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ASYMPTOTIC STABILITY OF A JUMP-DIFFUSION EQUATION AND ITS NUMERICAL APPROXIMATION*

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Abstract. Asymptotic linear stability is studied for stochastic differential equations (SDEs) that incorporate Poisson-driven jumps and their numerical simulations using theta-method discretizations. The property is shown to have a simple explicit characterization for the SDE, whereas for the discretization a condition is found that is amenable to numerical evaluation. This allows us to evaluate the asymptotic stability behavior of the methods. One surprising observation is that there exist problem parameters for which an explicit, forward Euler method has better stability properties than its trapezoidal and backward Euler counterparts. Other computational experiments indicate that all theta methods reproduce the correct asymptotic linear stability for sufficiently small step sizes. By using a recent result of Appleby, Berkolaiko, and Rodkina, we give a rigorous verification that both linear stability and instability are reproduced for small step sizes. This property is known not to hold for general, nonlinear problems.

Key words. asymptotic stability, backward Euler, Euler–Maruyama, jump-diffusion, Poisson process, stochastic differential equation, theta method, trapezoidal rule

AMS subject classifications. 65C30, 60H35

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1. Introduction.

Stability is an important property in any time-stepping scenario. For stochastic differential equations (SDEs), two very natural, but distinct, concepts are mean-square and asymptotic stabilities. Mean-square stability is more amenable to analysis, and hence this property dominates in the literature [3, 16, 21]. Asymptotic stability has received some attention in the case of nonjump SDEs [2, 16, 15, 20]. However, in the jump-SDE context, which is becoming increasingly important in mathematical finance [4, 7, 8, 6, 11, 12, 17, 19, 22], we are only aware of mean-square results [13, 14]. This motivates the work in this article, where asymptotic stability is studied for jump SDEs.

Our test model has the linear, scalar form

$$\mathrm{d}X(t) = \mu X(t^-)\,\mathrm{d}t + \sigma X(t^-)\,\mathrm{d}W(t) + \gamma X(t^-)\,\mathrm{d}N(t)\ , \quad X(0) = X_0, \tag{1.1}$$

for $t > 0$, where $X_0 \neq 0$ with probability one. We use $X(t^-)$ to denote $\lim_{s\uparrow t^-} X(s)$. Here, for $t \geq 0$, $W(t)$ is a scalar Brownian motion and $N(t)$ is a scalar Poisson process (independent of W) with jump intensity λ, both defined on an appropriate complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right-continuous, while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets); see [4, 8]. In addition to λ, this model involves three other constants: μ is the drift coefficient, σ is the diffusion coefficient, and γ is the jump coefficient. We assume throughout that $\lambda > 0$ and $\gamma \neq 0$ (if $\gamma = 0$, the problem reduces to a nonjump SDE). We may view the problem (1.1) in terms of the exponentially distributed jump times of the Poisson process. Between each jump, the solution evolves according to the nonjump version $\mathrm{d}X(t) = \mu X(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}W(t)$. At a jump time, the solution

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gets an instantaneous kick, and $X(t)$ is replaced by $(1+\gamma)X(t)$. For $\gamma > 0$ or $\gamma < -2$, this has the effect of increasing the solution size, and for $-2 < \gamma < 0$, the solution size is decreased.

The class (1.1) is important in its own right as a model in mathematical finance [4, 8, 19], but here we are using it as a natural extension to the linear test problem that has proved valuable in the analysis of numerical methods for ODEs [10] and SDEs [2, 3, 16, 20, 21]. It is known that (1.1) has the solution

$$X(t) = X_0 \, (1+\gamma)^{N(t)} \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t)\right]; \tag{1.2}$$

see, for example, [4, 5, 8].

2. Model stability. Following the standard definitions for nonjump SDEs (see, for example, [18]), given parameters μ, σ, γ, and λ, we define the *trivial solution* (alternatively *zero solution* or *equilibrium solution*) of the jump SDE (1.1) to be *stochastically asymptotically stable in the large* (hereafter, *asymptotically stable*) if it is stable in probability and, moreover, for all $X_0 \in \mathbb{R}$

$$\lim_{t\to\infty} |X(t)| = 0, \quad \text{with probability 1.} \tag{2.1}$$

We now give a lemma that characterizes asymptotic stability in terms of the problem parameters.

LEMMA 2.1. *Suppose $\gamma \neq -1$ in (1.1); then*

$$\lim_{t\to\infty} |X(t)| = 0 \ , \quad \text{with probability 1} \quad \Longleftrightarrow \quad \mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| < 0. \tag{2.2}$$

Proof. Taking the logarithms in (1.2) gives

$$\log|X(t)| = \log|X_0| + \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma W(t) + N(t)\log|1+\gamma|. \tag{2.3}$$

We know that

$$\lim_{t\to\infty} \frac{W(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{N(t)}{t} = \lambda \ , \quad \text{with probability 1,}$$

by the law of the iterated logarithm [18] and the strong law of large numbers [9]. Hence,

$$\lim_{t\to\infty} \frac{1}{t} \log|X(t)| = \mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| \ , \quad \text{with probability 1.} \tag{2.4}$$

We consider separately the cases where $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma|$ is positive, negative, and zero.

Case 1. For $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| < 0$, it follows immediately from (2.4) that $\log|X(t)| \to -\infty$ and thus $|X(t)| \to 0$ as $t \to \infty$, and so the zero solution is asymptotically stable.

Case 2. Similarly, for $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| > 0$, it follows immediately from (2.4) that $|X(t)| \to \infty$.

Case 3. For $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| = 0$, we return to (2.3) and introduce the compensated Poisson process $\widetilde{N}(t) := N(t) - \lambda t$ so that (2.3) becomes

$$\log|\widehat{X}(t)| = \sigma W(t) + \widetilde{N}(t)\log|1+\gamma|,$$

where $\widehat{X}(t) = X(t)/X_0$.

We note that W and $\widetilde{N}$ are independent and also that

$$\mathbb{E}\left[\sigma W(t) + \widetilde{N}(t)\log|1+\gamma|\right] = 0$$

and

$$\mathrm{Var}\left[\sigma W(t) + \widetilde{N}(t)\log|1+\gamma|\right] = \left(\sigma^2 + \lambda\big(\log|1+\gamma|\big)^2\right)t.$$

By choosing $\Delta = \frac{1}{\sigma^2+\lambda(\log|1+\gamma|)^2}$, we can construct the sequence

$$\xi_n = \log\left|\widehat{X}(n\Delta)\right| - \log\left|\widehat{X}\big((n-1)\Delta\big)\right|, \quad n \geq 1,$$

where the ξ_n are independent and identically distributed with mean 0 and variance 1. We can now apply the law of the iterated logarithm to $S_n = \sum_{i=1}^{n} \xi_i$, giving

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n\log\log n}} = 1\right] = 1,$$

which implies that

$$\mathbb{P}\left[\lim_{t\to\infty} \log|X(t)| = -\infty\right] = 0.$$

Hence, the zero solution is not asymptotically stable in this case. ∎

In the exceptional case where $\gamma = -1$, a jump kills the solution, so we have

$$X(t) = X_0 \exp\left[\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W(t)\right] \cdot \mathbf{1}_{\{N(t)=0\}}, \quad t \geq 0,$$

where $\mathbf{1}_A$ denotes the indicator function for A. So $\mathbb{P}[X(t) = 0] \geq 1 - e^{-\lambda t}$, and we conclude that, for any μ, σ, and λ, $\lim_{t\to\infty}|X(t)| = 0$, with probability one. We note that the condition (2.2) in Lemma 2.1 could be regarded as applying in the $\gamma = -1$ case if we view $\log(0)$ as $-\infty$.

We also note that the jump coefficient γ appears in (2.2) in the form $|1+\gamma|$, a term which is symmetric about $\gamma = -1$. This follows from the fact that the stability definition (2.1) involves only the modulus of the solution, and, in this sense, the effect of a jump with $\gamma = -1 + a$ is the same as for a jump with $\gamma = -1 - a$.

The stability characterization $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| < 0$ involves four parameters and hence is difficult to visualize. In Figure 1 we focus on the effect of the jump parameters λ and γ. Here, we have contoured the function $\lambda\log|1+\gamma|$. Any particular contour in the plot corresponds to a combination of fixed choices of μ and σ, the value of which is calculated as $\frac{1}{2}\sigma^2 - \mu$. For instance, a choice of $\mu = 1$ and $\sigma = 2$ would correspond to the contour at "height" 1. This contour represents the border between the stable region and the unstable one. If we focus on those pairs (μ, σ) corresponding to the contour at 1, we can see that a choice of $\lambda = 0.5, \gamma = 4$ (represented in Figure 1 by $\times$) yields an asymptotically stable equilibrium solution, whereas a choice of $\lambda = 0.75, \gamma = 4$ (represented in Figure 1 by $+$) would yield an unstable equilibrium solution.

In essence, crossing from above a contour to below it is equivalent to moving from an unstable zero solution to a stable one for a particular fixed choice of μ and σ by varying λ and/or γ.

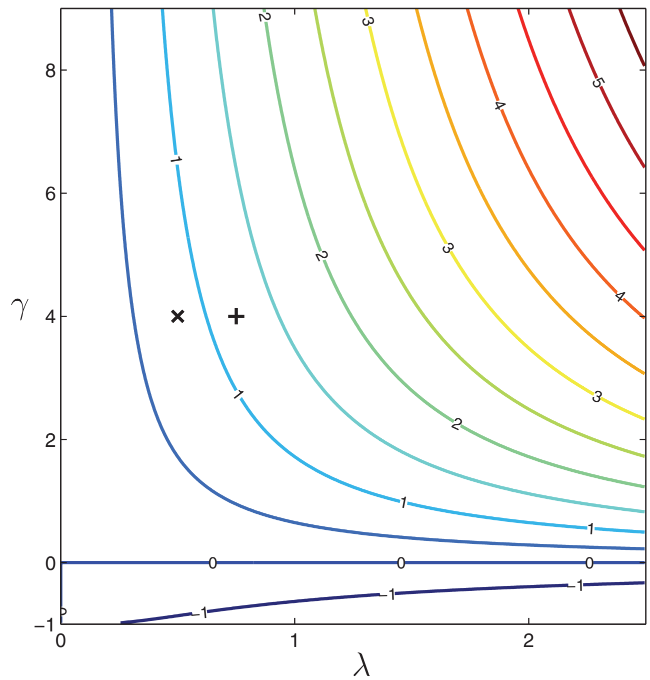

FIG. 1. *Contour plot of* $\lambda \log|1+\gamma|$ *illustrating the asymptotic stability of the trivial solution of* (1.1). *Markers* $\times$ *and* $+$ *represent stable and unstable, respectively, choices of the pair* (λ, γ) *for a fixed pair* (μ, σ) *corresponding to* $\frac{1}{2}\sigma^2 - \mu = 1$.

The broad features of the plot are intuitively reasonable. For $\gamma > 0$, increasing either the jump coefficient γ or the jump intensity λ makes the problem less stable. On the other hand, for $-1 < \gamma < 0$, where a jump reduces the solution magnitude, increasing the jump frequency λ makes the problem more stable. For $\gamma = 0$, we revert to the condition $\mu - \frac{1}{2}\sigma^2 < 0$ for the nonjump SDE. Figure 1 only shows the case $\gamma \geq -1$ because of the underlying symmetry that we mentioned earlier.

3. Theta-method stability. A generalization of the theta method to jump SDEs was introduced in [14] and studied in terms of strong convergence and linear mean-square stability, with further results for nonlinear problems appearing in [13]. Applied to the test equation (1.1) the method takes the form

$$Y_{n+1} = Y_n + (1-\theta)\mu Y_n \,\Delta t + \theta\mu Y_{n+1}\,\Delta t + \sigma Y_n\,\Delta W_n + \gamma Y_n\,\Delta N_n, \tag{3.1}$$

with $Y_0 = X_0$. Here, $Y_n \approx X(t_n)$, with $t_n = n\Delta t$, $\Delta W_n = W(t_{n+1}) - W(t_n)$ is the Brownian increment, $\Delta N_n = N(t_{n+1}) - N(t_n)$ is the Poisson increment, and $\theta \in [0,1]$ is a fixed parameter. We suppose that the step size Δt is fixed. For the implicit case $\theta > 0$, we require $\theta\mu\Delta t \neq 1$ in order for the method to be well defined. Given θ and Δt, we may write the recurrence (3.1) in the form

$$(1-\theta\mu\Delta t)Y_{n+1} = \big(1 + (1-\theta)\mu\,\Delta t + \sigma\sqrt{\Delta t}\,\xi_n + \gamma\,\Delta N_n\big)Y_n, \tag{3.2}$$

where the ξ_n are independent standard normal random variables and the ΔN_n are independent Poisson random variables with mean $\lambda\Delta t$ and variance $\lambda\Delta t$.

By analogy with the SDE definition (2.1), given parameters μ, σ, λ, and γ and values for θ and Δt, we say that the theta method is asymptotically stable if $\lim_{n\to\infty} |Y_n| = 0$, with probability one, for any X_0.

Using [16, Lemma 3.1], which is essentially an application of the strong law of large numbers, we find that a necessary and sufficient condition for asymptotic stability of the numerical method (3.2) is

$$\mathbb{E}\left[\log\left|\frac{1}{1-\theta\mu\,\Delta t}\Big(1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi_i+\gamma\,\Delta N_i\Big)\right|\right]<0. \tag{3.3}$$

Hence, the stability issue involves the expected value of the logarithm of a linear combination of independent normal and Poisson random variables. We are not aware of any useful analytical expression for this quantity.

To gain some computational insight, we may rearrange (3.3) into the form

$$\mathbb{E}\Big[\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma\,\Delta N\big|\Big]-\log\big|1-\theta\mu\,\Delta t\big|$$

and expand over the possible values of ΔN to get

$$\begin{aligned}
&\mathbb{E}\Big[\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma\,\Delta N\big|\Big]\\
&\quad=\sum_{k=0}^{\infty}\mathbb{P}\big(\Delta N_i=k\big)\mathbb{E}\Big[\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma k\big|\Big]\\
&\quad=\frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\int_{\mathbb{R}}\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,x+\gamma k\big|e^{-\frac{x^2}{2}}\,\mathrm{d}x\\
&\quad\simeq\frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\sum_{k=0}^{K}\frac{(\lambda\Delta t)^k}{k!}\int_{-R}^{R}\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,x+\gamma k\big|e^{-\frac{x^2}{2}}\,\mathrm{d}x\\
&\quad\simeq\frac{e^{-\lambda\Delta t}}{\sqrt{2\pi}}\Delta x\sum_{k=0}^{K}\frac{(\lambda\Delta t)^k}{k!}\left(\sum_{j=0}^{J}\log\big|1+(1-\theta)\mu\,\Delta t+\sigma\sqrt{\Delta t}\,x_j+\gamma k\big|\exp\left(-\frac{x_j^2}{2}\right)\right).
\end{aligned}$$

Here, we truncated the infinite sum to the range $0\le k\le K$, truncated each infinite integral to the range $-R\le x\le R$, and then applied a simple quadrature approximation to each integral, using a spacing Δx, with $J=\frac{2R}{\Delta x}-1$, $x_0=-R$, and $x_{j+1}=x_j+\Delta x$.

The plots in Figure 2 were produced with $K=10$, $R=10$, and $\Delta x=0.0004$. In each case, for fixed values of $\mu=0.25$ and $\sigma=0.5$, we show the range of γ and λ values for which the theta method is stable. Computations are given for $\theta=0,\ 0.25,\ 0.5,\ 0.75$, and 1. For reference the contour for the underlying test problem (as given in Figure 1) is also shown. The three pictures correspond to step sizes $\Delta t=0.1,\ 0.01$, and 0.001. The pictures suggest that varying theta has little effect on the asymptotic stability properties and also that all theta methods will reproduce the correct asymptotic stability for sufficiently small Δt. In section 4 we give a rigorous proof of the latter property.

The surface plot in Figure 3 gives another view, showing the expected value on the left-hand side of (3.3) for the fixed values $\mu=1$, $\sigma=2$, $\lambda=1.5$, and $\gamma=0.25$ as a function of θ and Δt. Here, $\mu-\frac{1}{2}\sigma^2+\lambda\log|1+\gamma|=-0.66$, so, by Lemma 2.1, the problem is stable. The black contour line, highlighted underneath the surface, shows where the expected value in (3.3) is zero. This is the critical value where

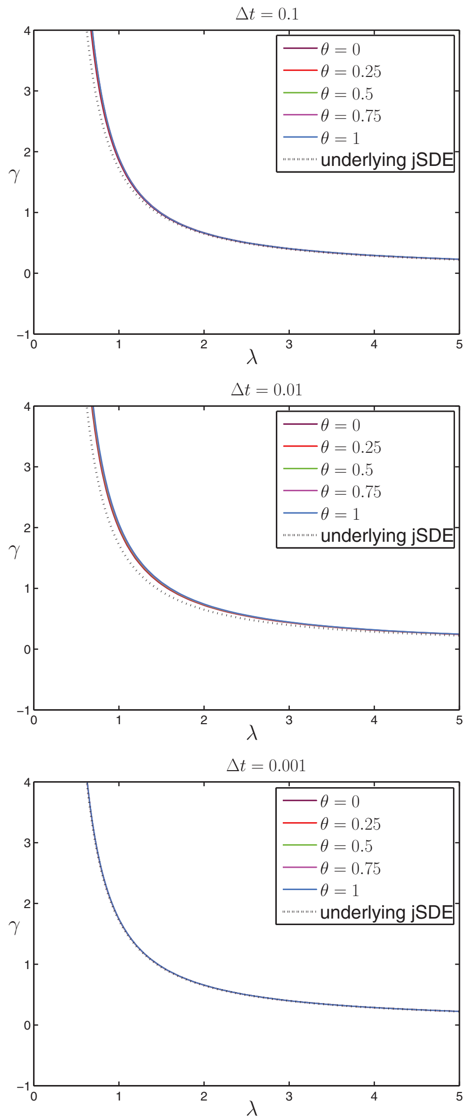

FIG. 2. *Asymptotic stability boundaries for the theta methods and the underlying jump-SDE (jSDE) zero solution, with fixed $\mu = 0.25$ and $\sigma = 0.5$.*

the method moves from instability to stability. The contour indicates that for these problem parameters the stability behavior, measured as the range of Δt values that reproduce asymptotic stability, is best for $\theta = 0$ and gets uniformly worse as θ increases. This effect is at odds with the behavior seen for deterministic problems [10] and for mean-square stability on SDEs and jump SDEs [16, 14, 21]. To confirm this visual observation, Table 3.1 computes the expected value in (3.3) in two different ways: one by the quadrature technique and the other by the Monte Carlo technique (with 95% confidence intervals shown), for $\theta = 0,\ 0.5$, and 1 with $\Delta t = 0.18$. Note that $\theta = 0.5$ corresponds to a generalization of the trapezoidal rule for ODEs. We see that the expected value increases with θ and that $\theta = 0$ yields a stable method, whereas $\theta = 1$ does not. As a final check, Figure 4 shows one path for each of the three

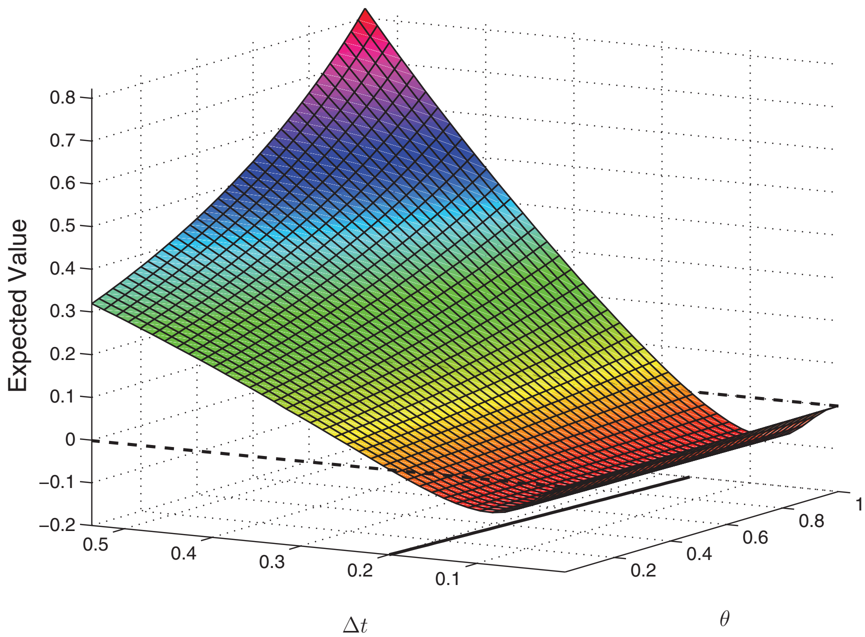

FIG. 3. *Left-hand side of* (3.3) *as a function of* θ *and* Δt*, illustrating conditions for asymptotic stability of the theta method* (3.1).

TABLE 3.1
Comparison of approximations of the expected value in the left-hand side of (3.3) *by the quadrature and the Monte Carlo simulations.*

$\Delta t = 0.18$	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
Quadrature	-0.0203	-0.0043	0.0188
Monte Carlo	-0.0156 ± 0.0082	-0.0027 ± 0.0086	0.0163 ± 0.0090

methods, with the vertical axis scaled logarithmically. The behavior for $\theta = 0$ and $\theta = 0.5$ is clearly consistent with asymptotic stability. For $\theta = 1$, the lower picture, which covers a longer time scale, reveals the asymptotic instability.

4. Euler–Maruyama for small step size.

The nonlinear SDE $\mathrm{d}X(t) = (X(t) - X(t)^3)\,\mathrm{d}t + 2X(t)\,\mathrm{d}W(t)$, with deterministic initial data, was studied in [15]. For this problem, $\limsup_{t\to\infty} \frac{1}{t} \log |X(t)| \le -1$, with probability one. However, given any arbitrarily small Δt, we can find deterministic initial data for which, with positive probability, the Euler–Maruyama solutions blow up at a geometric rate. This motivated a study of small step size asymptotic stability. It was shown in [15] that on linear, scalar SDEs, the theta method will preserve asymptotic stability for all sufficiently small Δt. In this section we extend this result to the case of the jump SDE (1.1). Further, we simultaneously cover both the stable and unstable regimes, obtaining positive results in both cases. The analysis makes use of a recent result by Appleby, Berkolaiko, and Rodkina [1].

For convenience, we focus on the $\theta = 0$ or extended Euler–Maruyama method for jump SDEs. As we show in Corollary 5.1, the result then extends readily to general θs.

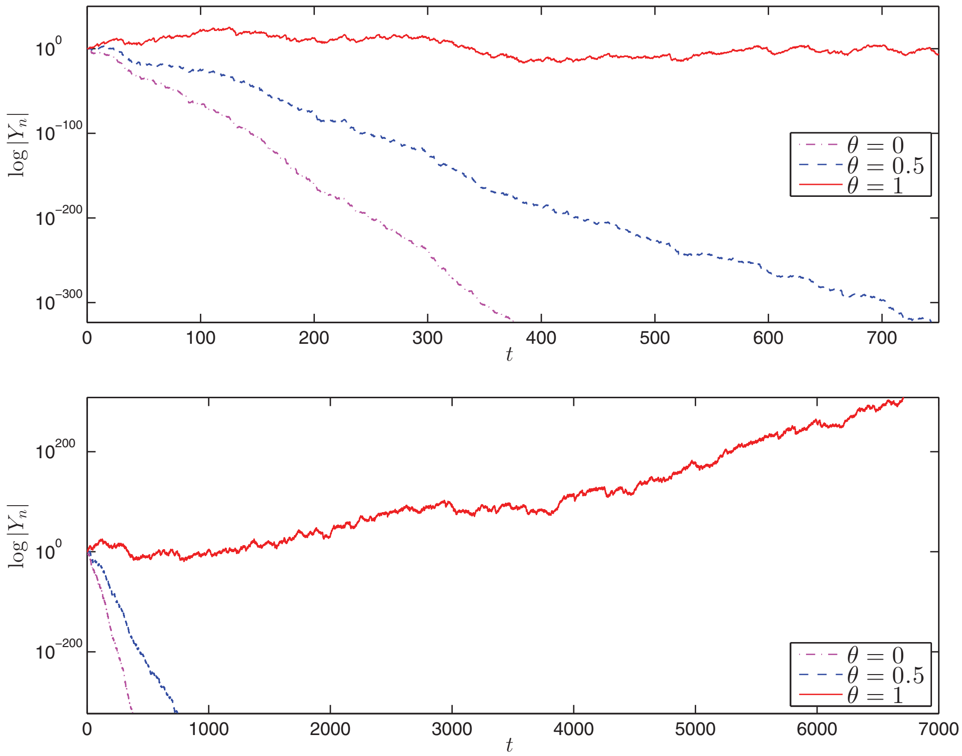

FIG. 4. *Medium (upper) and long (lower) time trajectories with fixed $\Delta t = 0.18$ showing asymptotic stability for $\theta = 0$ and 0.5 and instability for $\theta = 1$.*

With $\theta = 0$ the recurrence (3.1) reduces to

$$Y_{n+1} = Y_n(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi_n + \gamma\Delta N_n). \tag{4.1}$$

Lemma 3.1 of [16] then gives a necessary and sufficient condition for asymptotic stability of the form

$$\mathbb{E}\Big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi + \gamma\Delta N|\Big] < 0, \tag{4.2}$$

where ξ is standard normal and ΔN is Poisson with parameter $\lambda\Delta t$, respectively.

We make use of part of [1, Theorem 5] in the proof of Theorem 4.2. For completeness, we state this result here.

LEMMA 4.1 (Appleby, Berkolaiko, and Rodkina [1]). *Let ξ be a random variable with bounded third moment and density monotonically decreasing at $\pm\infty$, and let ψ be an integrable function on $\mathbb{R}$ which is $C^3\big((1-\delta, 1+\delta)\big)$. Then, for $\mu, \sigma \in \mathbb{R}$ and $\Delta t \to 0$, we have*

$$\mathbb{E}\Big[\psi(1 + \mu\Delta t + \sigma\sqrt{\Delta t}\xi)\Big] = \psi(1) + \psi'(1)\mu\Delta t + \frac{1}{2}\psi''(1)\sigma^2\Delta t + o(\Delta t). \tag{4.3}$$

THEOREM 4.2. *Given μ, σ, γ, and λ such that $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| < 0$ so that, by Lemma 2.1, the jump SDE (1.1) is asymptotically stable, there exists a $\Delta t^\star = \Delta t^\star(\mu, \sigma, \gamma, \lambda)$ such that the Euler–Maruyama method (4.1) is asymptotically stable for all $0 < \Delta t < \Delta t^\star$.*

Conversely, given μ, σ, γ, and λ such that $\mu - \frac{1}{2}\sigma^2 + \lambda \log|1+\gamma| > 0$ so that, by Lemma 2.1, *the jump SDE* (1.1) *is not asymptotically stable, there exists a $\Delta t^\star = \Delta t^\star(\mu,\sigma,\gamma,\lambda)$ such that the Euler–Maruyama method* (4.1) *is not asymptotically stable for any $0 < \Delta t < \Delta t^\star$.*

Proof. Multiplying the expected value in (4.2) by $e^{\lambda\Delta t}$ for convenience and expanding, we get

$$\begin{aligned}
&e^{\lambda\Delta t}\,\mathbb{E}\Big[\log\big|1+\mu\Delta t+\sigma\sqrt{\Delta t}\xi+\gamma\Delta N\big|\Big]\\
&\quad=\sum_{k=0}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\Big[\log\big|1+\gamma k+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big]\\
&\quad=\mathbb{E}\Big[\log\big|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big]\\
&\qquad+\lambda\Delta t\,\mathbb{E}\Big[\log\big|1+\gamma+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big]\\
&\qquad+\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\Big[\log\big|1+\gamma k+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\big|\Big].
\end{aligned}\tag{4.4}$$

We now consider three distinct cases, depending on the value of γ.

Case 1 ($\gamma \neq -1/k$). First, we deal with the generic case where $\gamma \neq -1/k$ for any integer $k \geq 1$. In this case, we may write (4.4) as

$$\begin{aligned}
&e^{\lambda\Delta t}\,\mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi+\gamma\Delta N|\Big]\\
&\quad=\mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\Big]\\
&\qquad+\lambda\Delta t\Big(\log|1+\gamma|+\mathbb{E}\Big[\log|1+\hat\mu\Delta t+\hat\sigma\sqrt{\Delta t}\,\xi|\Big]\Big)\\
&\qquad+\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\log|1+\mu\Delta t+\gamma k|\\
&\qquad+\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\,\mathbb{E}\Big[\log|1+r_k\,\xi|\Big],
\end{aligned}\tag{4.5}$$

where $\hat\mu = \frac{\mu}{1+\gamma}$, $\hat\sigma = \frac{\sigma}{1+\gamma}$, and $r_k = \frac{\sigma\sqrt{\Delta t}}{1+\mu\Delta t+\gamma k}$, for $k = 2, 3, \ldots$, and, for sufficiently small Δt, there is no issue of "division by zero" or "log of zero."

Now, using Lemma 4.1 with $\psi(\cdot) \equiv \log(\cdot)$, we find that

$$\mathbb{E}\Big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\Big] = \Big(\mu-\frac{1}{2}\sigma^2\Big)\Delta t + o(\Delta t) \tag{4.6}$$

and

$$\lambda\Delta t\Big(\log|1+\gamma|+\mathbb{E}\Big[\log|1+\hat\mu\Delta t+\hat\sigma\sqrt{\Delta t}\,\xi|\Big]\Big) = \lambda\Delta t\log|1+\gamma| + O(\Delta t^2). \tag{4.7}$$

By restricting Δt to, say, $\Delta t \leq \frac{1}{2}$, we may choose a constant K_1 such that $|\gamma K_1| \geq 1+\mu\Delta t$, and hence $|1+\mu\Delta t+\gamma k| \leq |2\gamma k K_1|$. Then

$$\log|1+\mu\Delta t+\gamma k| \leq \log|2\gamma k K_1| = \log|2\gamma K_1| + \log k$$

for $k \geq 2$. Furthermore, there exists some $\widehat{k} \geq 2$ such that $|1+\mu\Delta t+\gamma k| > 1$ for $k > \widehat{k}$.

We then have

$$
\begin{aligned}
&\left|\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\log|1+\mu\Delta t+\gamma k|\right| \\
&\le \left|\sum_{k=2}^{\widehat{k}}\frac{(\lambda\Delta t)^k}{k!}\log|1+\mu\Delta t+\gamma k|\right| + \left|\sum_{k=\widehat{k}+1}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\log|1+\mu\Delta t+\gamma k|\right| \\
&\le \sum_{k=2}^{\widehat{k}}\frac{(\lambda\Delta t)^k}{k!}\Big|\log|1+\mu\Delta t+\gamma k|\Big| + \sum_{k=\widehat{k}+1}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\log|1+\mu\Delta t+\gamma k| \\
&\le (\lambda\Delta t)^2\left(\sum_{k=2}^{\widehat{k}}\frac{(\lambda\Delta t)^{k-2}}{k!}\Big|\log|1+\mu\Delta t+\gamma k|\Big|\right. \\
&\qquad\qquad \left. + \sum_{k=\widehat{k}+1}^{\infty}\frac{(\lambda\Delta t)^{k-2}}{k!}\log|1+\mu\Delta t+\gamma k|\right) \\
&= (\lambda\Delta t)^2\left(K_2\widehat{k} + \sum_{k=\widehat{k}+1}^{\infty}\frac{(\lambda\Delta t)^{k-2}}{k!}\Big(\log|2\gamma K_1|+\log k\Big)\right) \\
&\le \lambda^2\Delta t^2\left(K_2\widehat{k}+\log|2\gamma K_1|\sum_{k=\widehat{k}+1}^{\infty}\frac{(\lambda\Delta t)^{k-2}}{k!}+\sum_{k=\widehat{k}+1}^{\infty}\frac{(\lambda\Delta t)^{k-2}}{k!}\log k\right) \qquad (4.8) \\
&= \Big(K_2\widehat{k}+\log|2\gamma K_1|K_3+K_4\Big)\lambda^2\Delta t^2 \\
&= O(\Delta t^2). \qquad (4.9)
\end{aligned}
$$

Here, $K_2 = \max_{\Delta t\le\frac{1}{2},\ 2\le k\le\widehat{k}} |\log|1+\mu\Delta t+\gamma k||(\lambda\Delta t)^{k-2}/(k!)$, and, taking Δt to satisfy $\lambda\Delta t<1$, constants K_3, K_4 are bounds (uniform in Δt) for the two convergent infinite series in (4.8).

To bound the final term in (4.5), we note that

$$
\begin{aligned}
\left|\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\,\mathbb{E}\Big[\log|1+r_k\,\xi|\Big]\right| &= \left|\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^k}{k!}\cdot\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\log|1+r_k\,x|\,e^{-x^2/2}\,\mathrm{d}x\right| \\
&= \frac{1}{\sqrt{2\pi}}(\lambda\Delta t)^2\left|\sum_{k=2}^{\infty}\frac{(\lambda\Delta t)^{k-2}}{k!}F(r_k)\right|, \qquad (4.10)
\end{aligned}
$$

where $F(r_k)=\int_{\mathbb{R}}\log|1+r_k\,x|\,e^{-x^2/2}\,\mathrm{d}x$. Making the substitution $r_{k+1}\,x=r_k\,y$, we have

$$
F(r_{k+1}) = \int_{\mathbb{R}}\log|1+r_k\,y|\exp\left(-\left(\frac{r_k}{r_{k+1}}\right)^2\frac{y^2}{2}\right)\cdot\frac{r_k}{r_{k+1}}\,\mathrm{d}y.
$$

Noting that $r_k/r_{k+1} > 1$ and taking absolute values, we find

$$\begin{aligned} |F(r_{k+1})| &= \left|\frac{r_k}{r_{k+1}}\right| \left|\int_{\mathbb{R}} \log|1 + r_k\, y| \exp\left(-\left(\frac{r_k}{r_{k+1}}\right)^2 \frac{y^2}{2}\right) \mathrm{d}y\right| \\ &\le \left|\frac{r_k}{r_{k+1}}\right| \left|\int_{\mathbb{R}} \log|1 + r_k\, y| \exp\left(-\frac{y^2}{2}\right) \mathrm{d}y\right| \\ &= \left|\frac{r_k}{r_{k+1}}\right| |F(r_k)|. \end{aligned}$$

Hence,

$$\frac{|F(r_{k+1})|}{|F(r_k)|} \le \left|\frac{r_k}{r_{k+1}}\right|.$$

We can now examine the convergence of the infinite series in (4.10). If we set

$$a_k = \left|\frac{(\lambda\Delta t)^{k-2} F(r_k)}{k!}\right|,$$

then

$$\begin{aligned} \frac{a_{k+1}}{a_k} &= \left|\frac{\lambda\Delta t\, F(r_{k+1})}{(k+1)F(r_k)}\right| \\ &\le \left|\frac{\lambda\Delta t}{k+1} \cdot \frac{r_k}{r_{k+1}}\right| \\ &= \left|\frac{\lambda\Delta t\,(1 + \mu\Delta t + \gamma(k+1))}{(k+1)(1 + \mu\Delta t + \gamma k)}\right| \quad \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

Hence, the series in (4.10) is absolutely convergent, and we have

$$\left|\sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \mathbb{E}\Big[\log|1 + r_k\, \xi|\Big]\right| = O(\Delta t^2). \tag{4.11}$$

Using (4.6), (4.7), (4.9), and (4.11) in (4.5) gives

$$e^{\lambda\Delta t}\, \mathbb{E}\Big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi + \gamma\Delta N|\Big] = \Big(\mu - \frac{1}{2}\sigma^2 + \lambda\log|1 + \gamma|\Big)\Delta t + o(\Delta t).$$

It follows that for sufficiently small Δt and $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma| \ne 0$, the sign of $\mathbb{E}[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi + \gamma\Delta N|]$ matches the sign of $\mu - \frac{1}{2}\sigma^2 + \lambda\log|1+\gamma|$; so by Lemma 2.1 and (4.2) the result follows.

Case 2 ($\gamma = -1$). When $\gamma = -1$, we know that the problem (1.1) is asymptotically stable for all values of μ, σ, and λ. Hence, we must show that the numerical method has the same property for all sufficiently small Δt.

In this case, (4.4) becomes

$$\begin{aligned} e^{\lambda\Delta t}\, \mathbb{E}\big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi - \Delta N|\big] &= \mathbb{E}\big[\log|1 + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi|\big] \\ &\quad + \lambda\Delta t\, \mathbb{E}\big[\log|\mu\Delta t + \sigma\sqrt{\Delta t}\,\xi|\big] \\ &\quad + \sum_{k=2}^{\infty} \frac{(\lambda\Delta t)^k}{k!} \mathbb{E}\big[\log|1 - k + \mu\Delta t + \sigma\sqrt{\Delta t}\,\xi|\big]. \end{aligned} \tag{4.12}$$

To analyze the second term in the expansion of (4.12), we write

$$\mathbb{E}\big[\log|\mu\Delta t + \sigma\sqrt{\Delta t}\xi|\big] = \log(\sqrt{\Delta t}) + \mathbb{E}\big[\log|\mu\sqrt{\Delta t} + \sigma\xi|\big],$$

and so

$$\mathbb{E}\big[\log|\mu\Delta t + \sigma\sqrt{\Delta t}\xi|\big] - \frac{1}{2}\log\Delta t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\log|\mu\sqrt{\Delta t} + \sigma x|e^{-x^2/2}\,\mathrm{d}x. \tag{4.13}$$

Now choosing some constant $K_\delta = \sigma(1+\delta)$, $0 < \delta < 1$, we have $\log|K_\delta\, x| \geq \log|\mu\sqrt{\Delta t} + \sigma x|$ for $x \in \big(-\infty, c_1\sqrt{\Delta t}\big] \cup \big[c_2\sqrt{\Delta t}, \infty\big)$, where

$$(c_1, c_2) = \begin{cases} \Big(-\mu/(\sigma - K_\delta), -\mu/(\sigma + K_\delta)\Big), & \mu < 0, \\ \Big(-\mu/(\sigma + K_\delta), -\mu/(\sigma - K_\delta)\Big), & \mu > 0. \end{cases}$$

Note that as $K_\delta > \sigma$, we have $c_1 \leq 0, c_2 \geq 0\ \forall\mu \in \mathbb{R}$. So splitting the integral up in the natural way, taking absolute values, and applying the triangle inequality, we have

$$\begin{aligned}\left|\int_{-\infty}^{\infty}\log|\mu\sqrt{\Delta t} + \sigma x|e^{-x^2/2}\,\mathrm{d}x\right| \leq &\left|\int_{-\infty}^{c_1\sqrt{\Delta t}}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right| \\ &+ \left|\int_{c_2\sqrt{\Delta t}}^{\infty}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right| \\ &+ \left|\int_{c_1\sqrt{\Delta t}}^{c_2\sqrt{\Delta t}}\log|\mu\sqrt{\Delta t} + \sigma x|e^{-x^2/2}\,\mathrm{d}x\right|.\end{aligned} \tag{4.14}$$

We deal with the first two integrals in (4.14) in the same manner. Using the triangle inequality we have

$$\left|\int_{-\infty}^{c_1\sqrt{\Delta t}}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right| \leq \left|\int_{-\infty}^{0}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right| + \left|\int_{c_1\sqrt{\Delta t}}^{0}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right|.$$

The first term on the right-hand side has an analytical expression. For the second term, we use $e^{-x^2/2} \leq 1$ so that

$$\begin{aligned}\left|\int_{c_1\sqrt{\Delta t}}^{0}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right| &\leq \left|\int_{c_1\sqrt{\Delta t}}^{0}\log|K_\delta\, x|\,\mathrm{d}x\right| \\ &= \left|\int_{c_1\sqrt{\Delta t}}^{0}\log(-K_\delta\, x)\,\mathrm{d}x\right| \\ &= \big|c_1\sqrt{\Delta t}\big(1 - \log(-K_\delta\, c_1\sqrt{\Delta t})\big)\big| \\ &\leq \sqrt{\Delta t}|c_1|\left(1 + |\log K_\delta| + |\log(-c_1)| + \frac{1}{2}|\log\Delta t|\right).\end{aligned}$$

So we have

$$\begin{aligned}\left|\int_{-\infty}^{c_1\sqrt{\Delta t}}\log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right| \leq &\frac{\sqrt{2\pi}}{4}\left(\epsilon + \left|\log\frac{2}{K_\delta^2}\right|\right) \\ &+ \sqrt{\Delta t}|c_1|\left(1 + |\log K_\delta| + |\log(-c_1)| + \frac{1}{2}|\log\Delta t|\right),\end{aligned}$$

where $\epsilon = -\int_0^\infty e^{-t}\log t\,\mathrm{d}t = \lim_{n\to\infty}\left(\sum_{k=1}^n \frac{1}{k} - \log n\right)$ is Euler's constant. Similarly,

$$\left|\int_{c_2\sqrt{\Delta t}}^{\infty} \log|K_\delta\, x| e^{-x^2/2}\,\mathrm{d}x\right| \le \frac{\sqrt{2\pi}}{4}\left(\epsilon + \left|\log\frac{2}{K_\delta^2}\right|\right) + \sqrt{\Delta t}\, c_2\left(1 + |\log K_\delta| + |\log c_2| + \frac{1}{2}|\log\Delta t|\right).$$

Taking $c_3 = \max(|c_1|, c_2)$, both integrals may therefore be bounded by

$$\max\left(\left|\int_{-\infty}^{c_1\sqrt{\Delta t}} \log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right|, \left|\int_{c_2\sqrt{\Delta t}}^{\infty} \log|K_\delta\, x|e^{-x^2/2}\,\mathrm{d}x\right|\right)$$
$$\le \frac{\sqrt{2\pi}}{4}\left(\epsilon + \left|\log\frac{2}{K_\delta^2}\right|\right) + \sqrt{\Delta t}\, c_3\left(1 + |\log K_\delta| + \left|\log c_3\right| + \frac{1}{2}\left|\log\Delta t\right|\right). \tag{4.15}$$

For the third component of (4.14), we note that our choice of K_δ means we avoid a "log of zero" over the interval $[c_1\sqrt{\Delta t}, c_2\sqrt{\Delta t}]$, and therefore we may bound this definite integral in modulus as

$$\left|\int_{c_1\sqrt{\Delta t}}^{c_2\sqrt{\Delta t}} \log|\mu\sqrt{\Delta t} + \sigma x|e^{-x^2/2}\,\mathrm{d}x\right| \le \left|\int_{c_1\sqrt{\Delta t}}^{c_2\sqrt{\Delta t}} \log|\mu\sqrt{\Delta t} + \sigma x|\,\mathrm{d}x\right| = \left|K_5\sqrt{\Delta t} + K_6\sqrt{\Delta t}\log\Delta t\right|,$$

where

$$K_5 = \frac{1}{\sigma}\Big((\mu+\sigma c_2)\big(\log|\mu+\sigma c_2| - 1\big) - (\mu+\sigma c_1)\big(\log|\mu+\sigma c_1| - 1\big)\Big),$$
$$K_6 = -\frac{K_\delta|\mu|}{\sigma^2 - K_\delta^2},$$

independent of Δt.

Since $\Delta t < 1$, we have $|\log\Delta t| = -\log\Delta t$, and so, using the bounds (4.15) in (4.14) and (4.13), we find that

$$\left|\mathbb{E}\big[\log|\mu\Delta t + \sigma\sqrt{\Delta t}\xi|\big] - \frac{1}{2}\log\Delta t\right| \le K_7$$

for some constant K_7 independent of Δt. Now the first term on the right-hand side of (4.12) was shown to be $O(\Delta t)$ in (4.6), and the third term can be shown to be $O(\Delta t^2)$ using the same technique that we used for the infinite series in Case 1. Hence, we conclude that, for all small Δt, $|e^{\lambda\Delta t}\mathbb{E}[\log|1+\mu\Delta t + \sigma\sqrt{\Delta t}\xi - \Delta N|] - \frac{1}{2}\log\Delta t|$ is uniformly bounded, showing that $\mathbb{E}[1+\log|\mu\Delta t + \sigma\sqrt{\Delta t}\xi - \Delta N|]$ is negative for small Δt, as required.

Case 3 ($\gamma = -1/k^\star$, for $k^\star \in \mathbb{N},\ k^\star > 1$). In this third case, (4.4) can be expanded as

$$\begin{aligned} &e^{\lambda\Delta t}\,\mathbb{E}\left[\log\left|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi-\frac{\Delta N}{k^\star}\right|\right] \\ &\quad= \mathbb{E}\big[\log|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big] \\ &\qquad+\lambda\Delta t\,\mathbb{E}\left[\log\left|1-\frac{1}{k^\star}+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\right|\right] \\ &\qquad+\frac{(\lambda\Delta t)^{k^\star}}{(k^\star)!}\mathbb{E}\big[\log|\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi|\big] \\ &\qquad+\sum_{k\neq k^\star}\frac{(\lambda\Delta t)^k}{k!}\mathbb{E}\left[\log\left|1-\frac{k}{k^\star}+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi\right|\right]. \end{aligned}$$

The first term on the right-hand side is dealt with by (4.6). The remaining terms can be analyzed using the arguments developed for Cases 1 and 2 in order to show that

$$e^{\lambda\Delta t}\,\mathbb{E}\left[\log\left|1+\mu\Delta t+\sigma\sqrt{\Delta t}\,\xi-\frac{\Delta N}{k^\star}\right|\right] = \left(\mu-\frac{1}{2}\sigma^2+\lambda\log\left|1-\frac{1}{k^\star}\right|\right)\Delta t+o(\Delta t),$$

and so the asymptotic stability result follows from Lemma 2.1 and (4.2). ∎

5. Theta method for small step size. Using an idea from [15, section 4.3], we may extend Theorem 4.2 to the case of the general theta method.

COROLLARY 5.1. *The statements in Theorem* 4.2 *for the Euler–Maruyama method* (4.1) *also apply to the general theta method* (3.1).

Proof. The result follows from Theorem 4.2 when we observe that the theta method (3.1) is equivalent to the Euler–Maruyama method (4.1) applied to the perturbed problem

$$\mathrm{d}X(t) = \frac{\mu}{1-\theta\mu\Delta t}X(t^-)\,\mathrm{d}t+\frac{\sigma}{1-\theta\mu\Delta t}X(t^-)\,\mathrm{d}W(t)+\frac{\gamma}{1-\theta\mu\Delta t}X(t^-)\,\mathrm{d}N(t),$$
$$X(0) = X_0. \qquad ∎$$

6. Discussion. The main conclusions of this work are that (a) a standard theta method discretization for jump SDEs will correctly preserve asymptotic stability for sufficiently small stepsizes, but (b) in general there is no benefit to using implicitness. This raises the open question of whether new methods can be devised that guarantee Δt-independent stability preservation and hence offer efficiency gains on stiff problems.

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REFERENCES

[1] J. Appleby, G. Berkolaiko, and A. Rodkina, *Non-exponential stability and decay rates in non-linear stochastic difference equation with unbounded noises*, Stoch. Stoch. Rep., to appear.

[2] A. Bryden and D. J. Higham, *On the boundedness of asymptotic stability regions for the stochastic theta method*, BIT, 43 (2003), pp. 1–6.

[3] E. Buckwar, R. Horvath-Bokor, and R. Winkler, *Asymptotic mean-square stability of two-step methods for stochastic ordinary differential equations*, BIT, 46 (2006), pp. 261–282.
[4] R. Cont and P. Tankov, *Financial Modelling with Jump Processes*, Chapman and Hall/CRC Press, Boca Raton, FL, 2004.
[5] S. Cyganowski, L. Grüne, and P. E. Kloeden, *MAPLE for jump-diffusion stochastic differential equations in finance*, in Programming Languages and Systems in Computational Economics and Finance, S. S. Nielsen, ed., Kluwer, Boston, 2002, pp. 441–460.
[6] P. Glasserman and N. Merener, *Numerical solution of jump-diffusion LIBOR market models*, Finance Stoch., 7 (2003), pp. 1–27.
[7] P. Glasserman and N. Merener, *Convergence of a discretization scheme for jump-diffusion processes with state-dependent intensities*, Proc. R. Soc. Lond. Math. Phys. Ser. A Eng. Sci., 460 (2004), pp. 111–127.
[8] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer-Verlag, Berlin, 2003.
[9] G. Grimmett and D. Stirzaker, *Probability and Random Processes*, 3rd ed., Oxford University Press, Oxford, 2001.
[10] E. Hairer and G. Wanner, *Solving Ordinary Differential Equations* II, *Stiff and Differential-Algebraic Problems*, 2nd ed., Springer-Verlag, Berlin, 1996.
[11] F. Hanson and J. Westman, *Optimal consumption and portfolio control for jump-diffusion stock process with log-normal jumps*, in Proceedings of the 2002 American Control Conference, Anchorage, AK, IEEE, 2002, pp. 4256–4261.
[12] F. Hanson and G. Yan, *Option pricing for a stochastic-volatility jump-diffusion model with log-uniform jump amplitudes*, in Proceedings of the 2006 American Control Conference, Minneapolis, MN, IEEE, 2006, pp. 1–7.
[13] D. J. Higham and P. E. Kloeden, *Numerical methods for nonlinear stochastic differential equations with jumps*, Numer. Math., 101 (2005), pp. 101–119.
[14] D. J. Higham and P. E. Kloeden, *Convergence and stability of implicit methods for jump-diffusion systems*, Int. J. Numer. Anal. Model., 3 (2006), pp. 125–140.
[15] D. J. Higham, X. Mao, and C. Yuan, *Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations*, SIAM J. Numer. Anal., 45 (2007), pp. 592–609.
[16] D. J. Higham, *Mean-square and asymptotic stability of the stochastic theta method*, SIAM J. Numer. Anal., 38 (2000), pp. 753–769.
[17] S. Kou, *A jump-diffusion model for option pricing*, Management Sci., 48 (2002), pp. 1086–1101.
[18] X. Mao, *Stochastic Differential Equations and Applications*, Horwood, Chichester, 1997.
[19] R. Merton, *Theory of rational option pricing*, Bell J. Econ. Management Sci., 4 (1973), pp. 141–183.
[20] Y. Saito and T. Mitsui, *T-stability of numerical scheme for stochastic differential equations*, World Sci. Ser. Appl. Anal., 2 (1993), pp. 333–344.
[21] Y. Saito and T. Mitsui, *Stability analysis of numerical schemes for stochastic differential equations*, SIAM J. Numer. Anal., 33 (1996), pp. 2254–2267.
[22] Z. Zhu and F. B. Hanson, *A Monte-Carlo option-pricing algorithm for log-uniform jump-diffusion model*, in Proceedings of the Joint 44nd IEEE Conference on Decision and Control and European Control Conference, Seville, Spain, 2005, pp. 5221–5226.