Chapter Sixteen. Binomial method

Outline Solutions to odd-numbered exercises from the book:
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16.1 Let Y_i be defined by (16.10). Given $S(t_i)$, we see that $S(t_{i+1})$ in (6.2) has two possible values

$$S(t_{i+1}) = \begin{cases} S(t_i) + \mu\delta t S(t_i) + \sigma\sqrt{\delta t}\left(\frac{u-1-\mu\delta t}{\sigma\sqrt{\delta t}}\right) S(t_i) = uS(t_i), & \text{with probability } p, \\ S(t_i) + \mu\delta t S(t_i) + \sigma\sqrt{\delta t}\left(\frac{d-1-\mu\delta t}{\sigma\sqrt{\delta t}}\right) S(t_i) = dS(t_i), & \text{with probability } 1-p. \end{cases}$$

16.3 Setting $p = \frac{1}{2}$ in (16.5) gives

$$\tfrac{1}{2}\log(u) + \tfrac{1}{2}\log(d) = \left(r - \tfrac{1}{2}\sigma^2\right)\delta t \qquad \text{(A)}$$

and in (16.6) gives

$$\log(u) - \log(d) = \sigma 2\sqrt{\delta t}. \qquad \text{(B)}$$

Then 2(A) + (B) gives

$$\begin{aligned} 2\log(u) &= 2\left\{\left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t}\right\} \\ \Rightarrow \quad \log u &= \left(r - \tfrac{1}{2}\sigma^2\right)\delta t + \sigma\sqrt{\delta t} \\ \Rightarrow \quad u &= e^{\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t}. \end{aligned}$$

Similarly, 2(A) - (B) gives

$$2\log(d) = 2\left\{\left(r - \tfrac{1}{2}\sigma^2\right)\delta t - \sigma\sqrt{\delta t}\right\}$$

So

$$d = e^{-\sigma\sqrt{\delta t} + (r - \frac{1}{2}\sigma^2)\delta t}.$$

16.5 Setting $\mu = r$ and $p = \frac{1}{2}$ we have (using (3.1))

$$0 = \mathbb{E}(Y_i) = \tfrac{1}{2}\left(\frac{u - 1 - r\delta t}{\sigma\sqrt{\delta t}}\right) + \tfrac{1}{2}\left(\frac{d - 1 - r\delta t}{\sigma\sqrt{\delta t}}\right).$$

Hence,

$$\left(\frac{u-1-r\delta t}{\sigma\sqrt{\delta t}}\right) = -\left(\frac{d-1-r\delta t}{\sigma\sqrt{\delta t}}\right). \qquad \text{(C)}$$

Now, since $\mathbb{E}(Y_i) = 0$, we have $\mathbb{V}ar(Y_i) = \mathbb{E}(Y_i^2)$, so we need

$$1 = \tfrac{1}{2}\left(\frac{u-1-r\delta t}{\sigma\sqrt{\delta t}}\right)^2 + \tfrac{1}{2}\left(\frac{d-1-r\delta t}{\sigma\sqrt{\delta t}}\right)^2.$$

Using (C), we see that this is equivalent to

$$\frac{u-1-r\delta t}{\sigma\sqrt{\delta t}} = 1, \quad \frac{d-1-r\delta t}{\sigma\sqrt{\delta t}} = -1$$

so that $u = 1 + \sigma\sqrt{\delta t} + r\delta t$ and $d = 1 - \sigma\sqrt{\delta t} + r\delta t$.

16.7 The required $i \times (i+1)$ matrix has the form

$$e^{-r\delta t}\begin{bmatrix} 1-p & p & & & & \\ & 1-p & p & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 1-p & p \end{bmatrix}$$