Chapter Eighteen. American options

Outline Solutions to odd-numbered exercises from the book:
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18.1 Let $C^{\mathrm{Am}}(S,t)$ denote the American call value. Arguing as in section 18.3, we find that

$$C^{\mathrm{Am}}(S,t) \geq \Lambda(S(t)), \qquad \text{for all } 0 \leq t \leq T,\ S \geq 0, \tag{1}$$

with $\Lambda(S(t)) = \max(S(t) - E, 0)$. Then

$$\frac{\partial C^{\mathrm{Am}}}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C^{\mathrm{Am}}}{\partial S^2} + rS\frac{\partial C^{\mathrm{Am}}}{\partial S} - rC^{\mathrm{Am}} \leq 0. \tag{2}$$

Then

$$\text{for each } S,t \text{ one of (1) or (2) is at equality.} \tag{3}$$

Then

$$C^{\mathrm{Am}}(S,T) = \Lambda(S(T)), \qquad \text{for all } S \geq 0, \tag{4}$$

$$C^{\mathrm{Am}}(S,t) \to 0, \quad \text{as } S \to 0, \quad \text{for all } 0 \leq t \leq T. \tag{5}$$

and

$$C^{\mathrm{Am}}(S,t) \approx S, \quad \text{as } S \to \infty, \quad \text{for all } 0 \leq t \leq T. \tag{6}$$

Let's see whether we can solve these conditons with $C^{\mathrm{Am}}(S,t) \equiv C(S,t)$, where $C(S,t)$ is the European option value. We know from Section 10.4 that $C(S,t)$ satisfies the Black–Scholes PDE, so (2) holds with equality. Hence, (3) is also true. We also know that $C(S,t)$ satisfies (4), (5) and (6)—see Exercise 8.3. Now the time-zero result (2.4) that we proved in Chapter 2 may be generalized to a time t result $C(S,t) \geq \max(S - Ee^{-r(T-t)}, 0)$, which implies (1).

Since the European value, $C(S,t)$, satisfies all the conditions needed for the Anerican value, we must have $C^{\mathrm{Am}}(S,t) \equiv C(S,t)$.

18.3 At any time t, it is clearly optimal not to exercise when $S(T) > E$ (since exercising would give zero payoff.) Hence all points (S,t) with $S > E$ must be above the exercise boundary, for all $0 \leq t \leq T$. At expiry it is clearly optimal to exercise if $S(T) < E$. Hence all points (S,T) with $S < E$ must be below the exercise boundary. It follows that (E, T) must be the location of the exercise boundary at expiry; that is, $S^\star(T) = E$.

To show that $S^\star(t)$ is a non-decreasing function of t, it is equivalent to show the following.

> For a fixed asset price $\widehat{S}$ and two times t_1 and t_2, with $0 \le t_1 < t_2 \le T$, if it is optimal to exercise at $(\widehat{S}, t_1)$ then it is optimal to exercise at $(\widehat{S}, t_2)$. (In other words, if $(\widehat{S}, t_1)$ is in the exercise region then $(\widehat{S}, t_2)$ is in the exercise region.)

Suppose that it is optimal to exercise at $(\widehat{S}, t_1)$. We can prove that it is therefore optimal to exercise at $(\widehat{S}, t_2)$ via the following sequence of arguments.

1. We have $P^{\mathrm{Am}}(\widehat{S}, t_1) \ge P^{\mathrm{Am}}(\widehat{S}, t_2)$; that is, the American put at time t_1 with asset price $\widehat{S}$ is worth more than the American put at time t_2 with asset price $\widehat{S}$. (This is because the former case gives all the opportunities of the latter, plus some extra opportunities.)
2. We have $P^{\mathrm{Am}}(\widehat{S}, t_1) = E - \widehat{S}$; that is, the value of the American put at time t_1 and asset price $\widehat{S}$ is that given by exercising.
3. We have $P^{\mathrm{Am}}(\widehat{S}, t_2) \le E - \widehat{S}$. This follows from points 1 and 2.
4. If the asset price is $\widehat{S}$ at time t_2, then we can obtain the value $E - \widehat{S}$ by exercising at that time. From point 3 it must therefore be optimal to exercise in this circumstance.

18.5 Strategy 1 is OK, as it uses a stopping time. (Decision to exercise at time $t^\star$ uses only information about $S(t)$ between $0 \le t \le t^\star$. Actually it only uses $S(t^\star)$.)

Strategy 2 is not OK, as it does not use a stopping time. (Decision to exercise at time $t^\star$ requires knowledge of $S(t)$ at times beyond $t^\star$.)

Strategy 3 is OK, as it uses a stopping time. (Decision to exercise at time $t^\star$ uses only information about $S(t)$ between $0 \le t \le t^\star$.)