$\mathbb{E}\left[\| y(kT) \|_2^2\right] \geq \left(\tfrac{D}{4}e^{\delta T}\right)^k \mathbb{E}\left[\| x_0 \|_2^2\right], \quad \text{for all } k \in \mathbb{Z}^+.$$

Since T was chosen to give (3.5), the result follows with $\gamma = \frac{1}{2}\delta$.

The remaining case, where $0 < \mathbb{E}[\| x_0 \|_2^2] < 1$, can be treated by noting from (2.13) that there exists a finite time $\widehat{T}$ such that $\mathbb{E}[\| x(\widehat{T}) \|_2^2] \geq 2$. Thus, from finite-time error analysis, we have $\mathbb{E}[\| y(\widehat{T}) \|_2^2] \geq 1$ for all sufficiently small Δt. The arguments from the first part of the proof may then be used, after increasing T to $\widehat{T}$ if necessary. □

COROLLARY 3.2. *Suppose* $\mathbb{E}[\| x_0 \|_2^2] \neq 0$. *Given sufficiently large* b, *there exist positive constants* $\gamma = \gamma(b)$, $T = T(b)$ *and* $\Delta t^\star = \Delta t^\star(b)$ *such that for any* $\Delta t \in (0, \Delta t^\star)$ *with the property that* T *is a multiple of* Δt, *the Euler–Maruyama recurrence* (3.1) *satisfies*

$$\liminf_{n\to\infty} \frac{1}{n\Delta t} \log\left(\mathbb{E}\left[\| y_n \|_2^2\right]\right) \geq \gamma.$$

PROOF. We first note from Theorem 3.1 that $\mathbb{E}[\| y_n \|_2^2] \neq 0$ for all $n \geq 0$; otherwise (3.3) is contradicted. Let $j = T/\Delta t$, which is an integer, and let i be any integer in $[0, j]$. It follows from Theorem 3.1 that

$$\mathbb{E}\big[\| y_{i+kj} \|_2^2\big] = \mathbb{E}\big[\| y(i\Delta t + kT) \|_2^2\big] \geq \mathbb{E}\big[\| y_i \|_2^2\big]e^{\gamma kj\Delta t}, \quad \forall k \in \mathbb{Z}^+.$$

Now, for any $n > j$, let k be the integer part of n/j, with $i = n - kj$. Then

$$\mathbb{E}\big[\| y_n \|_2^2\big] \geq \mathbb{E}\big[\| y_i \|_2^2\big]e^{\gamma kj\Delta t} \geq \Big(\min_{0\leq i\leq j} \mathbb{E}\big[\| y_i \|_2^2\big]\Big)e^{\gamma(n\Delta t - T)},$$

which implies the assertion. □

4 Discussion.

There are various directions in which this work could be extended.

- Rather than mean-square stability, asymptotic stability could be considered – here the stability criterion is $\lim_{t\to\infty} \| x(t) \|_2 = 0$ with probability one. Although at least as relevant in practice, this alternative seems to be harder to analyse.

- An alternative canonical way to introduce nonnormality is to fix b in the bidiagonal matrix

$$\begin{bmatrix} -1 & b & & & & \\ & -1 & b & & & \\ & & -1 & b & & \\ & & & \ddots & \ddots & \\ & & & & \ddots & b \\ & & & & & -1 \end{bmatrix} \in \mathbb{R}^{N \times N}$$

  and consider $N \to \infty$. In this case the question of finding an appropriate form of noise that vanishes as $N \to \infty$ and de-stabilises the system appears to be open.
- More speculatively, it would be of interest to formulate and prove a general result of the form "stable yet highly nonnormal implies de-stabilisable by small noise" in the continuous and discrete realms.

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