Chapter Seventeen. Cash-or-nothing options

17.1 Consider a portfolio consisting of a cash-or-nothing call and a cash-or-nothing put, with the same strike prices and expiry dates. Consider the possible asset prices at expiry.

If $S(T) < E$, the put pays A and the call pays nothing.

If $S(T) = E$, the put and call both pay $A/2$.

If $S(T) > E$, the call pays A and the put pays nothing.

Hence, the overall payoff at time T is guaranteed to be A. The no arbitrage principle implies that the portfolio must be worth the same as investing in bank. If it were valued at less than this on the market, an arbitrageur could buy the portfolio and make a guaranteed profit greater than that given by investing in a bank. Similarly, if it were valued at more than this on the market, an arbitrageur could sell the portfolio and make a guaranteed profit greater than that given by investing in a bank. Hence, discounting for interest, we must have

$$C^{\text{cash}}(S,t) + P^{\text{cash}}(S,t) = Ae^{-r(T-t)}.$$

17.3 Differentiating the expression for $C^{\text{cash}}(S,t)$ in (17.4) gives

$$\frac{\partial C^{\text{cash}}}{\partial S} = Ae^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial S} = \frac{Ae^{-r(T-t)}N'(d_2)}{\sigma S\sqrt{T-t}},$$

where we have used

$$\frac{\partial d_2}{\partial S} = \frac{1}{\sigma S\sqrt{T-t}}.$$

Now, noting that $N''(x) = -xN'(x)$ we may differentiate this result to get

$$\frac{\partial^2 C^{\text{cash}}}{\partial S^2} = Ae^{-r(T-t)}\left(\frac{N''(d_2)}{\sigma S\sqrt{T-t}}\frac{\partial d_2}{\partial S} - \frac{N'(d_2)}{S^2\sigma\sqrt{T-t}}\right)$$

$$= -Ae^{-r(T-t)}\left(\frac{d_2N'(d_2)}{S^2\sigma^2(T-t)} + \frac{N'(d_2)}{S^2\sigma\sqrt{T-t}}\right)$$

$$= -Ae^{-r(T-t)}\frac{N'(d_2)}{S^2\sigma^2(T-t)}\left(d_2 + \sigma\sqrt{T-t}\right)$$

$$= -Ae^{-r(T-t)}d_1\frac{N'(d_2)}{S^2\sigma^2(T-t)},$$

as required.

Finally, we have

$$\begin{aligned}
\frac{\partial d_2}{\partial t} &= \frac{\log(S/E)}{\sigma}(-\tfrac{1}{2})(T-t)^{-3/2}(-1) + \frac{r - \tfrac{1}{2}\sigma^2}{\sigma}(\tfrac{1}{2})(T-t)^{-\frac{1}{2}}(-1) \\
&= \frac{1}{2\sigma\sqrt{T-t}}\left(\frac{\log(S/E)}{T-t} - (r - \tfrac{1}{2}\sigma^2)\right) \\
&= \frac{1}{2(T-t)}\left(\frac{\log(S/E) + (r + \tfrac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} - \frac{2r(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{1}{2(T-t)}\left(d_1 - \frac{2r(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= \frac{d_1}{2(T-t)} - \frac{r}{\sigma\sqrt{T-t}}
\end{aligned}$$

So,

$$\frac{\partial C^{\mathrm{cash}}}{\partial t} = Are^{-r(T-t)}N(d_2) + Ae^{-r(T-t)}N'(d_2)\left\{\frac{d_1}{2(T-t)} - \frac{r}{\sigma\sqrt{T-t}}\right\}.$$

17.5 Using $N'(x) = (1/\sqrt{2\pi})e^{-\frac{1}{2}x^2}$ in (17.5), we have

$$\frac{\partial C^{\mathrm{cash}}}{\partial S} = \frac{Ae^{-r(T-t)}e^{-\frac{1}{2}d_2^2}}{\sqrt{2\pi}\sigma S\sqrt{T-t}},$$

Consider the limit $t \to T^-$.

Case 1: $S > E$ here $d_2 \to \infty$. The $e^{-\frac{1}{2}d_2^2}$ factor in the numerator dominates the $\sqrt{T-t}$ factor in the denominator to give $\partial C^{\mathrm{cash}}/\partial S \to 0$.

Case 2: $S = E$ here $d_2 \to 0$. The $e^{-\frac{1}{2}d_2^2}$ factor in the numerator tends to one, so the $\sqrt{T-t}$ factor in the denominator causes $\partial C^{\mathrm{cash}}/\partial S \to \infty$.

Case 3: $S < E$ here $d_2 \to -\infty$. The $e^{-\frac{1}{2}d_2^2}$ factor in the numerator dominates the $\sqrt{T-t}$ factor in the denominator to give $\partial C^{\mathrm{cash}}/\partial S \to 0$.

17.7 Payoff diagram is

Asset–or–Nothing Call
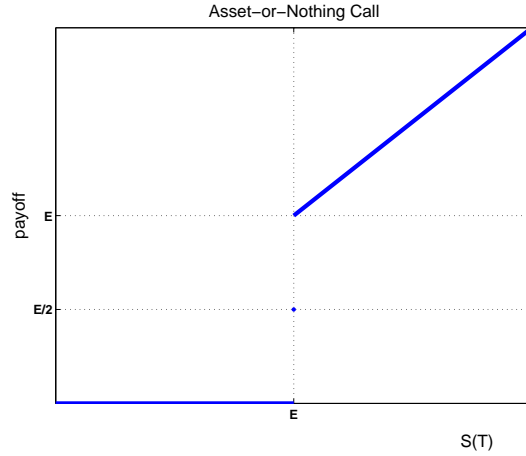
Setting $\mu = r$ and computing $e^{-r(T-t)}\mathbb{E}(\text{payoff from } S, t)$ gives

$$e^{-r(T-t)} \int_{-\infty}^{\infty} \Lambda(x) \frac{\exp\left(\frac{-(\log(x/S)-(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}\right)}{x\sigma\sqrt{2\pi(T-t)}} \, dx,$$

which becomes

$$e^{-r(T-t)} \int_{E}^{\infty} \frac{\exp\left(\frac{-(\log(x/S)-(r-\sigma^2/2)(T-t))^2}{2\sigma^2(T-t)}\right)}{\sigma\sqrt{2\pi(T-t)}} \, dx.$$

We obviously need to change variable. The trick is to spot that the lower limit of integration should be $-d_1$. Hence, let

$$y = \frac{\log(x/S) - (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

When $x = E$, we have $y = -d_1$, when $x \to \infty$, we have $y \to \infty$, and $dy/dx = 1/(x\sigma\sqrt{T-t})$. Also $x = Se^{\sigma\sqrt{T-t}y+(r+\frac{1}{2}\sigma^2)(T-t)}$. The integral becomes

$$e^{-r(T-t)} \int_{-d_1}^{\infty} \frac{e^{-\frac{1}{2}(y+\sigma\sqrt{T-t})^2}}{\sqrt{2\pi}} Se^{\sigma\sqrt{T-t}y+(r+\frac{1}{2}\sigma^2)(T-t)} \, dy = S \int_{-d_1}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} \, dy = SN(d_1).$$

An asset-or-nothing put option has payoff function $\Lambda(S(T))$ of the form

$$\Lambda(x) = \begin{cases} 0 & \text{for } x > E, \\ x/2 & \text{for } x = E, \\ x & \text{for } x < E. \end{cases}$$

We could value this via the risk-neutral approach above. It is quicker, however, to use put-call parity. A portfolio of one asset-or-nothing call and

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one asset-or-nothing put is equivalent to holding the asset. (Since the payoff is always $S(T)$.) Hence, from the no arbitrage principle,

$$\text{Call} + \text{Put} = S(t).$$

Hence, the asset-or-nothing put option has value at asset price S and time t given by $S(1 - N(d_1))$, or $SN(-d_1)$.