Chapter Ten. The Greeks

Outline Solutions to odd-numbered exercises from the book:
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10.1 By definition,

$$F'(x) = \lim_{\delta \to 0} \frac{F(x+\delta) - F(x)}{\delta}$$

so $F(x+\delta) \approx F(x) + \delta F'(x)$ for $|\delta|$ small.

[If $F \in C^2$ we can write $F(x+\delta) = F(x) + \delta F'(x) + O(\delta^2)$, from a Taylor series expansion.]

This relation tells us that if we change x by a small amount, δ, then F changes by an amount $\delta F'(x)$. So $F'(x)$ measures how much a change in x is amplified or diminished to become a change in F.

10.3 We have

$$
\begin{aligned}
\Theta &= \frac{\partial C}{\partial t} \\
&= SN'(d_1)\frac{\partial d_1}{\partial t} - Ere^{-r(T-t)}N(d_2) - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}.
\end{aligned}
$$

Using $\dfrac{\partial d_1}{\partial t} = \dfrac{\partial d_2}{\partial t} - \dfrac{1}{2}\dfrac{\sigma}{\sqrt{T-t}}$ (from (8.22)) and the identity (10.1), we find

$$\Theta = -Er\, e^{-r(T-t)}N(d_2) - \frac{SN'(d_1)\sigma}{2\sqrt{T-t}},$$

as required.

Now

$$\text{vega} = \frac{\partial C}{\partial \sigma} = SN'(d_1)\frac{\partial d_1}{\partial \sigma} - Ee^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial \sigma}.$$

Using $\dfrac{\partial d_1}{\partial \sigma} = \dfrac{\partial d_2}{\partial \sigma} + \sqrt{T-t}$ (from (8.22)) and the identity (10.1), we find

$$\text{vega} = SN'(d_1)\sqrt{T-t},$$

as required.

10.5 In Section 2.6 we saw that C is always a non-decreasing function of T. This means that $\partial C/\partial T \geq 0$. But in the Black–Scholes expression for $C(S,t)$, the parameter T always appears in the form $T-t$. Hence, $\partial C/\partial t = -\partial C/\partial T$, and we deduce that $\partial C/\partial t \leq 0$.

10.7 The put-call parity relation (8.23) says $P = C - S + Ee^{-r(T-t)}$.

So $\dfrac{\partial P}{\partial S} = \dfrac{\partial C}{\partial S} - 1 = N(d_1) - 1$.

(We could also write $\dfrac{\partial P}{\partial S} = -N(-d_1)$, since $N(-z) = 1 - N(z)$ for all z.)

Further,
$$\frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 C}{\partial S^2} = \frac{N'(d_1)}{S\sigma\sqrt{T-t}},$$

and
$$
\begin{aligned}
\frac{\partial P}{\partial r} &= \frac{\partial C}{\partial r} - (T-t)Ee^{-r(T-t)} \\
&= (T-t)Ee^{-r(T-t)}N(d_2) - (T-t)Ee^{-r(T-t)} \\
&= (T-t)Ee^{-r(T-t)}[N(d_2) - 1] \\
&= -(T-t)Ee^{-r(T-t)}N(-d_2),
\end{aligned}
$$

and
$$
\begin{aligned}
\frac{\partial P}{\partial t} &= \frac{\partial C}{\partial t} + Er\,e^{-r(T-t)} \\
&= \frac{-S\sigma}{2\sqrt{T-t}}N'(d_1) - rEe^{-r(T-t)}N(d_2) + Ere^{-r(T-t)} \\
&= \frac{-S\sigma\,N'(d_1)}{2\sqrt{T-t}} + rEe^{-r(T-t)}[1 - N(d_2)] \\
&= \frac{-S\sigma\,N'(d_1)}{2\sqrt{T-t}} + rEe^{-r(T-t)}N(-d_2),
\end{aligned}
$$

and
$$\frac{\partial P}{\partial \sigma} = \frac{\partial C}{\partial \sigma} = SN'(d_1)\sqrt{T-t},$$

and
$$\frac{\partial P}{\partial E} = \frac{\partial C}{\partial E} + e^{-r(T-t)} = e^{-r(T-t)}[1 - N(d_2)] = e^{-r(T-t)}N(-d_2).$$

We have $\dfrac{\partial P}{\partial S} < 0$. This makes sense, because an increase in asset price decreases the likely payoff.

We have $\dfrac{\partial P}{\partial r} < 0$. Increasing the interest rate, r, is equivalent to lowering the exercise price, E, This decreases the likely payoff.

We have $\dfrac{\partial P}{\sigma} > 0$. This can be understood by considering that an increase in volatility leads to a wider spread of asset prices. However, assets moving deeper out of the money have no effect on the option price (the payoff remains zero) while assets moving deeper into the money lead to a greater payoff. Because of this asymmetry, increasing σ has a net positive effect.

We have $\dfrac{\partial P}{E} > 0$. This is reasonable because increasing E increases the likely payoff.

We have

$$\frac{\partial P}{\partial t} = -\frac{SN'(d_1)\sigma}{2\sqrt{T-t}} + rEe^{-r(T-t)}N(-d_2).$$

First term is ≤ 0 and second term is ≥ 0. Overall sign may be positive or negative, e.g.

- if $\log(S/E) = 0$, $r = 0$, $\sigma = 1$, $T - t = 1$, then $d_1 > 0$ and we get $\partial P/\partial t < 0$.

- if $\log(S/E) = -1$, $r = 1$, $T - t = 1$, and σ is very small, then we get $d_2 \approx 0$ and $\partial P/\partial t > 0$.

More specifically,

- $S = E = 1$, $r = 0$, $\sigma = 1$, $T - t = 1$ gives $d_1 = \frac{1}{2}$, so $N'(d_1) = e^{-1/8}/\sqrt{2\pi}$ and
$$\frac{\partial P}{\partial t} = \frac{-1e^{-1/8}}{2\sqrt{2\pi}} < 0.$$

- $E = 1$, $S = e^{-1}$, $r = 1$, $T - t = 1$ gives $d_2 = -\frac{1}{2}\sigma$. Choosing $\sigma \approx 0$ we can make $N(-d_2) \ge \frac{1}{4}$, so
$$\frac{\partial P}{\partial t} = 1e^{-1}N(-d_2) \ge \frac{e^{-1}}{4} > 0.$$

Finally, check the Black–Scholes PDE:

$$
\begin{aligned}
\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rV &= \frac{-S\sigma\, N'(d_1)}{2\sqrt{T-t}} + rEe^{-r(T-t)}N(-d_2) + \tfrac{1}{2}\frac{S\sigma N'(d_1)}{\sqrt{T-t}} \\
&\quad + rS(N(d_1) - 1) - r\left(Ee^{-r(T-t)}N(-d_2) - SN(-d_1)\right) \\
&= 0.
\end{aligned}
$$