Chapter Twelve. Risk neutrality

Outline Solutions to odd-numbered exercises from the book:
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12.1 We have

$$W(S,t) = e^{-r(T-t)} \int_0^\infty \frac{\Lambda(x)}{x\sigma\sqrt{2\pi}} \frac{1}{\sqrt{T-t}} e^{-\frac{\{\log(x)-\log(S)-(r-\frac{1}{2}\sigma^2)(T-t)\}^2}{2\sigma^2(T-t)}} dx$$

We will use the shorthand $\{-\}$ to denote $\left\{\log(x)-\log(s)-\left(r-\frac{1}{2}\sigma^2\right)(T-t)\right\}$.

First, note that

$$\begin{aligned}
\frac{\partial}{\partial t}\left[\frac{\{-\}^2}{2\sigma^2(T-t)}\right] &= \frac{2\sigma^2(T-t)2\{-\}(r-\frac{1}{2}\sigma^2)+\{-\}^2 2\sigma^2}{(2\sigma^2(T-t))^2} \\
&= \frac{(r-\frac{1}{2}\sigma^2)\{-\}}{\sigma^2(T-t)} + \frac{\{-\}^2}{2\sigma^2(T-t)^2}
\end{aligned}$$

Also,

$$\frac{\partial}{\partial S}\left[\{-\}^2\right] = \frac{-2\{-\}}{S}.$$

Hence

$$\begin{aligned}
\frac{\partial W}{\partial t} &= rW + \frac{1}{2(T-t)}W - \frac{(r-\frac{1}{2}\sigma^2)\{-\}W}{\sigma^2(T-t)} - \frac{\{-\}^2 W}{2\sigma^2(T-t)^2} \\
\frac{\partial W}{\partial S} &= \frac{2\{-\}W}{S2\sigma^2(T-t)} = \frac{\{-\}W}{S\sigma^2(T-t)} \\
\frac{\partial^2 W}{\partial S^2} &= \frac{\{-\}^2}{S^2\sigma^4(T-t)^2} - \frac{\{-\}W}{S^2\sigma^2(T-t)} - \frac{W}{S^2\sigma^2(T-t)}
\end{aligned}$$

So

$$\frac{\partial W}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 W}{\partial S^2} + rS\frac{\partial W}{\partial S} - rW =$$

$$\left[r + \frac{1}{2(T-t)} - \frac{r\{-\}}{\sigma^2(T-t)} + \frac{\{-\}}{2(T-t)} - \frac{\{-\}^2}{2\sigma^2(T-t)^2} + \frac{1}{2}\,\frac{\{-\}^2}{\sigma^2(T-t)^2}\right.$$

$$-\frac{1}{2}\frac{\{-\}}{(T-t)}-\frac{1}{2}\frac{1}{(T-t)}+\frac{r\{-\}}{\sigma^2(T-t)}-r\Bigg]W$$
$$=0$$

12.3 Using the density function, (6.10), the probability that a European call option will be exercised is

$$\mathbb{P}(S(T)\geq E)=\int_E^\infty \frac{\exp\left(\frac{-(\log(x/S)-(\mu-\sigma^2/2)T)^2}{2\sigma^2 T}\right)}{x\sigma\sqrt{2\pi}}\,dx.$$

We obviously need to change variable. The trick is to spot that the lower limit of integration should be $-d_2$. Hence, let

$$y=\frac{\log(x/S)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}.$$

When $x=E$, we have

$$y=\frac{\log(E/S)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}=-\,d_2,$$

where we used the risk-neutrality condition, $\mu=r$. Also,

$$\frac{dy}{dx}=\frac{1}{\sigma\sqrt{T}}\frac{1}{x}.$$

The integral above then simplifies to

$$\frac{1}{\sqrt{2\pi}}\int_{-d_2}^\infty e^{-\frac{1}{2}y^2}\,dy=1-N(-d_2)=N(d_2).$$

12.5 To replicate the option, the portfolio must have value at time T given by Λ_{up} when $S(T)=S_{\text{up}}$, and Λ_{down} when $S(T)=S_{\text{down}}$. This gives two the equations

$$\begin{aligned} AS_{\text{up}}+Ce^{rT} &= \Lambda_{\text{up}}, && (1)\\ AS_{\text{down}}+Ce^{rT} &= \Lambda_{\text{down}}. && (2)\end{aligned}$$

Subtracting (2) from (1) gives (12.5). Substituting this into (2) then gives (12.6). The price of the portfolio at time $t=0$ is therefore AS_0+C, which, using (12.5)–(12.6), can be rerranged to (12.7). If the option is valued *above* this level, then at $t=0$ an arbitrageur could *sell* the option and *buy* the portfolio. This gives an instant, riskless, profit at $t=0$, because whichever of the two asset prices, S_{up} or S_{down}, prevailed at expiry, the arbitrageur can pay off the option holder using the funds in the portfolio. The arbitrageur has locked into a guaranteed, instananeous, riskless profit, which violates the no arbitrage principle. We conclude that the time-zero option value cannot exceed (12.7).

An analogous argument using the words *below*, *buy* and *sell* shows that the time-zero option value cannot be less than (12.7), and hence (12.7) is the fair price.

Now, if $q < 0$ then $S_0 e^{rT} < S_{\text{down}}$. This means that both of the possible asset values at expiry correspond to better performance than cash in the bank. Thus, there is an arbitrage opportunity: using loan from the bank to buy the asset (and selling the asset at expiry to replay the loan) guarantees a profit with no outlay. Hence $q < 0$ cannot hold. Similarly, if $q > 1$ then $S_0 e^{rT} > S_{\text{up}}$. This means that both of the possible asset values at expiry correspond to worse performance than cash in the bank. Thus, there is an arbitrage opportunity: short selling the asset and investing the proceeds in the bank (and buying the asset with that cash at expiry to cover the short sale) guarantees a profit with no outlay. Hence $q > 1$ cannot hold. Thus, by the no arbitrage principle we have $0 < q < 1$.

Using the definition of q, we may rearrange (12.7) into the form

$$e^{-rT}\left[(1-q)\Lambda_{\text{down}} + q\Lambda_{\text{up}}\right],$$

which is precisely the discounted expected payoff for an asset taking the values

$$\begin{aligned} S(T) &= S_{\text{up}} > S_0, && \text{with probability } q \\ S(T) &= S_{\text{down}} < S_0, && \text{with probability } 1-q. \end{aligned}$$

Although this question is based on an artificially simple scenario, a number of features ring bells from the Black–Scholes analysis.

1. The probability p does not affect the option value, just as the drift parameter μ does not appear in the Black–Scholes PDE. (So, two investors who agree on the two possible asset values S_{up} and S_{down}, but have wildly different views about the probability p, will agree on the option value. An analogous statement for the Back–Scholes case appeared in Chapter 11.)
2. With a little imagination, the expression (12.5) for the asset holding can be likened to the delta value $\partial C/\partial S$ that arose in the Black–Scholes hedging argument.
3. The time-zero option value is not simply the discounted expected payoff for the asset model, $p\Lambda_{\text{up}} + (1-p)\Lambda_{\text{down}}$. However, it **is** the discounted expected payoff for a different asset model that does not involve p. This chapter showed that an analogous statement is true in the Black–Scholes case.