Journal of Difference Equations and Applications

ISSN: 1023-6198 (Print) 1563-5120 (Online) Journal homepage: https://www.tandfonline.com/loi/gdea20

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To cite this article: Graeme D. Chalmers & Desmond J. Higham (2010) First and second moment reversion for a discretized square root process with jumps, Journal of Difference Equations and Applications, 16:2-3, 143-156, DOI: 10.1080/10236190802705719

To link to this article: https://doi.org/10.1080/10236190802705719

Journal of Difference Equations and Applications
Vol. 16, Nos. 2–3, February–March 2010, 143–156

First and second moment reversion for a discretized square root process with jumps

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(*Received 26 September 2008; final version received 16 December 2008*)

Dedicated to Peter Kloeden on the Occasion of his 60th Birthday

Mean-reversion is an important component of many financial models. When simulations are performed with numerical methods, it is therefore desirable to reproduce this qualitative property. Here, we study a square root process with jumps that has been used to model interest rates and volatilities, and we characterize the parameter regimes under which the first and second moments revert to steady state values. We then consider a class of implicit theta methods and investigate the same moment properties for the corresponding stochastic difference equation. We find that the theta method is unconditionally stable in first and second moment for theta values below a cutoff level. This cutoff level depends on the parameters governing the mean reversion and the jumps, but is always more favourable than the value of one half that arises in the deterministic setting. In the case of high jump intensity, large jump magnitude or slow mean reversion, it is even possible for the explicit Euler–Maruyama type method from this class to be unconditionally stable. We also establish upper and lower bounds for the second moment steady state that are close to that of the continuous-time process for small step-sizes. Numerical experiments are given to illustrate the results.

Keywords: implicit; interest rate; Ito lemma; Monte Carlo; stability; stochastic differential equation; variance; volatility

1. Introduction

We consider the following stochastic differential equation (SDE) with jumps:

$$\mathrm{d}X(t) = \xi(\mu - X(t^-))\mathrm{d}t + \sigma\sqrt{X(t^-)}\mathrm{d}W(t) + \gamma X(t^-)\mathrm{d}N(t). \tag{1}$$

Here, $X(0) = X_0 \neq 0$ (a.s.), and $W(t)$ and $N(t)$ are independent scalar Wiener and Poisson processes, respectively. The constant model parameters are

$\mu > 0$, which represents the long-term mean, in appropriate circumstances;
$\xi > 0$, which controls the rate of the mean reversion;
$\sigma > 0$, which represents the strength of the diffusion term;
$\gamma > 0$, which represents the relative jump size (here we consider upwards jumps) and
$\lambda > 0$, which is the intensity of the Poisson process.

We emphasize that all model parameters are assumed to be positive throughout our analysis.

The equation (1) plays an important role in mathematical finance. In the non-jump case, $\lambda = 0$, this is the classical mean-reverting square-root process, which was introduced

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ISSN 1023-6198 print/ISSN 1563-5120 online

DOI: 10.1080/10236190802705719
http://www.informaworld.com

and proposed as a potential model for interest rates by Cox et al. [9], it is therefore commonly referred to as the *CIR process*. The process has also been used as part of a so-called stochastic volatility model in Ref. [14]. Alternatively referred to as the *Heston model*, it comprises two coupled SDEs with a CIR process describing the volatility component, $V(t)$, of the asset price process, $X(t)$

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sqrt{V(t)}X(t)\mathrm{d}W^{x}(t),$$

$$\mathrm{d}V(t) = \alpha(b - V(t))\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W^{v}(t),$$

where (W^x, W^v) is a, perhaps correlated, two-dimensional Brownian motion. Existence and uniqueness theory for (1) follows directly from that of the non-jump case, which is discussed, for example, in Ref. [21].

Compared with the linear term that arises in standard geometric Brownian motion, the square root diffusion term in (1) produces a less dramatic variance when the solution is large, while continuing to exclude the possibility of negative solutions. It may therefore be regarded as a better reflection of financial reality [21]. It is well-known that the non-jump version of the square-root process has a well-defined non-negative solution [21] and, whilst a transition density of the process may be characterized, no general analytical solution has been found. Therefore, several authors have considered the issue of how to simulate the process numerically, focussing on convergence over finite time intervals, see, for instance, Refs. [2,3,13,17,20].

The jump term in (1) represents an attempt to account for unexpected, abrupt changes. This model is considered in Refs. [1,7] and is referred to as a jump-extended CIR model. A jump-extended version of the two-factor Heston model, called the *Bates model*

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sqrt{V(t)}X(t)\mathrm{d}W^{x}(t) + \gamma(t)X(t)\mathrm{d}N(t), \tag{2}$$

$$\mathrm{d}V(t) = \alpha(b - V(t))\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W^{v}(t), \tag{3}$$

is proposed in Ref. [6] and supported empirically in Refs. [4,5,22], where jumps (of random magnitude) are included in the asset price process, as opposed to the volatility process. It is further discussed in Ref. [8], with extensions to a more general class of Lévy models in the volatility process, which include models with jumps in the volatility component; such as the two factor model including correlated jumps in the asset and the volatility processes considered in the empirical study [11]. In this proposed model, the asset process is described as in (2), but the volatility component (3) is replaced with

$$\mathrm{d}V(t) = \alpha(b - V(t))\mathrm{d}t + \sigma\sqrt{V(t)}\mathrm{d}W^{v}(t) + \gamma^{v}(t)V(t)\mathrm{d}N^{v}(t),$$

where the jump process $N^v(t)$ and jump-magnitudes $\gamma^v(t)$ may be correlated with those governing (2).

Existence, uniqueness and finite-time numerical convergence theory extends readily to this jump case. The purpose of this work is to focus on the long-time, qualitative properties of mean-reversion for the first and second moment. In addition to giving insights about more general qualitative behaviour, this type of study is also relevant to the propagation of error in numerical simulations.

2. First and second moment reversion for the exact process

The following theorem characterizes first and second moment reversion for the continuous-time process.

THEOREM 1. *For the jump-SDE* (1), $\lim_{t\to\infty}\mathbb{E}[X(t)]$ *is finite if and only if* $\xi - \lambda\gamma > 0$, *in which case*

$$\lim_{t\to\infty}\mathbb{E}[X(t)] = \frac{\xi\mu}{\xi - \lambda\gamma}. \tag{4}$$

Similarly, $\lim_{t\to\infty}\mathbb{E}[X^2(t)]$ *is finite if and only if* $2\xi - \lambda\gamma(2 + \gamma) > 0$, *in which case*

$$\lim_{t\to\infty}\mathbb{E}[X^2(t)] = \frac{\xi\mu(2\xi\mu + \sigma^2)}{(\xi - \lambda\gamma)\big(2\xi - \lambda\gamma(2 + \gamma)\big)}. \tag{5}$$

Proof. Part 1: First Moment. We may rewrite (1) in integral form

$$X(t) = X(0) + \xi\int_0^t \mu - X(r^-)\mathrm{d}r + \sigma\int_0^t |X(r^-)|^{1/2}\mathrm{d}W(r) + \gamma\int_0^t X(r^-)\mathrm{d}N(r),$$

and take expectations, to get

$$\mathbb{E}[X(t)] = \mathbb{E}[X_0] + \int_0^t \xi\mu - (\xi - \lambda\gamma)\mathbb{E}[X(r)]\mathrm{d}r. \tag{6}$$

Case a: $\xi - \lambda\gamma = 0$.

When $\xi - \lambda\gamma = 0$, equation (6) becomes $\mathbb{E}[X(t)] = \mathbb{E}[X_0] + \xi\mu t$, and hence $\mathbb{E}[X(t)] \to\infty$ as $t\to\infty$.

Case b: $\xi - \lambda\gamma \neq 0$.

For $\xi - \lambda\gamma \neq 0$, we may solve the integral equation (6) and rearrange to show that the first moment of the solution of problem (1) is

$$\mathbb{E}[X(t)] = \frac{\xi\mu}{\xi - \lambda\gamma} + \mathbb{E}\left[X_0 - \frac{\xi\mu}{\xi - \lambda\gamma}\right]\mathrm{e}^{-(\xi-\lambda\gamma)t}, \tag{7}$$

which is clearly unbounded for $\xi - \lambda\gamma < 0$ as $t\to\infty$. Otherwise, for $\xi - \lambda\gamma > 0$ we have

$$\lim_{t\to\infty}\mathbb{E}[X(t)] = \frac{\xi\mu}{\xi - \lambda\gamma},$$

as required. Hence (4) is proved.

Part 2: Second Moment. Applying Itô's Lemma to the process $X^2(t)$, we get

$$X^2(t) = X_0^2 + (2\xi\mu + \sigma^2)\int_0^t X(r^-)\mathrm{d}r - 2\xi\int_0^t X^2(r^-)\mathrm{d}r + \sigma\int_0^t |X(r^-)|^{3/2}\mathrm{d}W(r)$$
$$+ \gamma(2 + \gamma)\int_0^t X^2(r^-)\mathrm{d}N(r).$$

Taking expectations we find

$$\mathbb{E}\big[X^2(t)\big] = \mathbb{E}\big[X_0^2\big] + (2\xi\mu + \sigma^2)\int_0^t \mathbb{E}[X(r)]\mathrm{d}r - \big(2\xi - \lambda\gamma(2+\gamma)\big)\int_0^t \mathbb{E}\big[X^2(r)\big]\mathrm{d}r. \quad (8)$$

Case a: $2\xi - \lambda\gamma(2 + \gamma) = 0$.

When $2\xi - \lambda\gamma(2 + \gamma) = 0$ it follows that $\xi - \lambda\gamma > 0$, and (8) reduces to

$$\mathbb{E}\big[X^2(t)\big] = \mathbb{E}\big[X_0^2\big] + (2\xi\mu + \sigma^2)\int_0^t \mathbb{E}[X(r)]\mathrm{d}r.$$

Then using (7), we have

$$\begin{aligned}\mathbb{E}\big[X^2(t)\big] &= \mathbb{E}\big[X_0^2\big] + (2\xi\mu + \sigma^2)\int_0^t \left(\mathbb{E}\left[X_0 - \frac{\xi\mu}{\xi - \lambda\gamma}\right]\mathrm{e}^{-(\xi-\lambda\gamma)r} + \frac{\xi\mu}{\xi-\lambda\gamma}\right)\mathrm{d}r \\ &= \mathbb{E}\big[X_0^2\big] - \frac{2\xi\mu+\sigma^2}{\xi-\lambda\gamma}\left(\mathbb{E}\left[X_0 - \frac{\xi\mu}{\xi-\lambda\gamma}\right]\mathrm{e}^{-(\xi-\lambda\gamma)t} - \xi\mu t\right).\end{aligned}$$

Therefore $\mathbb{E}\big[X^2(t)\big] \rightarrow \infty$ as $t \rightarrow \infty$.

Case b: $2\xi - \lambda\gamma(2 + \gamma) \neq 0$.

When $2\xi - \lambda\gamma(2 + \gamma) \neq 0$, we may use the expression for $\mathbb{E}[X(t)]$ from (7) in (8) to obtain

$$\begin{aligned}\mathbb{E}\big[X^2(t)\big] =& \mathbb{E}\big[X_0^2\big] + (2\xi\mu + \sigma^2)\int_0^t \left(\mathbb{E}\left[X_0 - \frac{\xi\mu}{\xi - \lambda\gamma}\right]\mathrm{e}^{-(\xi-\lambda\gamma)r} + \frac{\xi\mu}{\xi-\lambda\gamma}\right)\mathrm{d}r \\ &- \big(2\xi - \lambda\gamma(2+\gamma)\big)\int_0^t \mathbb{E}\big[X^2(r)\big]\mathrm{d}r.\end{aligned}$$

This solves to give

$$\begin{aligned}\mathbb{E}\big[X^2(t)\big] =& \frac{\xi\mu(2\xi\mu+\sigma^2)}{\big(\xi-\lambda\gamma\big)\big(2\xi-\lambda\gamma(2+\gamma)\big)} + \frac{2\xi\mu+\sigma^2}{\xi-\lambda\gamma(1+\gamma)}\mathbb{E}\left[X_0 - \frac{\xi\mu}{\xi-\lambda\gamma}\right]\mathrm{e}^{-(\xi-\lambda\gamma)t} \\ &+ \left(\mathbb{E}\big[X_0^2\big] - \frac{2\xi\mu+\sigma^2}{\xi-\lambda\gamma(1+\gamma)}\mathbb{E}[X_0] + \frac{\xi\mu(2\xi\mu+\sigma^2)}{\big(2\xi-\lambda\gamma(2+\gamma)\big)\big(\xi-\lambda\gamma(1+\gamma)\big)}\right) \\ &\times \mathrm{e}^{-(2\xi-\lambda\gamma(2+\gamma))t}.\end{aligned}$$

So, for $2\xi - \lambda\gamma(2 + \gamma) > 0$ (which implies $\xi - \lambda\gamma > 0$), we have

$$\lim_{t\rightarrow\infty}\mathbb{E}\big[X^2(t)\big] = \frac{\xi\mu(2\xi\mu+\sigma^2)}{\big(\xi-\lambda\gamma\big)\big(2\xi-\lambda\gamma(2+\gamma)\big)}. \quad (9)$$

Alternatively, for $2\xi - \lambda\gamma(2 + \gamma) < 0$, we see that $\mathbb{E}\big[X^2(t)\big] \rightarrow \infty$ as $t \rightarrow \infty$. □

3. Analysis of the Theta-method

3.1 Definition

Our aim is now to determine the extent to which a popular class of numerical methods can match the mean-reversion properties of the underlying problem. Following the standard, and practically useful, approach that began for deterministic ODEs [10,12] and has been carried through to SDEs [18] and jump-SDEs [15,16], we will focus in the following question

> Given parameters for which there is moment reversion in (1), what restriction, if any, must be placed on the step size in the numerical method in order to reproduce this behaviour?

We consider the class of theta methods which, when applied to (1), produce the stochastic difference equation

$$Y_{n+1} = Y_n + \big(\xi(1-\theta)(\mu - Y_n) + \xi\theta(\mu - Y_{n+1})\big)\Delta t + \sigma\sqrt{|Y_n|}\Delta W_n + \gamma Y_n \Delta N_n. \quad (10)$$

Here,

Y_n is the approximation to $X(n\Delta t)$, where $\Delta t > 0$ is a fixed step size, with $Y_0 = X(0)$,
$\Delta W_n := W((n+1)\Delta t) - W(n\Delta t)$ is the Brownian increment over a step,
$\Delta N_n := N((n+1)\Delta t) - N(n\Delta t)$ is the Poisson increment over a step,
$\theta \in [0,1]$ is a fixed parameter that defines the particular theta method.

Choosing $\theta = 0$, we have the explicit Euler–Maruyama method applied to (1). For the non-jump case, replication of moment behaviour was examined in Ref. [17] where also strong convergence (although no order of convergence) of the scheme was established. Typical convergence theorems for non-jump and jump SDEs, for example, see Refs. [15,19], restrict the models to global Lipschitz conditions on the coefficient functions. This is not applicable in the case of a square-root function for the diffusion term. More recent work has retrieved a strong order of convergence for numerical methods applied to square-root models without jumps, see Refs. [2,13].

3.2 First moment

Taking expectations in (10), using $\mathbb{E}[\Delta W_n] = 0$ and $\mathbb{E}[\Delta N_n] = \lambda\Delta t$, we find that

$$\mathbb{E}\left[Y_n - \frac{\xi\mu}{\xi - \lambda\gamma}\right] = \hat{r}^n \mathbb{E}\left[Y_0 - \frac{\xi\mu}{\xi - \lambda\gamma}\right], \quad (11)$$

where

$$\hat{r} := 1 - \frac{(\xi - \lambda\gamma)\Delta t}{1 + \xi\theta\Delta t}.$$

We conclude that $\lim_{n\to\infty}\mathbb{E}[Y_n - \xi\mu/(\xi - \lambda\gamma)] = 0$ for general initial data if and only if $|\hat{r}| < 1$, which is equivalent to the constraint

$$(\xi - \lambda\gamma)\Delta t\big[2 - \big(\xi(1 - 2\theta) - \lambda\gamma\big)\Delta t\big] > 0. \quad (12)$$

Now, suppose $\xi - \lambda\gamma > 0$, so that from Theorem 1, the problem (1) undergoes mean-reversion. It follows from (4) that there is a critical value

$$\theta^* := \frac{1}{2}\left(1 - \frac{\lambda\gamma}{\xi}\right) \quad (13)$$

with the property that

for $\theta \geq \theta^*$ the theta-method (10) replicates the mean-reversion for all $\Delta t > 0$, whereas for $\theta < \theta^*$ the mean-reversion is replicated if and only if the step size is restricted to

$$\Delta t < \frac{2}{\xi(1-2\theta) - \lambda\gamma}.$$

It is interesting to note that in the traditional deterministic ODE setting the value $\theta = 1/2$ gives the cutoff for unconditional stability [12]. However, for the problem (1), we see that although a cutoff θ^* exists, it depends on the problem parameters. In particular, we note two interesting consequences.

- Since $\theta^* < 1/2$, the theta method is *uniformly more stable* for (1) than in the ODE setting, in the sense that unconditional replication of stability arises for a larger range of theta values.
- If $\theta^* < 0$, then the theta method is always unconditionally stable; this includes the explicit $\theta = 0$ Euler–Maruyama based method. So, in this high jump intensity/large jump magnitude/slow reversion setting, implicitness does not offer any stability benefits.

3.3 Second moment

To analyse the second moment, we square and take expectations in (10), noting that $\mathbb{E}\left[\Delta W_n^2\right] = \Delta t$ and $\mathbb{E}\left[\Delta N_n^2\right] = \lambda\Delta t(1 + \lambda\Delta t)$, to get

$$\begin{aligned}(1+\xi\theta\Delta t)^2\mathbb{E}\left[Y_{n+1}^2\right] = &\Big(\lambda\gamma^2\Delta t + \big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)^2\Big)\mathbb{E}\left[Y_n^2\right] \\ &+ \big(2\xi\mu\big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)\big)\Delta t\mathbb{E}[Y_n] \\ &+ \sigma^2\Delta t\mathbb{E}\left[|Y_n|\right] + \xi^2\mu^2\Delta t^2. \end{aligned} \tag{14}$$

3.3.1 Second moment lower bound

Because of the modulus sign in (14), we do not seek an exact analytical expression for the second moment of the numerical solution. Instead, we will develop explicit upper and lower bounds. (A similar approach was taken for the non-jump case in Ref. [17].) We begin with a lower bound. Replacing $\mathbb{E}\left[|Y_n|\right]$ in (14) by $\mathbb{E}[Y_n]$, we obtain the sequence $\{z_n\}$ with $z_0 = \mathbb{E}\left[Y_0^2\right]$ and

$$\begin{aligned}(1+\xi\theta\Delta t)^2 z_{n+1} = &\Big(\lambda\gamma^2\Delta t + \big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)^2\Big)z_n \\ &+ \big(2\xi\mu\big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)+\sigma^2\big)\Delta t\mathbb{E}[Y_n] + \xi^2\mu^2\Delta t^2. \end{aligned} \tag{15}$$

Since $\mathbb{E}[Y_n] \leq \mathbb{E}\left[|Y_n|\right]$, it is clear that $\mathbb{E}\left[Y_n^2\right] \geq z_n$ for all n. Substituting for $\mathbb{E}[Y_n]$ from (11) into (15) we get

$$\begin{aligned}(1+\xi\theta\Delta t)^2 z_{n+1} = &\Big(\lambda\gamma^2\Delta t + \big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)^2\Big)z_n \\ &+ \big(2\xi\mu\big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)+\sigma^2\big)\Delta t\mathbb{E}\left[Y_0 - \frac{\xi\mu}{\xi-\lambda\gamma}\right]\hat{r}^n \\ &+ \frac{\xi\mu}{\xi-\lambda\gamma}\big(2\xi\mu\big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)+\sigma^2\big)\Delta t + \xi^2\mu^2\Delta t^2, \end{aligned}$$

which has the form

$$(1+\xi\theta\Delta t)^2 z_{n+1} = az_n + c\hat{r}^n + b, \tag{16}$$

where

$$a = \lambda\gamma^2\Delta t + \big(1 - (\xi(1-\theta) - \lambda\gamma)\Delta t\big)^2,$$

$$b = \frac{\xi\mu}{\xi-\lambda\gamma}\big(2\xi\mu\big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)+\sigma^2\big)\Delta t + \xi^2\mu^2\Delta t^2,$$

$$c = \big(2\xi\mu\big(1-(\xi(1-\theta)-\lambda\gamma)\Delta t\big)+\sigma^2\big)\Delta t\mathbb{E}\left[Y_0 - \frac{\xi\mu}{\xi-\lambda\gamma}\right].$$

We are interested in the case where the original problem has a reverting second moment, so, following Theorem 1, we assume henceforth that $2\xi - \lambda\gamma(2+\gamma) > 0$. Since $a > 0$, we require $a < (1+\xi\theta\Delta t)^2$ for generic convergence of the sequence $\{z_n\}$ in (16). This constraint may be written

$$\Delta t\big(\xi(1-2\theta)-\lambda\gamma\big) < \frac{2\xi-\lambda\gamma(2+\gamma)}{\xi-\lambda\gamma}, \tag{17}$$

and it leads to the limit

$$z_n \longrightarrow \frac{b}{(1+\xi\theta\Delta t)^2 - a}, \quad \text{as } n\longrightarrow\infty.$$

As in the first moment analysis, the parameter value θ^* in (13) is an important cutoff point. For $\theta \geq \theta^*$, the constraint (17) holds for all step sizes Δt, whereas for $\theta < \theta^*$ we have the problem-dependent constraint

$$\Delta t < \frac{2\xi-\lambda\gamma(2+\gamma)}{(\xi-\lambda\gamma)\big(\xi(1-2\theta)-\lambda\gamma\big)}.$$

Returning to our original variables, under the constraint (17) we have a lower bound on the long term second moment of the form

$$\begin{aligned}\liminf_{n\to\infty}\mathbb{E}\big[Y_n^2\big] &\geq \frac{\frac{2\xi^2\mu^2+\xi\mu\sigma^2}{\xi-\lambda\gamma} - \frac{\xi^2\mu^2}{\xi-\lambda\gamma}\big(\xi(1-2\theta)-\lambda\gamma\big)\Delta t}{2\xi-\lambda\gamma(2+\gamma)-\big(\xi(1-2\theta)-\lambda\gamma\big)\big(\xi-\lambda\gamma\big)\Delta t} \\ &=: L(\xi,\mu,\sigma,\lambda,\gamma,\theta;\Delta t).\end{aligned} \tag{18}$$

This lower bound is sharp in the sense that for small Δt it converges to the second moment steady state for the underlying problem:

$$\lim_{\Delta t\to 0} L(\xi,\mu,\sigma,\lambda,\gamma,\theta;\Delta t) = \frac{\xi\mu(2\xi\mu+\sigma^2)}{\big(\xi-\lambda\gamma\big)\big(2\xi-\lambda\gamma(2+\gamma)\big)}. \tag{19}$$

3.3.2 Second moment upper bound

For an upper bound, we note that for any $\beta > 0$

$$\mathbb{E}\big[|Y_n|\big] \le \frac{1}{2}\left(\frac{1}{\beta} + \beta\big(\mathbb{E}\big[|Y_n|\big]\big)^2\right) \le \frac{1}{2\beta} + \frac{1}{2}\beta\mathbb{E}\big[Y_n^2\big].$$

Substituting this into (14) yields

$$\begin{aligned}
\mathbb{E}\big[Y_{n+1}^2\big](1+\xi\theta\Delta t)^2 \le& \left(\lambda\gamma^2\Delta t + \big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)^2 + \frac{1}{2}\sigma^2\beta\Delta t\right)\\
&+ 2\xi\mu\big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)\Delta t\mathbb{E}[Y_n]\\
&+ \frac{1}{2\beta}\sigma^2\Delta t + \xi^2\mu^2\Delta t^2\\
=& \left(\lambda\gamma^2\Delta t + \big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)^2 + \frac{1}{2}\sigma^2\beta\Delta t\right)\mathbb{E}\big[Y_n^2\big]\\
&+ 2\xi\mu\big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)\Delta t\mathbb{E}\left[Y_0 - \frac{\xi\mu}{\xi-\lambda\gamma}\right]\hat{r}^n\\
&+ \frac{\xi\mu}{\xi-\lambda\gamma}2\xi\mu\big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)\Delta t + \frac{1}{2\beta}\sigma^2\Delta t + \xi^2\mu^2\Delta t^2.
\end{aligned}$$

This leads us to define a sequence $\{\hat{z}_n\}$ for which $\hat{z}_n \ge \mathbb{E}\big[Y_n^2\big]$ by $\hat{z}_0 = \mathbb{E}\big[Y_0^2\big]$ and

$$(1+\xi\theta\Delta t)^2\,\hat{z}_{n+1} = \tilde{a}\hat{z}_n + \tilde{c}\hat{r}^n + \tilde{b}, \tag{20}$$

where

$$\begin{aligned}
\tilde{a} &= \left(\lambda\gamma^2 + \frac{1}{2}\sigma^2\beta\right)\Delta t + \big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)^2,\\
\tilde{b} &= \frac{2\xi^2\mu^2}{\xi-\lambda\gamma}\big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)\Delta t + \frac{1}{2\beta}\sigma^2\Delta t + \xi^2\mu^2\Delta t^2,\\
\tilde{c} &= 2\xi\mu\big(1-\big(\xi(1-\theta)-\lambda\gamma\big)\Delta t\big)\Delta t\mathbb{E}\left[Y_0 - \frac{\xi\mu}{\xi-\lambda\gamma}\right].
\end{aligned}$$

Since $\tilde{a} > 0$, convergence of the sequence (20) is characterized by $\tilde{a} < (1+\xi\theta\Delta t)^2$; that is,

$$\Delta t\big(\xi(1-2\theta)-\lambda\gamma\big) < \frac{2\xi - \lambda\gamma(2+\gamma) - \frac{1}{2}\sigma^2\beta}{\xi-\lambda\gamma}. \tag{21}$$

Now, recall that we are assuming $2\xi - \lambda\gamma(2+\gamma) > 0$, so that the true second moment reverts (which implies $\xi - \lambda\gamma > 0$). We are free to choose any $\beta > 0$, and so by choosing sufficiently small β we can ensure that the right hand side in (21) is positive. In this case we see that $\theta \ge \theta^*$ guarantees convergence of the upper bound sequence to a finite limit for all

$\Delta t > 0$, whereas for $\theta < \theta^*$ we have convergence only for step sizes constrained by

$$\Delta t < \frac{2\xi - \lambda\gamma(2+\gamma) - \frac{1}{2}\sigma^2\beta}{(\xi - \lambda\gamma)\big(\xi(1-2\theta) - \lambda\gamma\big)}.$$

When $\{\hat{z}_n\}$ converges, we have the limit

$$\lim_{n\to\infty} \hat{z}_n = \frac{\tilde{b}}{(1+\xi\theta\Delta t)^2 - \tilde{a}},$$

giving a lim sup bound for $\mathbb{E}\left[Y_n^2\right]$ of the form

$$\limsup_{n\to\infty} \mathbb{E}\left[Y_n^2\right] \le \frac{\frac{2\xi^2\mu^2}{\xi-\lambda\gamma} + \frac{\sigma^2}{2\beta} - \frac{\xi^2\mu^2}{\xi-\lambda\gamma}\big(\xi(1-2\theta) - \lambda\gamma\big)\Delta t}{2\xi - \lambda\gamma(2+\gamma) - \frac{1}{2}\sigma^2\beta - \big(\xi - \lambda\gamma\big)\big(\xi(1-2\theta) - \lambda\gamma\big)\Delta t}$$

$$=: U(\xi, \mu, \sigma, \lambda, \gamma, \theta; \Delta t). \tag{22}$$

To get a feel for the sharpness of this bound, we may choose $\beta = (\xi - \lambda\gamma)/2\xi\mu$ and consider the limit as $\Delta t \to 0$, which gives

$$\lim_{\Delta t\to 0} U(\xi, \mu, \sigma, \lambda, \gamma, \theta; \Delta t) = \frac{\xi\mu(2\xi\mu + \sigma^2)}{(\xi - \lambda\gamma)\Big(2\xi - \lambda\gamma(2+\gamma) - \frac{1}{4}\sigma^2 \frac{\xi - \lambda\gamma}{\xi\mu}\Big)}.$$

This is close to the second moment limit for the true problem (and hence, from (19), to the corresponding lower bound at small step sizes) when the term $\sigma^2/(\xi - \lambda\gamma)$ is small.

4. Numerical results

To demonstrate the replication of mean and mean-square reversion of the approximation versus the model, we simulated 1 million sample paths for the following parameter set: $\xi = 0.3$, $\mu = 0.1$, $\sigma = 0.1$, $\gamma = 0.15$, and $\lambda = 0.05$ over various time intervals $[0, T]$ with an initial value of $X(0) = 0.111$. For this parameter combination, the critical value is $\theta^* = 0.4875$.

Figures 1 and 2 depict the successful replication of mean and mean-square reversion, respectively, by the theta-method, with $0.5875 = \theta > \theta^*$ for a fixed time-step chosen to be $\Delta t = 0.01$, where $T = 50$. We have included 99% confidence intervals, confirming that variance of the generated trajectories remains bounded.

In Figure 2, the shaded region illustrates the range given by the lim inf and lim sup bounds derived in Sections 3.3.1 and 3.3.2, respectively. We see that our numerically solved second moment lies within this region, in agreement with our analysis. In further concurrence, we observe that the lower bound is sharp for the small Δt used here.

In Figures 3 and 4 trajectories are simulated using $\Delta t = 10$ and $T = 2000$ to demonstrate that reversion is achieved even for large Δt. We observe successful replication of the mean and also stability of the mean-square. The fact that the scheme no longer closely approximates the theoretical mean-square path is consistent with the fact that the upper and lower bounds are sharp only for small time-steps.

For the case of $\theta < \theta^*$, we chose for simplicity $\theta = 0$, corresponding to the Euler–Maruyama scheme. Again we implement 1 million Monte Carlo simulations, this time for three types of step-size:

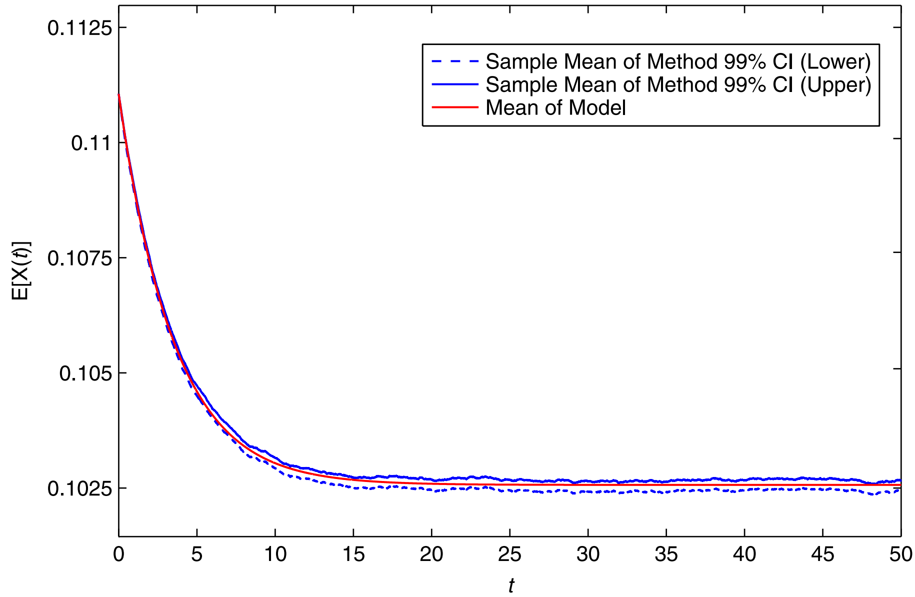

Figure 1. $\theta > \theta^*$: mean reversion for $\Delta t = 0.01$.

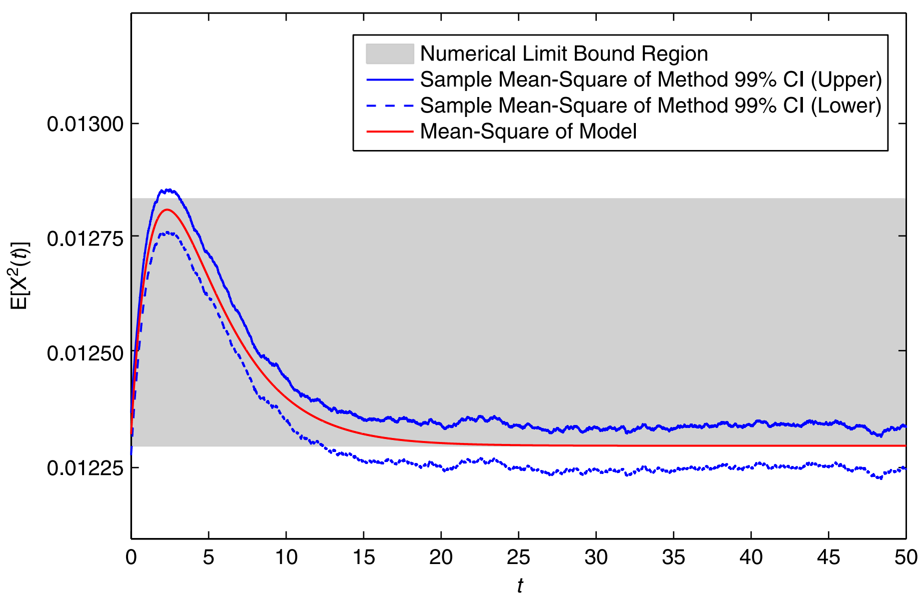

Figure 2. $\theta > \theta^*$: mean-square reversion for $\Delta t = 0.01$.

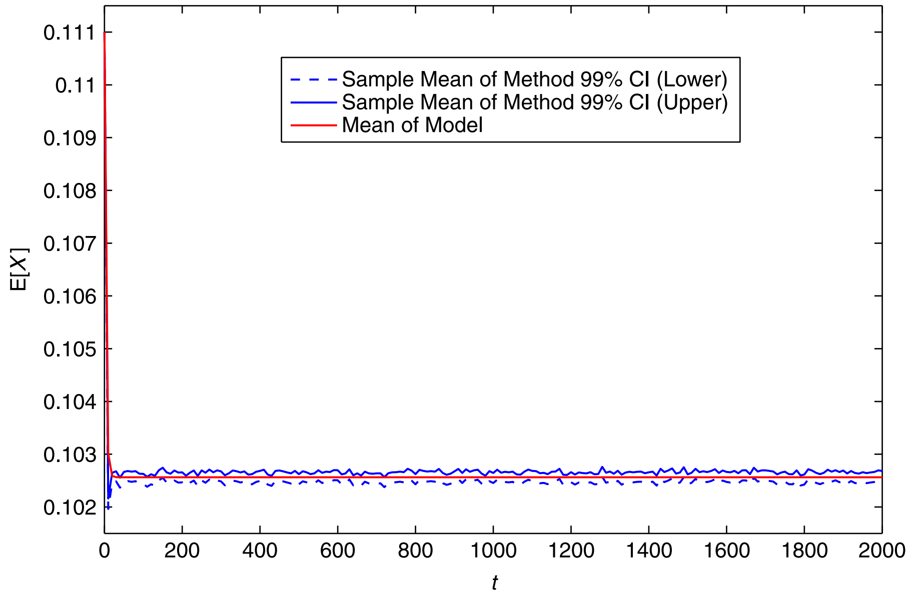

Figure 3. $\theta > \theta^*$: mean reversion for $\Delta t = 10$.

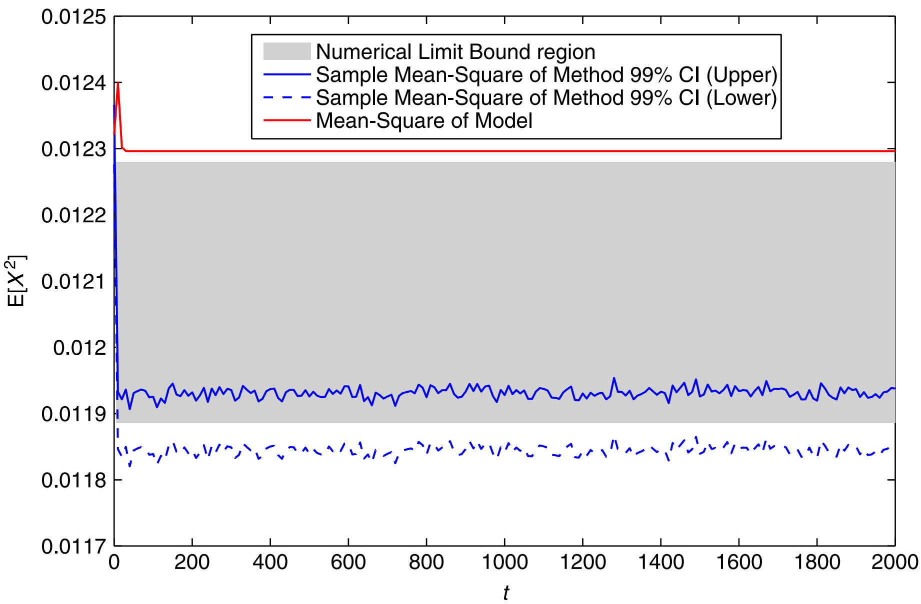

Figure 4. $\theta > \theta^*$: mean-square reversion for $\Delta t = 10$.

(a)

$$\Delta t < \frac{2\xi - \lambda\gamma(2+\gamma)}{(\xi - \lambda\gamma)^2}$$

(b)

$$\frac{2\xi - \lambda\gamma(2+\gamma)}{(\xi - \lambda\gamma)^2} < \Delta t < \frac{2}{\xi - \lambda\gamma}$$

(c)

$$\frac{2}{\xi - \lambda\gamma} < \Delta t,$$

in order to observe mean and mean-square replication in the case of (a); mean reversion but not mean-square reversion in the case of (b) and neither mean, nor mean-square reversion for the final case, (c).

Results are presented in Figure 5. In the upper pair of plots, corresponding to case (a), simulations are done at $\Delta t = 0.01$ for $T = 50$. Replication of the model's moment behaviour is achieved, supporting the preceding analysis.

The second pair of plots correspond to case (b), with a fixed step-size of $\Delta t \simeq 6.831$, which lies between the time-step constraints, and $T = 6000$. Whilst mean-reversion appears to be achieved, this is at the expense of sample variance which is observed to be blowing up (as reflected in the expanding confidence interval as time gets large). This is in agreement with the lack of mean-square reversion predicted by our analysis.

The final pair of plots in the figure correspond to case (c), for which we chose $\Delta t = 8$ and $T = 500$. In this case, we can see that both mean and mean-square rapidly become unbounded, in agreement with the analysis of Sections 3.2 and 3.3.

5. Concluding remarks

In this work, we examined the ability of the implicit theta-method to successfully replicate mean and mean-square reversion of fixed jump models featuring mean-reverting drift and square-root diffusion. We characterized the model parameters under which both reversion features occur and examined what further constraints, if any, must be placed on the step-size of the implicit method used.

A novel result of this analysis was that given a choice of implicitness parameter, the method is unconditionally stable for a larger range of θ than the traditional $\theta \geq 1/2$ found to hold in the deterministic ODE setting. There exists a critical value θ^*, dependent on the specified model parameters and defined by (13), for which if we choose $\theta \geq \theta^*$ any fixed time step-size $\Delta t > 0$ gives mean and mean-square reversion replication under the method.

In the case where $\theta < \theta^*$ we found a constraint on the choice of step-size below which we achieve replication of both the mean and mean-square reversion. There was also found to be an intermediate range of step-size, the upper limit coming from the first moment analysis, where a step-size within this range replicated mean but not mean-square reversion. In this regime, sample means would be unreliable, however, due to the large variances. Finally, choosing a time step above the constraint for mean-reversion, we observed the method's failure to replicate either mean or mean-square reversion.

It would be of interest to extend this analysis to the random jump-magnitude and to the cases of higher dimensional models such as those described in the introductory discussion.

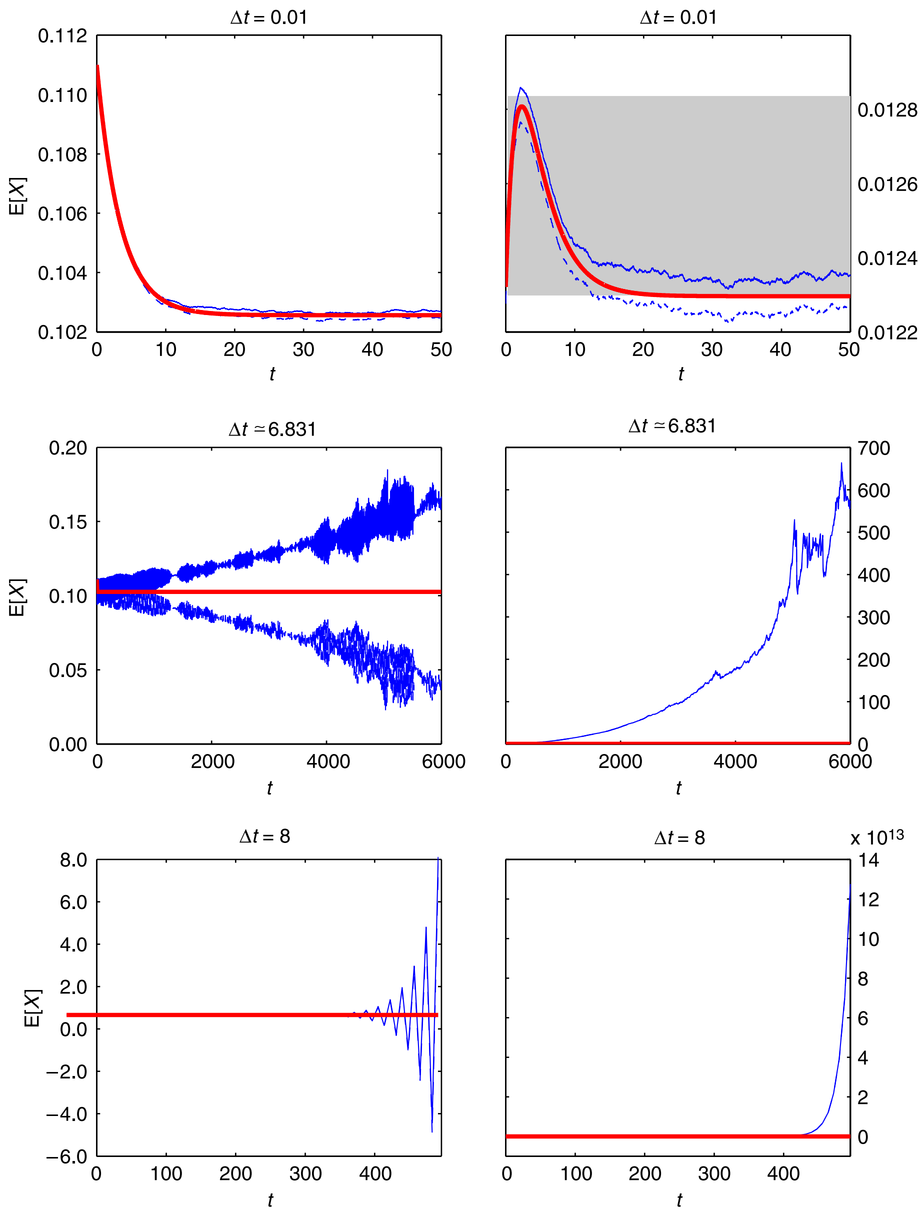

Figure 5. $\theta \leq \theta^*$: mean (left-hand plots) and mean-square (right-hand plots) behaviour for three cases of step-size: (a), (b) and (c), respectively.

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