

## Chapter Nineteen. Exotic options

Outline Solutions to odd-numbered exercises from the book:

*An Introduction to Financial Option Valuation:*

*Mathematics, Stochastics and Computation,*

by Desmond J. Higham, Cambridge University Press, 2004

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**19.1** We have

$$\frac{\partial \widehat{V}}{\partial t} = S^{1-2r/\sigma^2} \frac{\partial V}{\partial t} \left( \frac{X}{S}, t \right)$$

and

$$\frac{\partial \widehat{V}}{\partial S} = \left( 1 - \frac{2r}{\sigma^2} \right) S^{-2r/\sigma^2} V \left( \frac{X}{S}, t \right) - X S^{-1-2r/\sigma^2} \frac{\partial V}{\partial S} \left( \frac{X}{S}, t \right).$$

Then

$$\begin{aligned} \frac{\partial^2 \widehat{V}}{\partial S^2} &= \left( 1 - \frac{2r}{\sigma^2} \right) \left( \frac{-2r}{\sigma^2} \right) S^{-1-2r/\sigma^2} V \left( \frac{X}{S}, t \right) + \frac{\partial V}{\partial S} \left( \frac{X}{S}, t \right) \left( \frac{4Xr}{\sigma^2} \right) S^{-2-2r/\sigma^2} \\ &\quad + X^2 S^{-3-2r/\sigma^2} \frac{\partial^2 V}{\partial S^2} \left( \frac{X}{S}, t \right) \end{aligned}$$

So,

$$\begin{aligned} \frac{\partial \widehat{V}}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 \widehat{V}}{\partial S^2} + r S \frac{\partial \widehat{V}}{\partial S} - r \widehat{V} &= S^{1-2r/\sigma^2} \frac{\partial V}{\partial t} \left( \frac{X}{S}, t \right) \\ &\quad + \left( 1 - \frac{2r}{\sigma^2} \right) (-r) S^{1-2r/\sigma^2} V \left( \frac{X}{S}, t \right) \\ &\quad + 2Xr S^{-2r/\sigma^2} \frac{\partial V}{\partial S} \left( \frac{X}{S}, t \right) + \frac{1}{2} \sigma^2 X^2 S^{-1-2r/\sigma^2} \frac{\partial^2 V}{\partial S^2} \left( \frac{X}{S}, t \right) \\ &\quad + r \left( 1 - \frac{2r}{\sigma^2} \right) S^{1-2r/\sigma^2} V \left( \frac{X}{S}, t \right) - r X S^{-2r/\sigma^2} \frac{\partial V}{\partial S} \left( \frac{X}{S}, t \right) \\ &\quad - r S^{1-2r/\sigma^2} V \left( \frac{X}{S}, t \right) \\ &= S^{1-\frac{2r}{\sigma^2}} \left\{ \frac{\partial V}{\partial t} \left( \frac{X}{S}, t \right) + \frac{1}{2} \sigma^2 \left( \frac{X}{S} \right)^2 \frac{\partial^2 V}{\partial S^2} \left( \frac{X}{S}, t \right) \right. \\ &\quad \left. + r \frac{X}{S} \frac{\partial V}{\partial S} \left( \frac{X}{S}, t \right) - r V \left( \frac{X}{S}, t \right) \right\}. \end{aligned}$$

Since  $V$  solves the Black-Scholes PDE, the term inside the curly braces is zero.

**19.3** For each asset path, exactly one of the ‘in’ and the ‘out’ options will be active at expiry. Hence, holding the sum, ‘in’ plus ‘out’, is equivalent to holding the European.

**19.5** Putting  $S = B$  we find that  $f_1 = -e_2$ ,  $f_2 = -e_1$ ,  $g_1 = -d_2$  and  $g_2 = -d_1$ , so that (19.5) becomes

$$B(N(d_1) - N(e_1) - [N(-e_1) + N(-d_1)]) \\ - Ee^{-r(T-t)}(N(d_2) - N(e_2) - [N(-e_2) + N(-d_2)]) .$$

This collapses to zero when we apply the identity  $N(\alpha) + N(-\alpha) = 1$ ; see Exercise 3.9.

**19.7** With  $S_0$  given, letting  $M$  denote the number of asset paths that we sample and  $N$  denote the timestep length, with  $\Delta t = T/N$ , we have:

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for i = 1 to M
  for j = 0 to N-1
    compute a N(0, 1) sample  $\xi_j$ 
    set  $S_{j+1} = S_j e^{(r - \frac{1}{2}\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\xi_j}$ 
  end
  set  $S_{\min} = \min_{0 \leq j \leq N} S_j$ 
   $V_i = e^{-rT} \max(S_N - S_{\min}, 0)$ 
end
set  $a_M = \frac{1}{M} \sum_{i=1}^M V_i$ 
set  $b_M^2 = \frac{1}{M-1} \sum_{i=1}^M (V_i - a_M)^2$ 

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The result gives an approximate option price  $a_M$  and an approximate 95% confidence interval (15.5).