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Preserving exponential mean-square stability in the simulation of hybrid stochastic differential equations

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Abstract Positive results are derived concerning the long time dynamics of fixed step size numerical simulations of stochastic differential equation systems with Markovian switching. Euler–Maruyama and implicit theta-method discretisations are shown to capture exponential mean-square stability for all sufficiently small time-steps under appropriate conditions. Moreover, the decay rate, as measured by the second moment Lyapunov exponent, can be reproduced arbitrarily accurately. New finite-time convergence results are derived as an intermediate step in this analysis. We also show, however, that the mean-square A-stability of the theta method does not carry through to this switching scenario. The proof techniques are quite general and hence have the potential to be applied to other numerical methods.

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1 Introduction

Stochastic differential equation (SDE) models are now widely used in many application areas. Recently, models that switch between different SDE systems according to

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an independent Markov chain have been proposed. These hybrid SDEs are designed to account for circumstances where an abrupt change may take place in the nature of a physical process. In particular, important examples arise in mathematical finance, where a market may switch between two or more distinct modes (nervous, confident, cautious,...). For examples of such *regime switching* or *Markov-modulated dynamics* models, see, for example [9,18,20] and the references therein.

Generally, hybrid SDEs cannot be solved analytically and hence numerical methods must be used. Although it is intuitively straightforward to adapt existing SDE methods to the hybrid case, the traditional numerical analysis issues associated with the resulting methods have only recently received attention. Finite time convergence analysis of an Euler–Maruyama type method is given in [20]. In this work, we consider long time dynamics, and in particular focus on exponential mean-square stability. Our work therefore builds on the well known and highly informative analysis for deterministic problems and its more recent extension to SDEs [2,4–8,10,11,14–16].

The issue that we address is: can a numerical method reproduce the stability behaviour of the underlying hybrid SDE? In the general nonlinear case for (non-hybrid) SDEs it is known that Euler–Maryuma cannot guarantee to preserve exponential mean-square stability, even for arbitrarily small step sizes; see [8, Lemma 4.1]. Hence, in studying hybrid SDEs, we look for conditions under which positive results can be derived in the small step size setting. As further motivation for the small step size analysis, we also point out (in Sect. 6.2) that a mean-square generalization of A-stability, that is, unconditional numerical stability on stable problems, does not carry through to the hybrid setting.

Section 2 sets up the hybrid SDE and Euler–Maruyama method. In Sect. 3 we look at the case of a scalar, linear problem and obtain a precise result. Section 4 then derives results for nonlinear systems. The analysis in Sect. 4 makes use of two properties of the Euler–Maruyama method: (a) a finite-time convergence property and (b) a flow property. To emphasize the generality of the analysis, we show in Sects. 5 and 6 that the results extend to a class of stochastic theta methods. A by-product of this work is a finite-time strong convergence analysis of implicit methods for hybrid SDEs.

2 Hybrid SDEs and the Euler–Maruyama method

Throughout this paper, we let $B(t) = (B_1(t), \ldots, B_m(t))^T$ be an m-dimensional Brownian motion. Also we let $r(t)$ be a right-continuous Markov chain taking values in a finite state space $\mathbb{S} = \{1, 2, \ldots, N\}$ and independent of the Brownian motion $B(\cdot)$. The corresponding generator is denoted $\Gamma = (\gamma_{ij})_{N\times N}$, so that

$$\mathbb{P}\{r(t+\delta) = j \mid r(t) = i\} = \begin{cases} \gamma_{ij}\delta + o(\delta) & : \text{ if } i \neq j, \\ 1 + \gamma_{ij}\delta + o(\delta) & : \text{ if } i = j, \end{cases}$$

where $\delta > 0$. Here γ_{ij} is the transition rate from i to j and $\gamma_{ij} > 0$ if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}$. We note that almost every sample path of $r(\cdot)$ is a right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$.

In this paper, we need to work on the probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). To construct such a filtration, we denote by $\mathcal{N}$ the collection of $\mathbb{P}$-null sets, i.e. $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$. For each $t \geq 0$, define $\mathcal{F}_t = \sigma(\bar{\mathcal{F}}_t \cup \mathcal{N})$, where $\bar{\mathcal{F}}_t$ is the σ-algebra generated by the Brownian motion and the Markov chain, namely $\bar{\mathcal{F}}_t = \sigma(B(s), r(s) : 0 \leq s \leq t\}$. In other words, the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ we will work on is the augmentation under $\mathbb{P}$ of the natural filtration $\{\bar{\mathcal{F}}_t\}_{t\geq 0}$ generated by the Brownian motion and the Markov chain.

We will use $|\cdot|$ to denote the Euclidean norm of a vector and the trace norm of a matrix. We will denote the indicator function of a set G by I_G. For $x \in \mathbb{R}$, $\text{int}(x)$ denotes the integer part of x. We let $L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^n, \mathbb{S})$ denote the family of $\mathcal{F}_t$-measurable random variables of the form (ξ, ρ), where ξ is $\mathbb{R}^n$-valued with $\mathbb{E}|\xi|^2 < \infty$ and ρ is discrete and $\mathbb{S}$-valued.

We consider n-dimensional hybrid Itô SDEs having the form

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t) \tag{1}$$

on $t \geq 0$ with initial data $x(0) = x_0$ and $r(0) = r_0$ such that $(x(0), r(0)) \in L^2_{\mathcal{F}_0}(\Omega; \mathbb{R}^n, \mathbb{S})$. We assume that

$$f : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^{n\times m}$$

are sufficiently smooth for the existence and uniqueness of the solution (see, e.g. [13,19]). We also assume that

$$f(0, i) = 0 \quad \text{and} \quad g(0, i) = 0 \quad \forall i \in \mathbb{S}, \tag{2}$$

so equation (1) admits the zero solution, $x(t) \equiv 0$, whose stability is the issue under consideration.

We now introduce an Euler–Maruyama based computational method, which was shown in [20] to be strongly convergent. The method makes use of the following lemma (see [1]).

Lemma 1 *Given $\Delta > 0$, let $r^\Delta_k = r(k\Delta)$ for $k \geq 0$. Then $\{r^\Delta_k,\ k = 0, 1, 2, \ldots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$P(\Delta) = (P_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}. \tag{3}$$

Given a fixed step size $\Delta > 0$ and the one-step transition probability matrix $P(\Delta)$ in (3), the discrete Markov chain $\{r^\Delta_k,\ k = 0, 1, 2, \ldots\}$ can be simulated as follows. Let $r^\Delta_0 = i_0$ and compute a pseudo-random number ξ_1 from the uniform $(0, 1)$ distribution. Define

$$r^\Delta_1 = \begin{cases} i & \text{if } i \in \mathbb{S} - \{N\} \text{ such that} \\ & \quad \sum_{j=1}^{i-1} P_{r^\Delta_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i} P_{r^\Delta_0,j}(\Delta), \\ N & \text{if } \sum_{j=1}^{N-1} P_{r^\Delta_0,j}(\Delta) \leq \xi_1, \end{cases}$$

where we set $\sum_{i=1}^{0} P_{r_0^\Delta,j}(\Delta) = 0$ as usual. In other words, we ensure that the probability of state s being chosen is given by $\mathbb{P}(r_1^\Delta = s) = P_{r_0^\Delta,s}(\Delta)$. Generally, having computed $r_0^\Delta, r_1^\Delta, r_2^\Delta, \dots, r_k^\Delta$, we compute r_{k+1}^Δ by drawing a uniform (0, 1) pseudo-random number ξ_{k+1} and setting

$$r_{k+1}^\Delta = \begin{cases} i & \text{if } i \in \mathbb{S} - \{N\} \text{ such that} \\ & \quad \sum_{j=1}^{i-1} P_{r_k^\Delta,j}(\Delta) \le \xi_{k+1} < \sum_{j=1}^{i} P_{r_k^\Delta,j}(\Delta), \\ N & \text{if } \sum_{j=1}^{N-1} P_{r_k^\Delta,j}(\Delta) \le \xi_{k+1}. \end{cases}$$

This procedure can be carried out independently to obtain more trajectories.

Having explained how to simulate the discrete Markov chain, we now define the Euler–Maruyama (EM) approximation for the hybrid SDE (1). The discrete approximation $X_k \approx x(t_k)$, with $t_k = k\Delta$, is formed by simulating from $X_0 = x_0,\ r_0^\Delta = r_0$ and, generally,

$$X_{k+1} = X_k + f(X_k, r_k^\Delta)\Delta + g(X_k, r_k^\Delta)\Delta B_k, \tag{4}$$

where $\Delta B_k = B(t_{k+1}) - B(t_k)$. In words, r_k^Δ defines which of the N SDEs is currently active, and we apply EM to this SDE. Compared with the numerical analysis of standard SDEs, a new source of error arises in the method (4); the switching can only occur at discrete time points $\{t_k\}$, whereas for the underlying continuous-time problem (1) the Markov chain can produce a switch at any point in time.

Strong convergence of this EM method was studied in [20]. In this paper, our emphasis is on (a) analysing mean-square stability and (b) deriving and analysing appropriate implicit methods.

3 Scalar linear problems

We begin our study with the special but important case of scalar linear hybrid SDEs of the form

$$dx(t) = \mu(r(t))x(t)dt + \sigma(r(t))x(t)dB(t), \qquad x(0) \neq 0 \quad a.s., \tag{5}$$

on $t \ge 0$. Here, to avoid complicated notations, we let $B(t)$ be a scalar Brownian motion while μ and σ are mappings from $\mathbb{S} \to \mathbb{R}$. One motivation for studying this class is that sharp stability results can be derived, allowing for comparison with the more general nonlinear results that we derive in subsequent sections. However, we also note that volatility-switching geometric Brownian motion is a realistic model in mathematical finance [9] and hence the qualitative behaviour of numerical methods on this model is of independent interest.

It is known that the linear hybrid SDE (5) has the explicit solution

$$x(t) = x_0 \exp\left\{ \int_0^t [\mu(r(s)) - \tfrac{1}{2}\sigma^2(r(s))]ds + \int_0^t \sigma(r(s))dB(s) \right\}. \tag{6}$$

Making use of this explicit form we are able to discuss mean square exponential stability precisely.

As a standing hypothesis, we assume moreover in this section that the Markov chain is *irreducible*. This is equivalent to the condition that for any $i, j \in \mathbb{S}$, we can find $i_1, i_2, \ldots, i_k \in \mathbb{S}$ such that

$$\gamma_{i,i_1}\gamma_{i_1,i_2}\cdots\gamma_{i_k,j} > 0.$$

Note that Γ always has an eigenvalue 0. The algebraic interpretation of irreducibility is $\operatorname{rank}(\Gamma) = N - 1$. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \pi_2, \ldots, \pi_N) \in \mathbb{R}^{1\times N}$ which can be determined by solving $\pi\Gamma = 0$, subject to $\sum_{j=1}^N \pi_j = 1$ and $\pi_j > 0$ for all $j \in \mathbb{S}$. The following theorem gives a necessary and sufficient condition for the SDE (5) to be exponentially stable in mean square.

Theorem 1 *The second moment Lyapunov exponent of the SDE* (5) *is*

$$\lim_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^2) = \sum_{j\in\mathbb{S}} \pi_j(2\mu_j + \sigma_j^2), \tag{7}$$

where we write $\mu(j) = \mu_j$ and $\sigma(j) = \sigma_j$. Hence the SDE (5) *is exponentially stable in mean square if and only if*

$$\sum_{j\in\mathbb{S}} \pi_j(2\mu_j + \sigma_j^2) < 0. \tag{8}$$

Proof It is well known (see, e.g., [1]) that almost every sample path of the Markov chain $r(\cdot)$ is a right continuous step function with a finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$. Hence there is a sequence of finite stopping times $0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$ such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) I_{[\tau_k, \tau_{k+1})}(t), \quad t \geq 0.$$

For any integer $z > 0$, it then follows from (6) that

$$|x(t\wedge\tau_z)|^2 = |x_0|^2 \exp\left\{ \int_0^{t\wedge\tau_z} [2\mu(r(s)) - \sigma^2(r(s))]ds \right.$$
$$\left. + \int_0^{t\wedge\tau_z} 2\sigma(r(s))dB(s) \right\}$$

$$
\begin{aligned}
&= \xi(t \wedge \tau_z) \exp \left\{ - \int_0^{t\wedge\tau_z} 2\sigma^2(r(s))ds \right. \\
&\qquad \left. + \int_0^{t\wedge\tau_z} 2\sigma(r(s))dB(s) \right\} \\
&= \xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k,
\end{aligned}
$$

where

$$
\begin{aligned}
\xi(t \wedge \tau_z) &= |x_0|^2 \exp \left\{ \int_0^{t\wedge\tau_z} [2\mu(r(s)) + \sigma^2(r(s))]ds \right\}, \\
\zeta_k &= \exp \Big\{ -2\sigma^2(r(t \wedge \tau_k))(t \wedge \tau_{k+1} - t \wedge \tau_k) + 2\sigma(r(t \wedge \tau_k)) \\
&\quad \times [B(t \wedge \tau_{k+1}) - B(t \wedge \tau_k)] \Big\}.
\end{aligned}
$$

Let $\mathcal{G}_t = \sigma(\{r(u)\}_{u\geq 0}, \{B(s)\}_{0\leq s\leq t})$, that is, the σ-algebra generated by $\{r(u)\}_{u\geq 0}$ and $\{B(s)\}_{0\leq s\leq t}$. Compute

$$
\begin{aligned}
\mathbb{E}|x(t \wedge \tau_z)|^2 &= \mathbb{E}\left(\xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k \right) \\
&= \mathbb{E}\left\{ \mathbb{E}\left(\xi(t \wedge \tau_z) \prod_{k=0}^{z-1} \zeta_k \middle| \mathcal{G}_{t\vee\tau_{z-1}} \right) \right\} \\
&= \mathbb{E}\left\{ \left[\xi(t \wedge \tau_z) \prod_{k=0}^{z-2} \zeta_k \right] \mathbb{E}\left(\zeta_{z-1} \middle| \mathcal{G}_{t\vee\tau_{z-1}} \right) \right\}.
\end{aligned} \tag{9}
$$

Define

$$
\zeta_{z-1}(i) = \exp \left\{ -2\sigma_i^2(t \wedge \tau_z - t \wedge \tau_{z-1}) + 2\sigma_i[B(t \wedge \tau_z) - B(t \wedge \tau_{z-1})] \right\}, \quad i \in \mathbb{S}.
$$

By the well-known exponential martingale of a Brownian motion we have $\mathbb{E}\zeta_{z-1}(i) = 1$, for all $i \in \mathbb{S}$. Then

$$
\begin{aligned}
\mathbb{E}\left(\zeta_{z-1} \middle| \mathcal{G}_{t\vee\tau_{z-1}} \right) &= \mathbb{E}\left(\sum_{i\in\mathbb{S}} I_{\{r(t\wedge\tau_{z-1})=i\}} \zeta_{z-1}(i) \middle| \mathcal{G}_{t\vee\tau_{z-1}} \right) \\
&= \sum_{i\in\mathbb{S}} I_{\{r(t\wedge\tau_{z-1})=i\}} \mathbb{E}\left(\zeta_{z-1}(i) \middle| \mathcal{G}_{t\vee\tau_{z-1}} \right).
\end{aligned}
$$

But, noting that $t \wedge \tau_z - t \wedge \tau_{z-1}$ is $\mathcal{G}_{t \vee \tau_{z-1}}$-measurable while $B(t \wedge \tau_z) - B(t \wedge \tau_{z-1})$ is independent of $\mathcal{G}_{t \vee \tau_{z-1}}$, we have

$$\mathbb{E}\left(\zeta_{z-1}(i)\Big|\mathcal{G}_{t \vee \tau_{z-1}}\right) = \mathbb{E}\zeta_{z-1}(i) = 1,$$

whence $\mathbb{E}\left(\zeta_{z-1}\Big|\mathcal{G}_{t \vee \tau_{z-1}}\right) = 1$. Substituting this into (9) yields

$$\mathbb{E}|x(t \wedge \tau_z)|^2 = \mathbb{E}\left[\xi(t \wedge \tau_z)\prod_{k=0}^{z-2}\zeta_k\right].$$

Repeating this procedure implies $\mathbb{E}|x(t \wedge \tau_z)|^2 = \mathbb{E}\xi(t \wedge \tau_z)$. Letting $z \to \infty$ we obtain

$$\mathbb{E}|x(t)|^2 = \mathbb{E}\xi(t) = \mathbb{E}\left[|x_0|^2 \exp\left[\int_0^t [2\mu(r(s)) + \sigma^2(r(s))]ds\right]\right]. \tag{10}$$

Now, by the ergodic property of the Markov chain (see, e.g. [1]), we have

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t [2\mu(r(s)) + \sigma^2(r(s))]ds = \sum_{j\in\mathbb{S}}\pi_j(2\mu_j + \sigma_j^2) := \gamma \quad a.s. \tag{11}$$

Let $\varepsilon > 0$ be arbitrary. It follows from (10) that

$$e^{-(\gamma-\varepsilon)t}\mathbb{E}|x(t)|^2 = \mathbb{E}\left[|x_0|^2 \exp\left[-(\gamma-\varepsilon)t + \int_0^t [2\mu(r(s)) + \sigma^2(r(s))]ds\right]\right].$$

By (11),

$$\lim_{t\to\infty}\exp\left[-(\gamma-\varepsilon)t + \int_0^t [2\mu(r(s)) + \sigma^2(r(s))]ds\right] = \infty \quad a.s.$$

Hence

$$\lim_{t\to\infty} e^{-(\gamma-\varepsilon)t}\mathbb{E}|x(t)|^2 = \infty,$$

which implies

$$\mathbb{E}|x(t)|^2 \geq e^{(\gamma-\varepsilon)t} \quad \text{for all sufficiently large } t.$$

So,

$$\liminf_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \geq \gamma - \varepsilon.$$

Similarly, we can show

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq \gamma + \varepsilon.$$

Since ε is arbitrary, the assertion (7) follows. □

We remark that for a single linear SDE of the form $dx(t) = \mu x(t)dt + \sigma x(t)dB(t)$, where μ and σ are constants, Theorem 1 reproduces the well-known mean-square stability characterisation $2\mu + \sigma^2 < 0$. On this problem class, it is known that certain implicit methods, such as the trapezoidal rule, have an A-stability type property—they match the exponential mean-square stability/instability of the SDE for all $\Delta > 0$ [6,15,16]. In the more general hybrid case (5), Theorem 1 tells us that the same quantity, $2\mu + \sigma^2 < 0$, *appropriately averaged over the states of the Markov chain*, determines the stability. Intuitively, even though a numerical method such as the trapezoidal rule can match the stability properties of a single linear SDE for all $\Delta > 0$, it is much more demanding to ask a method to maintain this behaviour over all possible averages, especially those involving a mixture of individually stable and unstable problems. In Sect. 6.2 we spell out the details of this argument on a particular example, and show that the A-stability analogue does not hold for a natural trapezoidal method. This gives further motivation for the focus in this work on the $\Delta \to 0$ regime.

Given a step size $\Delta > 0$, the EM method (4) applied to (5) gives $X(0) = x_0$ and

$$X_{k+1} = X_k \left[1 + \mu(r_k^\Delta)\Delta + \sigma(r_k^\Delta)\Delta B_k\right], \quad k \geq 1. \tag{12}$$

We then have the following theorem.

Theorem 2 *The EM approximation* (12) *satisfies*

$$\lim_{n\to\infty} \frac{1}{n\Delta} \log\left(\mathbb{E}[X_n^2]\right) = \sum_{j\in\mathbb{S}} \pi_j(2\mu_j + \sigma_j^2) + \Delta \sum_{j\in\mathbb{S}} \pi_j \left(\tfrac{1}{2}\sigma_j^2 - (\mu_j + \sigma_j^2)^2\right) + O(\Delta^2) \tag{13}$$

as $\Delta \to 0$. Hence, the numerical method matches the exponential mean-square stability or instability of the SDE, for sufficiently small Δ.

Proof For any integer z, it follows from (12) that

$$|X_{z+1}|^2 = |x_0|^2 \prod_{k=0}^{z} Y_k, \qquad \text{where} \quad Y_k = \left[1 + \mu(r_k^\Delta)\Delta + \sigma(r_k^\Delta)\Delta B_k\right]^2.$$

Using the σ-algebra $\mathcal{G}_t$ defined in the proof of Theorem 1, we compute

$$\mathbb{E}|X_{z+1}|^2 = \mathbb{E}\left(\mathbb{E}\left[|x_0|^2 \prod_{k=0}^{z} Y_k \middle| \mathcal{G}_{t_z}\right]\right) = \mathbb{E}\left(|x_0|^2 \prod_{k=0}^{z-1} Y_k \,\mathbb{E}\left[Y_z \middle| \mathcal{G}_{t_z}\right]\right).$$

But

$$\begin{aligned}
&\mathbb{E}\left[Y_z \middle| \mathcal{G}_{t_z}\right] \\
&\quad = \mathbb{E}\left[(1+\mu(r_z^\Delta)\Delta)^2 + 2(1+\mu(r_z^\Delta)\Delta)\sigma(r_z^\Delta)\Delta B_k + \sigma^2(r_z^\Delta)\Delta B_k^2 \middle| \mathcal{G}_{t_z}\right] \\
&\quad = (1+\mu(r_z^\Delta)\Delta)^2 + 2(1+\mu(r_z^\Delta)\Delta)\sigma(r_z^\Delta)\mathbb{E}(\Delta B_z|\mathcal{G}_{t_z}) + \sigma^2(r_z^\Delta)\mathbb{E}(\Delta B_z^2|\mathcal{G}_{t_z}) \\
&\quad = (1+\mu(r_z^\Delta)\Delta)^2 + 2(1+\mu(r_z^\Delta)\Delta)\sigma(r_z^\Delta)\mathbb{E}(\Delta B_z) + \sigma^2(r_z^\Delta)\mathbb{E}(\Delta B_z^2) \\
&\quad = (1+\mu(r_z^\Delta)\Delta)^2 + \sigma^2(r_z^\Delta)\Delta.
\end{aligned}$$

Hence

$$\mathbb{E}|X_{z+1}|^2 = \mathbb{E}\left(|x_0|^2[(1+\mu(r_z^\Delta)\Delta)^2 + \sigma^2(r_z^\Delta)\Delta]\prod_{k=0}^{z-1} Y_k\right).$$

We compute furthermore that

$$\mathbb{E}|X_{z+1}|^2 = \mathbb{E}\left(|x_0|^2[(1+\mu(r_z^\Delta)\Delta)^2 + \sigma^2(r_z^\Delta)\Delta]\prod_{k=0}^{z-2} Y_k \,\mathbb{E}\left[Y_{z-1} \middle| \mathcal{G}_{t_{z-1}}\right]\right).$$

But, in the same way as before, we can show that

$$\mathbb{E}\left[Y_{z-1} \middle| \mathcal{G}_{t_{z-1}}\right] = (1+\mu(r_{z-1}^\Delta)\Delta)^2 + \sigma^2(r_{z-1}^\Delta)\Delta.$$

Hence

$$\mathbb{E}|X_{z+1}|^2 = \mathbb{E}\left(|x_0|^2\left[\prod_{k=z-1}^{z}[(1+\mu(r_k^\Delta)\Delta)^2 + \sigma^2(r_k^\Delta)\Delta]\right]\prod_{k=0}^{z-2} Y_k\right).$$

Repeating this procedure we obtain

$$\mathbb{E}|X_{z+1}|^2 = \mathbb{E}\left(|x_0|^2\prod_{k=0}^{z}[(1+\mu(r_k^\Delta)\Delta)^2 + \sigma^2(r_k^\Delta)\Delta]\right),$$

which we re-write as

$$\mathbb{E}|X_{z+1}|^2 = \mathbb{E}\left\{|x_0|^2 \exp\left[\sum_{k=0}^{z}\log\left((1+\mu(r_k^\Delta)\Delta)^2 + \sigma^2(r_k^\Delta)\Delta\right)\right]\right\}. \tag{14}$$

By the ergodic property of the Markov chain, we have

$$\lim_{z\to\infty}\frac{1}{z+1}\sum_{k=0}^{z}\log\Big((1+\mu(r_k^{\Delta})\Delta)^2+\sigma^2(r_k^{\Delta})\Delta\Big)$$
$$=\sum_{j\in\mathbb{S}}\pi_j\log\Big((1+\mu_j\Delta)^2+\sigma_j^2\Delta\Big)$$
$$=\sum_{j\in\mathbb{S}}\pi_j\Big((2\mu_j+\sigma_j^2)\Delta+\mu_j^2\Delta^2-\tfrac{1}{2}(2\mu_j+\sigma_j^2)^2\Delta^2+O(\Delta^3)\Big).$$

Using this in (14) gives the required result (13). □

4 Nonlinear systems

To study exponential mean-square stability for EM (4) on nonlinear systems (1), we find it convenient to extend the numerical method to continuous time. Thus, we let

$$\bar{X}(t)=X_k,\quad \bar{r}(t)=r_k^{\Delta},\qquad \text{for } t\in[t_k,t_{k+1}),\tag{15}$$

and take our continuous-time EM approximation to be

$$X(t)=x_0+\int_0^t f(\bar{X}(s),\bar{r}(s))ds+\int_0^t g(\bar{X}(s),\bar{r}(s))dB(s).\tag{16}$$

Note that $X(t_k)=\bar{X}(t_k)=X_k$, that is, $X(t)$ and $\bar{X}(t)$ interpolate the discrete numerical solution.

Following the standard definition for SDEs, [3,12], we define exponential stability in mean square for the hybrid SDE and continuous time numerical method as follows.

Definition 3 The hybrid SDE (1) is said to be *exponentially stable in mean square* if there is a pair of positive constants λ and M such that, for all initial data $(x_0,r(0))\in L^2_{\mathcal{F}_0}(\Omega;\mathbb{R}^n,\mathbb{S})$,

$$\mathbb{E}|x(t)|^2\le M\mathbb{E}|x_0|^2e^{-\lambda t},\quad \forall t\ge 0.\tag{17}$$

We refer to λ as a *rate constant* and M as a *growth constant*.

Definition 4 For a given step size $\Delta>0$, the EM method (16) is said to be *exponentially stable in mean square* on the hybrid SDE (1) if there is a pair of positive constants γ and H such that for all initial data $(x_0,r_0)\in L^2_{\mathcal{F}_0}(\Omega;\mathbb{R}^n,\mathbb{S})$

$$\mathbb{E}|X(t)|^2\le H\mathbb{E}|x_0|^2e^{-\gamma t},\quad \forall t\ge 0.\tag{18}$$

We refer to γ as a *rate constant* and H as a *growth constant*.

Building on Sect. 3, our aim is to find conditions under which the numerical method reproduces the stability behaviour of the underlying problem, for sufficiently small Δ. In order to do this, we introduce some conditions and perform preliminary analysis that establishes second moment boundedness and an appropriate form of strong convergence under a global Lipschitz assumption. The results and proofs in this section are an extension of those in [8] from the standard SDE case.

Assumption 5 (Global Lipschitz) There is a positive constant K such that

$$|f(x,i) - f(y,i)|^2 \vee |g(x,i) - g(y,i)|^2 \le K|x-y|^2 \tag{19}$$

for all $(x, y, i) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S}$.

Recalling (2) we observe from this assumption that the linear growth condition

$$|f(x,i)|^2 \vee |g(x,i)|^2 \le K|x|^2 \tag{20}$$

holds for all $(x, i) \in \mathbb{R}^n \times \mathbb{S}$.

Let us now present a number of lemmas that will lead to our "if and only if" result. First, we derive a basic growth bound.

Lemma 2 *If* (20) *holds, then for all sufficiently small* Δ *the continuous EM approximate solution* (16) *satisfies, for any* $T > 0$*,*

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^2 \le B_{x_0,T}, \tag{21}$$

where $B_{x_0,T} = 3\mathbb{E}|x_0|^2 e^{3(T+1)KT}$*. Moreover, the true solution of* (1) *also obeys*

$$\sup_{0\le t\le T} \mathbb{E}|x(t)|^2 \le B_{x_0,T}. \tag{22}$$

Proof By (20), we derive from (16) that, for $0 \le t \le T$,

$$\begin{aligned}\mathbb{E}|X(t)|^2 &\le 3\mathbb{E}|x_0|^2 + 2T\mathbb{E}\int_0^t |f(\bar{X}(s), \bar{r}(s)|^2 ds + 2\mathbb{E}\int_0^t |g(\bar{X}(s), \bar{r}(s)|^2 ds \\ &\le 3\mathbb{E}|x_0|^2 + 3(T+1)K\int_0^t \mathbb{E}|\bar{X}(s)|^2 ds.\end{aligned}$$

Since the right-hand side term is non-decreasing in t, we have

$$\sup_{0\le t\le t_1} \mathbb{E}|X(t)|^2 \le 3\mathbb{E}|x_0|^2 + 3(T+1)K\int_0^{t_1} \mathbb{E}|\bar{X}(s)|^2 ds$$
$$\le 3\mathbb{E}|x_0|^2 + 3(T+1)K\int_0^{t_1} \left(\sup_{0\le t\le s} \mathbb{E}|X(t)|^2\right) ds,$$

for any $t_1 \in [0, T]$. The continuous Gronwall inequality [12] hence yields

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^2 \le 3\mathbb{E}|x_0|^2 e^{3(T+1)KT},$$

which is the required assertion (21). Similarly, we can show (22). □

The next lemma bounds the effect of replacing the right-continuous Markov chain by the interpolant of the discrete time Markov chain.

Lemma 3 *If* (20) *holds, then for all sufficiently small* Δ, $\bar{X}(t)$ *in* (15) *obeys*

$$\mathbb{E}\int_0^T |f(\bar{X}(s), r(s)) - f(\bar{X}(s), \bar{r}(s))|^2 ds \le \beta_T \Delta \sup_{0\le t\le T} \mathbb{E}|\bar{X}(t)|^2 \tag{23}$$

and

$$\mathbb{E}\int_0^T |g(\bar{X}(s), r(s)) - g(\bar{X}(s), \bar{r}(s))|^2 ds \le \beta_T \Delta \sup_{0\le t\le T} \mathbb{E}|\bar{X}(t)|^2 \tag{24}$$

for any $T > 0$, *where* $\beta_T = 4KTN[1 + \max_{1\le i\le N}(-\gamma_{ii})]$.

Proof Let $j = \text{int}(T/\Delta)$. Then

$$\mathbb{E}\int_0^T |f(\bar{X}(s), \bar{r}(s)) - f(\bar{X}(s), r(s))|^2 ds$$
$$= \sum_{k=0}^{j} \mathbb{E}\int_{t_k}^{t_{k+1}} |f(\bar{X}(t_k), r(t_k)) - f(\bar{X}(t_k), r(s))|^2 ds \tag{25}$$

with, for convenience, t_{j+1} being redefined as T. By (20), we compute

$$
\begin{aligned}
&\mathbb{E}\int_{t_k}^{t_{k+1}} |f(\bar{X}(t_k), r(t_k)) - f(\bar{X}(t_k), r(s))|^2 ds \\
&\le 2\mathbb{E}\int_{t_k}^{t_{k+1}} \left[|f(\bar{X}(t_k), r(t_k))|^2 + |f(\bar{X}(t_k), r(s))|^2\right] I_{\{r(s)\neq r(t_k)\}} ds \\
&\le 4K\mathbb{E}\int_{t_k}^{t_{k+1}} |\bar{X}(t_k)|^2 I_{\{r(s)\neq r(t_k)\}} ds \\
&\le 4K\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\mathbb{E}\left[|\bar{X}(t_k)|^2 I_{\{r(s)\neq r(t_k)\}}|r(t_k)\right]\right] ds \\
&= 4K\int_{t_k}^{t_{k+1}} \mathbb{E}\left[\mathbb{E}\left[|\bar{X}(t_k)|^2|r(t_k)\right]\mathbb{E}\left[I_{\{r(s)\neq r(t_k)\}}|r(t_k)\right]\right] ds, \qquad (26)
\end{aligned}
$$

where in the last step we used the fact that $\bar{X}(t_k)$ and $I_{\{r(s)\neq r(t_k)\}}$ are conditionally independent with respect to the σ-algebra generated by $r(t_k)$. But, by the Markov property,

$$
\begin{aligned}
\mathbb{E}\left[I_{\{r(s)\neq r(t_k)\}}|r(t_k)\right] &= \sum_{i\in\mathbb{S}} I_{\{r(t_k)=i\}}\mathbb{P}(r(s)\neq i|r(t_k)=i) \\
&= \sum_{i\in\mathbb{S}} I_{\{r(t_k)=i\}} \sum_{j\neq i}(\gamma_{ij}(s-t_k)+o(s-t_k)) \\
&\le \left(\max_{1\le i\le N}(-\gamma_{ii})\Delta + o(\Delta)\right)\sum_{i\in\mathbb{S}} I_{\{r(t_k)=i\}} \\
&\le \hat{\gamma}\Delta, \qquad (27)
\end{aligned}
$$

where $\hat{\gamma} = N[1+\max_{1\le i\le N}(-\gamma_{ii})]$. So, in (26)

$$
\mathbb{E}\int_{t_k}^{t_{k+1}} |f(\bar{X}(t_k), r(t_k)) - f(\bar{X}(t_k), r(s))|^2 ds \le 4K\hat{\gamma}\Delta\int_{t_k}^{t_{k+1}} \mathbb{E}|\bar{X}(t_k)|^2 ds.
$$

Substituting this into (25) gives

$$
\mathbb{E}\int_0^T |f(\bar{X}(s), \bar{r}(s)) - f(\bar{X}(s), r(s))|^2 ds \le 4K\hat{\gamma}\Delta\sum_{k=0}^{j}\int_{t_k}^{t_{k+1}} \mathbb{E}|\bar{X}(t_k)|^2 ds
$$

$$\leq 4KT\hat{\gamma}\Delta \sup_{0\leq t\leq T} \mathbb{E}|\bar{X}(t)|^2,$$

which is the required assertion (23). We can show (24) similarly. □

The next lemma shows that EM has strong finite-time convergence order of at least $\frac{1}{2}$. This basic property was already derived in [20, Theorem 3.1]. However, Lemma 4 below establishes a 'squared error constant' that is linearly proportional to $\sup_{0\leq t\leq T} \mathbb{E}|X(t)|^2$, and clarifies the dependence of C upon T—these features are important in the subsequent analysis.

Lemma 4 *Under* (2) *and Assumption* 5*, for all sufficiently small* Δ *the continuous EM approximation* $X(t)$ *and true solution* $x(t)$ *obey*

$$\sup_{0\leq t\leq T} \mathbb{E}|X(t)-x(t)|^2 \leq \left(\sup_{0\leq t\leq T} \mathbb{E}|X(t)|^2\right) C_T\Delta \tag{28}$$

for any $T > 0$*, where*

$$C_T = 4(T+1)[\beta_T + K2T(1+2K)]e^{8K(T+1)T}$$

and β_T *has been defined in Lemma* 3.

Proof We compute from (1) and (16) that, for $0 \leq t \leq T$,

$$\begin{aligned}
\mathbb{E}|X(t)-x(t)|^2 &\leq 2T\mathbb{E}\int_0^t |f(\bar{X}(s),\bar{r}(s)) - f(x(s),r(s))|^2 ds \\
&\quad +2\mathbb{E}\int_0^t |g(\bar{X}(s),\bar{r}(s)) - g(x(s),r(s))|^2 ds \\
&\leq 4K(T+1)\int_0^t \mathbb{E}|\bar{X}(s)-x(s)|^2 ds \\
&\quad +4T\mathbb{E}\int_0^t |f(\bar{X}(s),\bar{r}(s)) - f(\bar{X}(s),r(s))|^2 ds \\
&\quad +4\mathbb{E}\int_0^t |g(\bar{X}(s),\bar{r}(s)) - g(\bar{X}(s),r(s))|^2 ds \\
&\leq 4K(T+1)\int_0^t \mathbb{E}|\bar{X}(s)-x(s)|^2 ds \\
&\quad +4(T+1)\beta_T\Delta\left(\sup_{0\leq t\leq T} \mathbb{E}|\bar{X}(t)|^2\right),
\end{aligned} \tag{29}$$

where Lemma 3 has been used. Moreover, note

$$\mathbb{E}|\bar{X}(s) - x(s)|^2 \le 2\mathbb{E}|\bar{X}(s) - X(s)|^2 + 2\mathbb{E}|X(s) - x(s)|^2. \tag{30}$$

Let $k = k(s) = \text{int}(s/\Delta)$, so $k\Delta \le s < (k+1)\Delta$. It then follows from (16) that

$$X(s) - \bar{X}(s) = X(s) - X_k = f(X_k, r_k^\Delta)(s - k\Delta) + g(X_k, r_k^\Delta)(B(s) - B(k\Delta)).$$

Thus, for $\Delta < 1/(2K)$,

$$\mathbb{E}|X(s) - \bar{X}(s)|^2 \le 2(\Delta^2 + \Delta)K\mathbb{E}|X_k|^2 \le (1 + 2K)\Delta\left(\sup_{0\le t\le T} \mathbb{E}|X(t)|^2\right). \tag{31}$$

Combining (29)–(31) yields

$$\begin{aligned}\mathbb{E}|X(t) - x(t)|^2 \le\ & 8K(T+1)\int_0^t \mathbb{E}|X(s) - x(s)|^2 ds \\ & +4(T+1)[\beta_T + K2T(1+2K)]\Delta\left(\sup_{0\le t\le T} \mathbb{E}|X(t)|^2\right).\end{aligned}$$

The continuous Gronwall inequality hence implies that, for any $t \in [0, T]$,

$$\mathbb{E}|X(t) - x(t)|^2 \le \left(\sup_{0\le t\le T} \mathbb{E}|X(t)|^2\right)4(T+1)[\beta_T + K2T(1+2K)]\Delta e^{8K(T+1)T},$$

which is the required assertion. □

Using the bounds from Lemmas 2–4, we now derive two results that relate the exact and numerical stability behaviour. They can be proved by adapting the proofs in [8, Lemmas 2.4 and 2.5] to allow for the Markovian switching. For completeness, we give proofs in the Appendix.

Lemma 5 *Let* (2) *and Assumption* 5 *hold. Assume that the hybrid SDE* (1) *is exponentially stable in mean square, satisfying* (17)*. Then there exists a* $\Delta^\star > 0$ *such that for every* $0 < \Delta \le \Delta^\star$ *the EM method is exponentially stable in mean square on the SDE* (1) *with rate constant* $\gamma = \frac{1}{2}\lambda$ *and growth constant* $H = 2Me^{\frac{1}{2}\lambda[1+(4\log M)/\lambda]}$.

Proof See the Appendix. □

The next lemma gives a result in the opposite direction.

Lemma 6 *Let* (2) *and Assumption* (5) *hold. Assume that for a step size* $\Delta > 0$*, the numerical method is exponentially stable in mean square with rate constant* γ *and growth constant* H*. If* Δ *satisfies*

$$C_{2T}e^{\gamma T}(\Delta + \sqrt{\Delta}) + 1 + \sqrt{\Delta} \le e^{\frac{1}{4}\gamma T} \quad \text{and} \quad C_T\Delta \le 1, \tag{32}$$

where $T := 1 + (4 \log H)/\gamma$*, then the hybrid SDE* (1) *is exponentially stable in mean square with rate constant* $\lambda = \frac{1}{2}\gamma$ *and growth constant* $M = 2He^{\frac{1}{2}\gamma T}$.

Proof See the Appendix. □

Lemmas 5 and 6 lead to the following equivalence result.

Theorem 6 *Under* (2) *and Assumption* 5*, the hybrid SDE* (1) *is exponentially stable in mean square if and only if there exists a* $\Delta > 0$ *such that the EM method is exponentially stable in mean square with rate constant* γ*, growth constant* H*, step size* Δ *and global error constant* C_T *for* $T := 1 + (4 \log H)/\gamma$ *satisfying* (32).

Proof The result follows almost immediately from Lemmas 5 and 6. □

Theorem 6 is a positive result, showing that the underlying problem and the EM discretisation have equivalent stability behaviour for sufficiently small step sizes.

Lemmas 5 and 6 produce new rate constants that are within a factor $\frac{1}{2}$ of the given ones. From the proofs, it is clear that we could match the rate constants more closely at the expense of larger growth factors. Our analysis may thus be interpreted as showing that upper bounds on the second moment Lyapunov exponents of the exact and numerical processes can be made arbitrarily close. These ideas could be formalized by copying directly the approach for non-hybrid SDEs in [8], and hence we omit the details.

5 Generalised results

Theorem 6 applies to the EM method. However, by examining the proofs of Lemmas 5 and 6 we see that the specific form of the numerical method was not exploited—the results presented there will hold for any numerical method applied to the SDE (1) as long as the corresponding continuous approximate solution $X(t)$ obeys the strong convergence property (28) and the "flow property" defined below. This observation leads to the more general treatment below.

We suppose that a numerical method is available which, given a step size $\Delta > 0$, computes discrete approximations $X_k \approx x(k\Delta)$, with $X_0 = x_0$. We also suppose that there is a well-defined interpolation process that extends the discrete approximation $\{X_k\}_{k\geq 1}$ to a continuous-time approximation $\{X(t)\}_{t\in\mathbb{R}_+}$, with $X(k\Delta) = X_k$. We require the numerical method to obey the following flow property.

Definition 7 The numerical method is said to obey the *flow property* if for any T that is a multiple of Δ, the continuous-time approximation $X(t)$ restricted to $[T, \infty)$ is the same as that when the numerical method is applied to the SDE (1) on $t \geq T$ with initial data $X(T)$ and $r(T)$.

In other words, under the flow property, if we apply the numerical method to the SDE

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t) \quad \text{on } t \geq T = n\Delta,$$

with $x(T) = X_n$ and $r(T) = r_n^{\Delta}$, producing a continuous-time approximation denoted by $Y(t)$, then $X(t) = Y(t)$ for $t \geq T$.

Next, we formalize the required strong convergence condition.

Condition 8 *For all sufficiently small Δ the continuous approximation $X(t)$ satisfies, for any $T > 0$,*

$$\sup_{0\leq t\leq T} \mathbb{E}|X(t)|^2 < \infty \tag{33}$$

and

$$\sup_{0\leq t\leq T} \mathbb{E}|X(t) - x(t)|^2 \leq \left(\sup_{0\leq t\leq T} \mathbb{E}|X(t)|^2\right) C_T \Delta \tag{34}$$

where C_T depends on T but not on x_0, r_0 and Δ.

It is useful to remark that Condition 8 guarantees a finite second moment of the true solution, that is,

$$\sup_{0\leq t\leq T} \mathbb{E}|x(t)|^2 < \infty, \quad \forall T \geq 0.$$

Extending Definition 4 to a more general numerical method in the natural way, we then have the following general result.

Theorem 9 *If Condition 8 holds, then the assertion of Theorem 6 follows for a numerical method that has the flow property.*

Motivated by Theorem 9, in the next section we give a class of methods that have the flow property and obey Condition 8.

6 Stochastic theta method

6.1 Definition

In this section we introduce the class of stochastic theta methods (STMs) and show that they fit into the framework of Theorem 9. We note that establishing strong finite-time convergence (Lemma 10) in this hybrid setting is of interest in its own right.

Given a step size $\Delta > 0$, with $X_0 = x_0$ and $r_0^{\Delta} = r_0$ the STM is defined for $k = 0, 1, 2, \ldots$ by

$$X_{k+1} = X_k + [(1-\theta) f(X_k, r_k^{\Delta}) + \theta f(X_{k+1}, r_k^{\Delta})]\Delta + g(X_k, r_k^{\Delta})\Delta B_k, \tag{35}$$

where $\theta \in [0, 1]$ is a fixed parameter. Note that with the choice $\theta = 0$, (35) reduces to the EM method. In this case we have an explicit equation that defines X_{k+1}. However, (35) generally represents a nonlinear system that is to be solved for X_{k+1} given X_k. The following lemma concerns the existence of a solution to the implicit equation.

Lemma 7 *Under Assumption 5, if Δ is sufficiently small that $\Delta\theta\sqrt{K} < 1$, then equation (35) can be solved uniquely for X_{k+1} given X_k, with probability* 1.

Proof Define, for $u \in \mathbb{R}^n$

$$F(u) = X_k + [(1-\theta) f(X_k, r_k^\Delta) + \theta f(u, r_k^\Delta)]\Delta + g(X_k, r_k^\Delta)\Delta B_k.$$

Then (35) can be written as $X_{k+1} = F(X_{k+1})$. Using (19), we have

$$|F(u) - F(v)| = |\theta f(u, r_k^\Delta)\Delta - \theta f(v, r_k^\Delta)\Delta| \le \theta\Delta\sqrt{K}|u - v|, \quad \forall u, v \in \mathbb{R}^n.$$

By the classical Banach contraction mapping theorem [17], $F(u)$ has a unique fixed point, which is X_{k+1}. □

6.2 Linear stability of the stochastic theta method

In this subsection we make a slight digression in order to study the linear stability behaviour of the stochastic theta method. Applied to (5) the method (35) gives

$$X_{k+1} = X_k + (1-\theta)\Delta\mu(r_k^\Delta)X_k + \theta\Delta\mu(r_k^\Delta)X_{k+1} + \sigma(r_k^\Delta)X_k\Delta B_k.$$

In the case of a single SDE, where $N = 1$, and $\mu(1) = \mu$ and $\sigma(1) = \sigma$ can be regarded as constants, if $1 - \theta\Delta\mu \neq 0$ then after rearranging, squaring, and taking expectations we have

$$\mathbb{E}\left[X_k^2\right] = \left(\frac{(1 + (1-\theta)\Delta\mu)^2 + \Delta\sigma^2}{(1 - \theta\Delta\mu)^2}\right)\mathbb{E}\left[X_{k-1}^2\right].$$

It follows immediately that

$$\lim_{k\to\infty} \mathbb{E}\left[X_k^2\right] = 0 \quad \Longleftrightarrow \quad \frac{(1 + (1-\theta)\Delta\mu)^2 + \Delta\sigma^2}{(1 - \theta\Delta\mu)^2} < 1.$$

As shown in [6,15,16] we may conclude that for $\theta \ge \frac{1}{2}$ the method has the "A-stability" property

$$\text{problem stable} \quad \Longrightarrow \quad \text{method stable for all } \Delta,$$

and for $\theta = \frac{1}{2}$ we have perfect stability/instability preservation

$$\text{problem stable} \quad \Longleftrightarrow \quad \text{method stable for all } \Delta.$$

For the general hybrid version (5), however, the theta method cannot maintain this excellent behaviour. For simplicity, consider a two-state ($N = 2$) problem where $\gamma_{12} = \gamma_{21} > 0$. In this case the stationary distribution has $\pi_1 = \pi_2 = \frac{1}{2}$. If we let

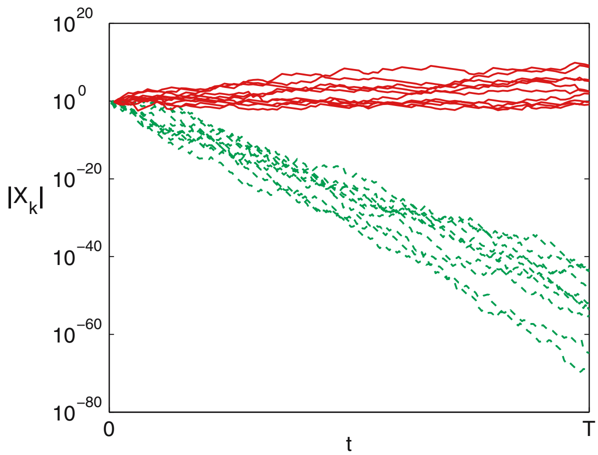

Fig. 1 Light curves *lower* 10 paths of $|X_k|$ for the $\theta = \frac{1}{2}$ method on the stable 2-state problem (5) with $x_0 = 1$, $\gamma_{12} = \gamma_{21} = 1$, $\mu_1 = -1$, $\sigma_1 = 0$, $\mu_2 = 0$ and $\sigma_2 = 1$, using step size $\Delta = 4$. Dark curves *upper* as above with $\Delta = 10$

$\mu_1 = -1$, $\sigma_1 = 0$, $\mu_2 = 0$ and $\sigma_2 = 2$, so that state 1 corresponds to deterministic exponential decay and state 2 corresponds to exponential Brownian motion, then

$$\sum_{j=1}^{2} \pi_j (2\mu_j + \sigma_j^2) = -\tfrac{1}{2}.$$

So by Theorem 1 we have $\lim_{t\to\infty} \mathbb{E}[x(t)^2] = 0$. However, analysis similar to that in the proof of Theorem 2 shows that for the theta method with $\theta = \frac{1}{2}$

$$\lim_{k\to\infty} \mathbb{E}\left[X_k^2\right] = 0 \quad \Longleftrightarrow \quad \tfrac{1}{2}\log\left(\frac{(1-\frac{1}{2}\Delta)^2}{(1+\frac{1}{2}\Delta)^2}\right) + \tfrac{1}{2}\log(1+\Delta) < 0,$$

and the condition on the right simplifies to

$$\frac{(1-\frac{1}{2}\Delta)^2(1+\Delta)}{(1+\frac{1}{2}\Delta)^2} < 1.$$

This inequality holds for sufficiently small Δ (and such behaviour could also be deduced from the analysis in Sect. 6.3) but fails for $\Delta \geq\approx 4.8$. In summary, there is a stable problem of the form (5) for which the $\theta = \frac{1}{2}$ method loses stability for sufficiently large Δ, showing that the A-stability property does not carry through.

We illustrate this behaviour in Fig. 1. Note that the vertical axis is scaled logarithmically. Here, with $x_0 = 1$ we computed 10 paths of the $\theta = \frac{1}{2}$ numerical solution over $[0, 10^3]$. The light curves (lower) show $|X_k|$ for $\Delta = 4$ and the dark curves (upper) for $\Delta = 10$. The results are consistent with a change from mean-square stability to mean-square instability.

6.3 Nonlinear systems

In the remainder of this section, we always let Δ be sufficiently small for the stochastic theta method to be well defined. Let us now define the continuous approximation by

$$X(t) = x_0 + \int_0^t [(1-\theta) f(z_1(s), \bar{r}(s)) + \theta f(z_2(s), \bar{r}(s))] ds + \int_0^t g(z_1(s), \bar{r}(s)) dB(s), \tag{36}$$

where

$$z_1(t) = X_k,\ z_2(t) = X_{k+1} \text{ and } \bar{r}(t) = r_k^{\Delta} \text{ for } t \in [k\Delta, (k+1)\Delta).$$

Note that $X(k\Delta) = X_k$, and hence $X(t)$ is an interpolant to the discrete stochastic theta method solution. We also note that $z_1(k\Delta) = z_2((k-1)\Delta) = X_k$.

It is clear that the stochastic theta method defined in this way has the flow property. Working towards a proof that Condition 8 is satisfied, we now develop some second moment bounds.

Lemma 8 *Under (20), for all sufficiently small Δ ($< 1/(2+2K)$ at least), the continuous approximation $X(t)$ defined by (36) satisfies*

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^2 \le \alpha_T \mathbb{E}|x_0|^2, \quad \forall T \ge 0, \tag{37}$$

where $\alpha_T = 3 + 12K(T+1)e^{2(3+4K)(T+1)}$.

Proof It follows from (35) that

$$\mathbb{E}|X_{k+1}|^2 = \mathbb{E}|X_k|^2 + 2\mathbb{E}\left(X_k^T[(1-\theta) f(X_k, r_k^{\Delta}) + \theta f(X_{k+1}, r_k^{\Delta})]\Delta\right) + \mathbb{E}\Big|[(1-\theta) f(X_k, r_k^{\Delta}) + \theta f(X_{k+1}, r_k^{\Delta})]\Delta + g(X_k, r_k^{\Delta})\Delta B_k\Big|^2.$$

By the elementary inequalities

$$2u^T v \le |u|^2 + |v|^2 \quad \text{and} \quad |(1-\theta)u + \theta v|^2 \le |u|^2 + |v|^2, \quad \forall u, v \in \mathbb{R}^n,$$

as well as (20), we then compute

$$\mathbb{E}|X_{k+1}|^2 \le \mathbb{E}|X_k|^2 + \Delta\mathbb{E}\left[((1-\theta)^2+\theta^2)|X_k|^2 + |f(X_k, r_k^{\Delta})|^2 + |f(X_{k+1}, r_k^{\Delta})|^2)\right] + 2\mathbb{E}\left(|f(X_k, r_k^{\Delta})|^2\Delta^2 + |f(X_{k+1}, r_k^{\Delta})|^2\Delta^2 + |g(X_k, r_k^{\Delta})|^2|\Delta B_k|^2\right)$$

$$\leq \mathbb{E}|X_k|^2 + \Delta\mathbb{E}\left[|X_k|^2 + K|X_k|^2 + K|X_{k+1}|^2)\right]$$
$$+2K\mathbb{E}\left(|X_k|^2\Delta^2 + |X_{k+1}|^2\Delta^2 + |X_k|^2\Delta\right)$$
$$\leq \mathbb{E}|X_k|^2 + (2+3K)\Delta\mathbb{E}|X_k|^2 + (1+K)\Delta\mathbb{E}|X_{k+1}|^2, \tag{38}$$

where we have noted that $2K\Delta < 1$. Let M be any positive integer such that $M \leq \text{int}(T/\Delta) + 1$. Summing the inequality above for k from 0 to $M-1$, we obtain

$$\mathbb{E}|X_M|^2 \leq \mathbb{E}|X_0|^2 + (2+3K)\Delta\sum_{k=0}^{M-1}\mathbb{E}|X_k|^2 + (1+K)\Delta\sum_{k=0}^{M-1}\mathbb{E}|X_{k+1}|^2$$
$$\leq \mathbb{E}|x_0|^2 + (3+4K)\Delta\sum_{k=0}^{M-1}\mathbb{E}|X_k|^2 + (1+K)\Delta\mathbb{E}|X_M|^2.$$

Noting that $(1+K)\Delta \leq 1/2$, we have

$$\mathbb{E}|X_M|^2 \leq 2\mathbb{E}|x_0|^2 + 2(3+4K)\Delta\sum_{k=0}^{M-1}\mathbb{E}|X_k|^2.$$

Using the discrete Gronwall inequality (see, for example, [12]) and recalling that $M\Delta \leq T+1$, we obtain

$$\mathbb{E}|X_M|^2 \leq 2\mathbb{E}|x_0|^2 e^{2(3+4K)\Delta M} \leq \bar{\alpha}_T\mathbb{E}|x_0|^2,$$

where $\bar{\alpha}_T = 2e^{2(3+4K)(T+1)}$. Recalling the definitions of $z_1(t)$ and $z_2(t)$ we see

$$\sup_{0\leq t\leq T}\mathbb{E}|z_j(t)|^2 \leq \bar{\alpha}_T\mathbb{E}|x_0|^2, \quad j = 1, 2. \tag{39}$$

It can be shown easily from (36) and (20) that, for $0 \leq t \leq T$,

$$\mathbb{E}|X(t)|^2 \leq 3\mathbb{E}|x_0|^2 + 3K(T+1)\int_0^t[\mathbb{E}|z_1(s)|^2 + \mathbb{E}|z_2(s)|^2]ds.$$

By (39) we have

$$\mathbb{E}|X(t)|^2 \leq [3 + 6K\bar{\alpha}_T(T+1)]\mathbb{E}|x_0|^2, \quad \forall t \in [0, T],$$

which is the required assertion (37). □

Lemma 9 *Under* (20)*, for all sufficiently small Δ ($< 1/(4+6K)$ at least),*

$$\mathbb{E}|X_{k+1}|^2 \leq 2\mathbb{E}|X_k|^2, \quad \forall k \geq 0.$$

Proof It follows from (38) that

$$\mathbb{E}|X_{k+1}|^2 \le \mathbb{E}|X_k|^2 + \frac{1}{2}\mathbb{E}|X_k|^2 + \frac{1}{4}\mathbb{E}|X_{k+1}|^2$$

and hence the assertion follows.

Lemma 10 *Under (20), for all sufficiently small Δ ($< 1/(4+6K)$ at least), the continuous approximation $X(t)$ defined by (36) satisfies*

$$\sup_{0\le t\le T}\left\{\mathbb{E}|X(t)-z_1(t)|^2 \vee \mathbb{E}|X(t)-z_2(t)|^2\right\} \le 2(K+1)\Delta \sup_{0\le t\le T}\mathbb{E}|X(t)|^2, \quad (40)$$

for all $T > 0$.

□

Proof Given any $0 \le t \le T$, let $k = \text{int}(T/\Delta)$, so $k\Delta \le t < (k+1)\Delta$. It follows from (36) that

$$\begin{aligned} X(t) - z_1(t) = &\left[(1-\theta)f(X_k, r_k^\Delta) + \theta f(X_{k+1}, r_k^\Delta)\right](t - k\Delta) \\ &+ g(X_k, r_k^\Delta)[B(t) - B(k\Delta)], \end{aligned} \quad (41)$$

and

$$\begin{aligned} z_2(t) - X(t) = &\left[(1-\theta)f(X_k, r_k^\Delta) + \theta f(X_{k+1}, r_k^\Delta)\right]((k+1)\Delta - t) \\ &+ g(X_k, r_k^\Delta)[B((k+1)\Delta) - B(t)]. \end{aligned} \quad (42)$$

By (20) and Lemma 9, we compute from (41) that

$$\begin{aligned} \mathbb{E}|X(t) - z_1(t)|^2 &\le 2\Delta^2 K(\mathbb{E}|X_k|^2 + \mathbb{E}|X_{k+1}|^2) + 2\Delta K\mathbb{E}|X_k|^2 \\ &\le (6\Delta^2 K + 2\Delta K)\mathbb{E}|X_k|^2 \\ &\le (2K+1)\Delta \sup_{0\le t\le T}\mathbb{E}|X(t)|^2, \end{aligned}$$

where we have used the condition that $6\Delta K < 1$. Similarly, we can show the same upper bound for $\mathbb{E}|z_2(t) - X(t)|^2$ and hence the assertion (40) follows. □

Lemma 11 *If (20) holds, then for all sufficiently small Δ,*

$$\mathbb{E}\int_0^T |\varphi(z_1(s), r(s)) - \varphi(z_1(s), \bar{r}(s))|^2 ds \le \beta_T \Delta \sup_{0\le t\le T}\mathbb{E}|z_1(t)|^2 \quad (43)$$

and

$$\mathbb{E}\int_0^T |f(z_2(s), r(s)) - f(z_2(s), \bar{r}(s))|^2 ds \le \beta_T \Delta \sup_{0\le t\le T}\mathbb{E}|z_2(s)|^2 \quad (44)$$

for any $T > 0$, where φ is either f or g and β_T is defined in Lemma 3.

Proof Assertion (43) can be proved in the same way that Lemma 3 was proved because $z_1(t)$ is an $\mathcal{F}_t$-adapted step process like $\bar{X}(t)$. However, $z_2(t)$ is not $\mathcal{F}_t$-adapted so assertion (44) requires a more careful treatment.

Let $j = \mathrm{int}(T/\Delta)$. Then

$$\mathbb{E}\int_0^T |f(z_2(s), \bar{r}(s)) - f(z_2(s), r(s))|^2 ds$$
$$= \sum_{k=0}^{j} \mathbb{E}\int_{t_k}^{t_{k+1}} |f(X_{k+1}, r(t_k)) - f(X_{k+1}, r(s))|^2 ds, \tag{45}$$

with t_{j+1} being now set to T. By (20), it is easy to show that

$$\mathbb{E}\int_{t_k}^{t_{k+1}} |f(X_{k+1}, r(t_k)) - f(X_{k+1}, r(s))|^2 ds$$
$$\leq 2K \int_{t_k}^{t_{k+1}} \mathbb{E}\left[|X_{k+1}|^2 I_{\{r(s)\neq r(t_k)\}}\right] ds. \tag{46}$$

But, by the Markov property,

$$\mathbb{E}\left[|X_{k+1}|^2 I_{\{r(s)\neq r(t_k)\}}\right]$$
$$= \int_{\mathbb{R}^n} \sum_{i\in\mathbb{S}} \mathbb{E}\left[|X_{k+1}|^2 I_{\{r(s)\neq i\}} \big| X_k = x, r(t_k) = i\right] \mathbb{P}\{X_k = dx, r(t_k) = i\}.$$

Given that $X_k = x$ and $r(t_k) = i$, we see from (35) that

$$X_{k+1} = x + [(1-\theta)f(x,i) + \theta f(X_{k+1}, i)]\Delta + g(x,i)\Delta B_k, \tag{47}$$

whence X_{k+1} depends on ΔB_k which is independent of the Markov chain. In other words, X_{k+1} and $I_{\{r(s)\neq i\}}$ are independent given $X_k = x$ and $r(t_k) = i$. Hence

$$\mathbb{E}\left[|X_{k+1}|^2 I_{\{r(s)\neq r(t_k)\}}\right] = \int_{\mathbb{R}^n} \sum_{i\in\mathbb{S}} \mathbb{E}\left[|X_{k+1}|^2 \big| X_k = x, r(t_k) = i\right]$$
$$\times \mathbb{P}\{r(s) \neq i \big| X_k = x, r(t_k) = i\} \mathbb{P}\{X_k = dx, r(t_k) = i\}. \tag{48}$$

We compute that

$$\mathbb{P}\{r(s) \neq i \big| X_k = x, r(t_k) = i\} = \frac{\mathbb{P}\{r(s) \neq i, X_k = x, r(t_k) = i\}}{\mathbb{P}\{X_k = x, r(t_k) = i\}} = \frac{\mathbb{P}\{r(s) \neq i, X_k = x | r(t_k) = i\}}{\mathbb{P}\{X_k = x | r(t_k) = i\}}. \tag{49}$$

Noting that given $r(t_k) = i$, the events $r(s) \neq i$ and $X_k = x$ are independent, we have

$$\mathbb{P}\{r(s) \neq i, X_k = x | r(t_k) = i\} = \mathbb{P}\{r(s) \neq i | r(t_k) = i\}\mathbb{P}\{X_k = x | r(t_k) = i\}.$$

Putting this into (49) and then recalling (27) we obtain

$$\mathbb{P}\{r(s) \neq i \big| X_k = x, r(t_k) = i\} = \mathbb{P}\{r(s) \neq i | r(t_k) = i\} \leq \hat{\gamma}\Delta. \tag{50}$$

Using this in (48) yields

$$\begin{aligned}\mathbb{E}\left[|X_{k+1}|^2 I_{\{r(s)\neq r(t_k)\}}\right] &\leq \hat{\gamma}\Delta \int_{\mathbb{R}^n} \sum_{i\in\mathbb{S}} \mathbb{E}\left[|X_{k+1}|^2 \big| X_k = x, r(t_k) = i\right] \\ &\qquad \mathbb{P}\{X_k = dx, r(t_k) = i\} \\ &= \hat{\gamma}\Delta\mathbb{E}|X_{k+1}|^2.\end{aligned}$$

Substituting this into (46) implies

$$\mathbb{E}\int_{t_k}^{t_{k+1}} |f(X_{k+1}), r(t_k)) - f(X_{k+1}), r(s))|^2 ds \leq 2K\hat{\gamma}\Delta^2\mathbb{E}|X_{k+1}|^2.$$

Using this in (45) we obtain

$$\begin{aligned}\mathbb{E}\int_0^T |f(z_2(s), \bar{r}(s)) - f(z_2(s), r(s))|^2 ds &\leq 2K\hat{\gamma}\Delta^2 \sum_{k=0}^{j} \mathbb{E}|X_{k+1}|^2 \\ &\leq 2KT\hat{\gamma}\Delta \sup_{0\leq t\leq T} \mathbb{E}|z_2(t)|^2,\end{aligned}$$

which is the required assertion (44). □

We are now in a position to establish Condition 8.

Theorem 10 *Under* (2) *and Assumption* 5 *the stochastic theta method defined by* (35) *with continuous extension* (36) *satisfies Condition* 8. *Since this method also has the flow property, Theorem* 9 *applies.*

Proof The bound (33) is given by Lemma 8. Hence, it remains to show (34).

It follows from (1) and (36) that for any $0 \le t \le T$,

$$X(t) - x(t) = \int_0^t ((1-\theta)[f(z_1(s), \bar{r}(s)) - f(x(s), r(s))] \\ +\theta[f(z_2(s), \bar{r}(s)) - f(x(s), r(s))])\, ds \\ + \int_0^t (g(z_1(s), \bar{r}(s)) - g(x(s), r(s)))\, dB(s).$$

We hence compute that

$$\begin{aligned}
\mathbb{E}|X(t) - x(t)|^2 \le 2T\mathbb{E} \int_0^t \Big(& |f(z_1(s), \bar{r}(s)) - f(x(s), r(s))|^2 \\
& + |f(z_2(s), \bar{r}(s)) - f(x(s), r(s))|^2 \Big)\, ds \\
& + 2\mathbb{E} \int_0^t |g(z_1(s), \bar{r}(s)) - g(x(s), r(s))|^2 ds \\
& \le 4TK\mathbb{E} \int_0^t \Big(|z_1(s) - x(s)|^2 + |z_2(s) - x(s)|^2 \Big)\, ds \\
& + 4K\mathbb{E} \int_0^t |z_1(s) - x(s)|^2 ds + J(T) \\
& \le J(T) + 8K(2T+1) \int_0^t \mathbb{E}|X(s) - x(s)|^2 ds \\
& + 8K(T+1) \int_0^t \Big(\mathbb{E}|X(s) - z_1(s)|^2 + \mathbb{E}|X(s) - z_2(s)|^2 \Big)\, ds \\
& \le J(T) + 8K(2T+1) \int_0^t \mathbb{E}|X(s) - x(s)|^2 ds \\
& + 32K(K+1)(T+1)T\Delta \sup_{0 \le t \le T} \mathbb{E}|X(t)|^2,
\end{aligned}$$

where Lemma 10 has been used in the last step while

$$J(T) := 4T\mathbb{E} \int_0^T \Big(|f(z_1(s), \bar{r}(s)) - f(z_1(s), r(s))|^2$$

$$+|f(z_2(s),\bar{r}(s)) - f(z_2(s),r(s))|^2\Big)\,ds$$
$$+4\mathbb{E}\int_0^T |g(z_1(s),\bar{r}(s)) - g(z_1(s),r(s))|^2 ds.$$

But, by Lemma 11

$$J(T) \le 4(T+1)\beta_T\Delta\left(\sup_{0\le t\le T}\mathbb{E}|z_1(t)|^2\right) + 4T\beta_T\Delta\left(\sup_{0\le t\le T}\mathbb{E}|z_2(t)|^2\right).$$

However, clearly

$$\sup_{0\le t\le T}\mathbb{E}|z_1(t)|^2 \le \sup_{0\le t\le T}\mathbb{E}|X(t)|^2$$

while, by Lemma 9,

$$\sup_{0\le t\le T}\mathbb{E}|z_2(t)|^2 \le 2\sup_{0\le t\le T}\mathbb{E}|z_1(t)|^2.$$

So

$$J(T) \le 4(3T+1)\beta_T\Delta\left(\sup_{0\le t\le T}\mathbb{E}|X(t)|^2\right).$$

We therefore have

$$\mathbb{E}|X(t)-x(t)|^2 \le \bar{C}_T\Delta\left(\sup_{0\le t\le T}\mathbb{E}|X(t)|^2\right)$$
$$+8K(2T+1)\int_0^t \mathbb{E}|X(s)-x(s)|^2 ds,$$

where $\bar{C}_T = 4(3T+1)\beta_T + 32K(K+1)(T+1)T$. An application of the continuous Gronwall inequality gives a bound of the form

$$\mathbb{E}|X(t)-x(t)|^2 \le \left(\sup_{0\le t\le T}\mathbb{E}|X(t)|^2\right)C_T\Delta,$$

where $C_T = \bar{C}_T e^{8K(2T+1)T}$. Since this is true for any $t \in [0,T]$, the result follows. □

7 Discussion

Our aim in this work was to give some rigorous justification for the numerical simulation of regime switching SDE models. We have added to the existing literature [20] by (a) showing that nonlinear exponential mean square stability can be preserved for small step sizes and (b) showing that stable, convergent implicit methods exist. The numerical analysis of this important problem class is still in its infancy, and hence many open equations remain regarding issues such as: quantifying the benefits of implicitness, searching for analogues of A-stability and deriving customized methods for particular applications.

A Appendix: Proofs of Lemmas 5 and 6

Proof of Lemma 5 Set $T = [\mathrm{int}(4\log M/(\lambda\Delta)) + 1]\Delta$, so that $4\log M/\lambda \le T \le 4\log M/\lambda + 1$ and

$$Me^{-\lambda T} \le e^{-\frac{3}{4}\lambda T}. \tag{51}$$

For any $\alpha > 0$,

$$\mathbb{E}|X(t)|^2 \le (1+\alpha)\mathbb{E}|X(t) - x(t)|^2 + (1 + 1/\alpha)\mathbb{E}|x(t)|^2. \tag{52}$$

So, using Lemma 4,

$$\sup_{0\le t\le 2T} \mathbb{E}|X(t)|^2 \le (1+\alpha) \sup_{0\le t\le 2T} \mathbb{E}|X(t)|^2 C_{2T}\Delta + (1+1/\alpha)M\mathbb{E}|x_0|^2.$$

For Δ sufficiently small, this rearranges to

$$\sup_{0\le t\le 2T} \mathbb{E}|X(t)|^2 \le \frac{(1+1/\alpha)M\mathbb{E}|x_0|^2}{1-(1+\alpha)C_{2T}\Delta}. \tag{53}$$

Now, taking the supremum over $[T, 2T]$ in (52), using Lemma 4 and the bound (53), and also the stability condition (17), gives

$$\sup_{T\le t\le 2T} \mathbb{E}|X(t)|^2 \le \sup_{T\le t\le 2T} \mathbb{E}|X(t)|^2 \le R(\Delta)\mathbb{E}|x_0|^2, \tag{54}$$

where

$$R(\Delta) := \frac{(1+\alpha)(1+1/\alpha)}{1-(1+\alpha)C_{2T}\Delta} C_{2T}\Delta M + (1+1/\alpha)Me^{-\lambda T}.$$

Taking $\alpha = 1/\sqrt{\Delta}$ and using (51) we see that for sufficiently small Δ

$$R(\Delta) \le 2\sqrt{\Delta}C_{2T}M + (1+\sqrt{\Delta})e^{-\frac{3}{4}\lambda T}.$$

By taking Δ sufficiently small we may ensure that

$$R(\Delta) \leq e^{-\frac{1}{2}\lambda T}, \tag{55}$$

which, in (54), gives

$$\sup_{T \leq t \leq 2T} \mathbb{E}|X(t)|^2 \leq e^{-\frac{1}{2}\lambda T}\mathbb{E}|x_0|^2 \leq e^{-\frac{1}{2}\lambda T} \sup_{0 \leq t \leq T} \mathbb{E}|X(t)|^2.$$

Now, let $\hat{x}(t)$ be the solution to the SDE (1) for $t \in [T, \infty)$ with the initial value $\hat{x}(T) = X(T)$ and with the Markov chain starting from $r(T)$ at $t = T$ (so no change in the Markov chain); that is,

$$d\hat{x}(t) = X(T) + \int_T^t f(\hat{x}(s), r(s))ds + \int_T^t g(\hat{x}(s), r(s))dB(s), \quad t \geq T. \tag{56}$$

This is the same as the SDE (1) except the time is switched by T, so shifting (17) we obtain

$$\mathbb{E}|\hat{x}(t)|^2 \leq M\mathbb{E}|X(T)|^2 e^{-\lambda(t-T)}, \quad \forall t \geq T. \tag{57}$$

On the other hand, (16) gives

$$X(t) = X(T) + \int_T^t f(\bar{X}(s), \bar{r}(s))ds + \int_T^t g(\bar{X}(s), \bar{r}(s))dB(s), \quad t \geq T. \tag{58}$$

Since T is a multiple of Δ, we see that $X(t)$ on $t \geq T$ is the continuous EM approximate solution to equation (56). Hence, applying Lemma 4 and the time shift, we have that

$$\sup_{T \leq t \leq 3T} \mathbb{E}|X(t) - \hat{x}(t)|^2 \leq \left(\sup_{0 \leq t \leq T} \mathbb{E}|X(t)|^2\right) C_{2T}\Delta, \quad \forall t \geq T. \tag{59}$$

Then, analogously to (54), we have

$$\sup_{2T \leq t \leq 3T} \mathbb{E}|X(t)|^2 \leq R(\Delta)\mathbb{E}|X(T)|^2.$$

Continuing this approach and using (55) gives

$$\sup_{(i+1)T \leq t \leq (i+2)T} \mathbb{E}|X(t)|^2 \leq e^{-\frac{1}{2}\lambda T}\mathbb{E}|X(iT)|^2, \quad \text{for } i \geq 0,$$

and hence

$$\begin{aligned}\sup_{(i+1)T\le t\le(i+2)T}\mathbb{E}|X(t)|^2 &\le e^{-\frac{1}{2}\lambda T}e^{-\frac{1}{2}\lambda T}\sup_{(i-1)T\le t\le iT}\mathbb{E}|X(t)|^2\\ &\vdots\\ &\le e^{-\frac{1}{2}\lambda T(i+1)}\sup_{0\le t\le T}\mathbb{E}|X(t)|^2. \end{aligned}\tag{60}$$

With $\alpha=1/\sqrt{\Delta}$ in (53), for sufficiently small Δ we see that

$$\sup_{0\le t\le T}\mathbb{E}|X(t)|^2\le 2M\mathbb{E}|x_0|^2.\tag{61}$$

From (60) and (61) we have

$$\begin{aligned}\sup_{(i+1)T\le t\le(i+2)T}\mathbb{E}|X(t)|^2 &\le e^{-\frac{1}{2}\lambda T(i+1)}2M\mathbb{E}|x_0|^2\\ &=2Me^{\frac{1}{2}\lambda T}\mathbb{E}|x_0|^2e^{-\frac{1}{2}\lambda T(i+2)}\\ &\le 2Me^{\frac{1}{2}\lambda T_1}\mathbb{E}|x_0|^2e^{-\frac{1}{2}\lambda T(i+2)},\end{aligned}$$

where $T_1=1+(4\log M)/\lambda\ge T$, and the result follows. $\square$

Proof of Lemma 6 Using Lemma 4 and (18) we have, for any $\alpha>0$,

$$\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2\le\sup_{T\le t\le 2T}(1+\alpha)\mathbb{E}|X(t)-x(t)|^2+(1+1/\alpha)\mathbb{E}|X(t)|^2\tag{62}$$

$$\begin{aligned}&\le(1+\alpha)\sup_{T\le t\le 2T}\mathbb{E}|X(t)-x(t)|^2+(1+1/\alpha)\sup_{T\le t\le 2T}\mathbb{E}|X(t)|^2\\ &\le(1+\alpha)C_{2T}\Delta\sup_{0\le t\le 2T}\mathbb{E}|X(t)|^2+(1+1/\alpha)\sup_{T\le t\le 2T}\mathbb{E}|X(t)|^2\\ &\le(1+\alpha)C_{2T}\Delta H\mathbb{E}|x_0|^2+(1+1/\alpha)H\mathbb{E}|x_0|^2e^{-\gamma T}\\ &\le\left[(1+\alpha)C_{2T}\Delta e^{\gamma T}+(1+1/\alpha)\right]H\mathbb{E}|x_0|^2e^{-\gamma T}.\end{aligned}\tag{63}$$

Taking $\alpha=1/\sqrt{\Delta}$ gives

$$\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2\le\left[C_{2T}e^{\gamma T}(\Delta+\sqrt{\Delta})+1+\sqrt{\Delta}\right]H\mathbb{E}|x_0|^2e^{-\gamma T}.\tag{64}$$

Since $e^{-\frac{3}{4}\gamma T}H\le e^{-\frac{1}{2}\gamma T}$, using (32) we then have

$$\sup_{T\le t\le 2T}\mathbb{E}|x(t)|^2\le e^{-\frac{3}{4}\gamma T}H\mathbb{E}|x_0|^2\le e^{-\frac{1}{2}\gamma T}\mathbb{E}|x_0|^2\le e^{-\frac{1}{2}\gamma T}\sup_{0\le t\le T}\mathbb{E}|x(t)|^2.\tag{65}$$

Now let $\hat{X}(t)$ for $t \in [T, \infty)$ denote the continuous EM approximation that arises from applying the EM method with initial data $x(T)$ and $r(T)$ at time $t = T$. Then, analogously to (65),

$$\sup_{2T \le t \le 3T} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}\gamma T} \sup_{T \le t \le 2T} \mathbb{E}|x(t)|^2.$$

Continuing these arguments we may show that

$$\sup_{iT \le t \le (i+1)T} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}\gamma T} \sup_{(i-1)T \le t \le iT} \mathbb{E}|x(t)|^2, \quad i \ge 1,$$

and so,

$$\sup_{iT \le t \le (i+1)T} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}\gamma iT} \sup_{0 \le t \le T} \mathbb{E}|x(t)|^2. \tag{66}$$

Now, using (32),

$$\begin{aligned}\sup_{0 \le t \le T} \mathbb{E}|x(t)|^2 &\le \sup_{0 \le t \le T} \mathbb{E}|X(t) - x(t)|^2 + \sup_{0 \le t \le T} \mathbb{E}|X(t)|^2 \\ &\le (C_T \Delta + 1) H \mathbb{E}|x_0|^2 \\ &\le 2H\mathbb{E}|x_0|^2.\end{aligned}$$

In (66) this gives

$$\sup_{iT \le t \le (i+1)T} \mathbb{E}|x(t)|^2 \le e^{-\frac{1}{2}\gamma(i+1)T} e^{\frac{1}{2}\gamma T} 2H\mathbb{E}|x_0|^2,$$

which completes the proof. □

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