Chapter Twenty. Historical volatility

Outline Solutions to odd-numbered exercises from the book:
An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation,
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20.1 From Section 6.5 we know that if $a_M \sim N(a, b^2/M)$ then

$$\left[a_M - \frac{1.96b}{\sqrt{M}}, \; a_M + \frac{1.96b}{\sqrt{M}} \right]$$

is a 95% confidence interval for a. We have $a = \left(r - \frac{\sigma^2}{2}\right) \Delta t = \left(r - \frac{\sigma^2}{2}\right) \frac{t^*}{M}$ and $\frac{b^2}{M} = \sigma^2 \frac{\Delta t}{M}$; hence $b^2 = \sigma^2 \Delta t$. So our 95% confidence interval is

$$\left[a_M - \frac{1.96\sigma\sqrt{t^*}}{M}, \; a_M + \frac{1.96\sigma\sqrt{t^*}}{M} \right].$$

The length of the confidence interval is $2 \times \frac{1.96\sigma\sqrt{t^*}}{M}$.

The quantity we are estimating, a, is also proportional to $\frac{1}{M}$. Hence, in this case the amount of uncertainty in the result is of the same order as the exact result itself.

20.3 We have $\mathbb{E}(Y) = \mathbb{E}(\alpha + \beta Z) = \alpha + \beta \mathbb{E}(Z) = \alpha$.

Hence,

$$Y - \mathbb{E}(Y) = \beta Z$$

and

$$\mathbb{V}ar((Y - \mathbb{E}(Y))^2) = \mathbb{V}ar((\beta Z)^2) = \beta^4 \, \mathbb{V}ar(Z^2)$$

Now

$$\mathbb{V}ar(Z^2) = \mathbb{E}(Z^4) - (\mathbb{E}(Z^2))^2 = 3 - 1^2 = 2.$$

Hence, $\mathbb{V}ar((Y - \mathbb{E}(Y))^2) = 2\beta^4$.

Since $\widehat{U}_i \sim \left(r - \frac{1}{2}\sigma^2\right) \Delta t + \sigma Z$, the result above applies with $\beta = \sigma$ to give

$$\mathbb{V}ar((\widehat{U}_i - \mathbb{E}(\widehat{U}_i))^2) = 2\sigma^4.$$

20.5 Let $\widehat{U}_i = U_i/\sqrt{\Delta t}$.

Let $z = \sigma^2$. We want to maximise $G(z) := \prod_{i=1}^{M} \frac{1}{\sqrt{2\pi z}} \exp\left(-\frac{\widehat{U}^2_{n+1-i}}{2z}\right)$ for $z \geq 0$. Equiv. to maximising

$$\log G(z) = -\frac{1}{2}\left[M \log z + \sum_{i=1}^{M} \frac{\widehat{U}^2_{n+1-i}}{z}\right] + \text{constant}$$

Equiv. to **minimising** $F(z) := M \log z + \sum_{i=1}^{M} \frac{\widehat{U}^2_{n+1-i}}{z}$.

Well $\frac{dF}{dz} = \frac{M}{z} - \sum_{i=1}^{M} \frac{\widehat{U}^2_{n+1-i}}{z^2}$, so $\frac{dF}{dz} = 0$ when $z = \frac{1}{M} \sum_{i=1}^{M} \widehat{U}^2_{n+1-i}$.

Since $F(z) \to \infty$ when $z \to 0$ and when $z \to \infty$, this is the global minimum required.

20.7 Going back as far as $t_0 = t_{n+1} - (n+1)\Delta t$, we have

$$\begin{aligned}
\Delta t \sigma^{*^2}_{n+1} &= \omega\, \Delta t \sigma^{*^2}_{n} + (1-\omega)U^2_{n+1} \\
&= \omega[\omega\, \Delta t \sigma^{*^2}_{n-1} + (1-\omega)U^2_{n}] + (1-\omega)U^2_{n+1} \\
&= \omega[\omega[\Delta t \sigma^{*^2}_{n-2} + (1-\omega)U^2_{n-1}] + (1-\omega)U^2_{n}] + (1-\omega)U^2_{n+1} \\
&\vdots \\
&= \omega^{n+1} \Delta t \sigma^{*^2}_{0} + \sum_{i=0}^{n} (1-\omega)\omega^i\, U^2_{n+1-i}
\end{aligned}$$

First term on RHS is exponentially small. Second term has geometrically declining weights. (But note that M is not fixed.)