Chapter Twentyone.
Monte Carlo part II: variance reduction by antithetic variates

Outline Solutions to odd-numbered exercises from the book:
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21.1 Starting from (21.1) and using (3.6) and (3.7), we have

$$\begin{aligned}
\mathsf{cov}(X,Y) &= \mathbb{E}\left[(X-\mathbb{E}(X))(Y-\mathbb{E}(Y))\right] \\
&= \mathbb{E}\left[XY - X\mathbb{E}(Y) - \mathbb{E}(X)Y + \mathbb{E}(X)\mathbb{E}(Y))\right] \\
&= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(X)\mathbb{E}(Y) \\
&= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).
\end{aligned}$$

If X ad Y are indepedent then $\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0$ (see (3.9)), so $\mathsf{cov}(X,Y) = 0$.

21.3 Using (3.6) and (3.10), we have

$$\begin{aligned}
\mathbb{V}ar(X+Y) &= \mathbb{E}((X+Y)^2) - (\mathbb{E}(X+Y))^2 \\
&= \mathbb{E}((X+Y)^2) - (\mathbb{E}(X)+\mathbb{E}(Y))^2 \\
&= \mathbb{E}(X^2+2XY+Y^2) - \mathbb{E}(X)^2 - 2\mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)^2 \\
&= \mathbb{E}(X^2) + 2\mathbb{E}(XY) + \mathbb{E}(Y^2) - \mathbb{E}(X)^2 - 2\mathbb{E}(X)\mathbb{E}(Y) - \mathbb{E}(Y)^2 \\
&= \mathbb{E}(X^2) - \mathbb{E}(X)^2 + \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 + 2\left(\mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)\right) \\
&= \mathbb{V}ar(X) + \mathbb{V}ar(Y) + 2\mathsf{cov}(X,Y).
\end{aligned}$$

21.5 Consider the circumstance where f and g are both monotonic increasing. We look at two cases.

Case 1: If $x \ge y$ then $f(x) - f(y) \ge 0$ and $g(x) - g(y) \ge 0$, so $(f(x) - f(y))(g(x) - g(y)) \ge 0$.

Case 2: If $y \ge x$ then $f(x) - f(y) \le 0$ and $g(x) - g(y) \le 0$, so $(f(x) - f(y))(g(x) - g(y)) \ge 0$. (We could also just refer to Case 1 and say "by symmetry".)

A similar argument works for the monotonic decreasing case.

21.7 For this f, when $U \sim \mathsf{U}(0,1)$ we have $f(U_i)+f(1-U_i) = \alpha U_1+\beta+\alpha(1-U_i)+\beta = \alpha+2\beta$, so the sum in (21.13) reduces to $\frac{1}{2}\alpha+\beta$ (which is non-random). The true expected value $\mathbb{E}(f(U))$ is $\mathbb{E}(\alpha U + \beta) = \alpha\mathbb{E}(U) + \beta = \frac{1}{2}\alpha + \beta$. Hence the estimate (21.13) is exact for this f. Each computed sample of $\frac{1}{2}(f(U_i) + f(1 - U_i))$ has the same, non-random, value, so the sample variance is zero. The confidence interval then collapses to the point $\frac{1}{2}\alpha+\beta$; this correctly reflects the fact that the sample mean is exact.

Similarly, when $U \sim \mathsf{N}(0,1)$ we have $f(U_i)+f(-U_i) = \alpha U_1+\beta+\alpha(-U_i)+\beta = 2\beta$, so the sum in (21.19) reduces to β (which is non-random). The true expected value $\mathbb{E}(f(U))$ is $\mathbb{E}(\alpha U+\beta) = \alpha\mathbb{E}(U)+\beta = \beta$. Hence the estimate (21.19) is exact for this f. Each computed sample of $\frac{1}{2}(f(U_i)+f(-U_i))$ has the same, non-random, value, so the sample variance is zero. The confidence interval then collapses to the point β; this correctly reflects the fact that the sample mean is exact.