Chapter Twentythree. Finite difference methods

Outline Solutions to odd-numbered exercises from the book:
An Introduction to Financial Option Valuation:
Mathematics, Stochastics and Computation,
by Desmond J. Higham, Cambridge University Press, 2004
ISBN 0521 83884 3 (hardback)
ISBN 0521 54757 1 (paperback)

23.1 We have

$$\Delta\nabla y_m = \Delta(y_m - y_{m-1}) = y_{m+1} - y_m - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1}$$

and

$$\nabla\Delta y_m = \nabla(y_{m+1} - y_m) = y_{m+1} - y_m - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1}.$$

So $\Delta\nabla = \nabla\Delta$.

Also,

$$(\Delta - \nabla)(y_m) = (y_{m+1} - y_m) - (y_m - y_{m-1}) == y_{m+1} - 2y_m + y_{m-1}.$$

So $\Delta\nabla = \Delta - \nabla$.

Also,

$$\delta^2 y_m = \delta(\delta y_m) = \delta(y_{m+\frac{1}{2}} - y_{m-\frac{1}{2}}) = y_{m+1} - y_m - (y_m - y_{m-1}) = y_{m+1} - 2y_m + y_{m-1}).$$

So $\delta^2 = \Delta\nabla$.

Also,

$$\mu\delta y_m = \mu(\delta y_m) = \mu(y_{m+\frac{1}{2}} - y_{m-\frac{1}{2}}) = \tfrac{1}{2}(y_{m+1} - y_m + (y_m - y_{m-1})) = \tfrac{1}{2}(y_m - y_{m-1}).$$

So $\mu\delta = \Delta_0$.

Similarly,

$$\delta\mu y_m = \delta(\mu y_m) = \delta(\tfrac{1}{2}(y_{m+\frac{1}{2}} + y_{m-\frac{1}{2}})) = \tfrac{1}{2}(y_{m+1} + y_m - (y_m + y_{m-1})) = \tfrac{1}{2}(y_m - y_{m-1}).$$

So $\delta\mu = \Delta_0$.

Also,

$$\Delta^2 y_m = \Delta(\Delta y_m) = \Delta(y_{m+1} - y_{m-1}) = y_{m+2} - y_{m+1} - (y_{m+1} - y_m) = y_{m+2} - 2y_{m+1} + y_m,$$

and

$$\delta^2 E y_m = \delta^2 (E y_m) = \delta^2 (y_{m+1}) = y_{m+2} - 2y_{m+1} + y_m.$$

So $\Delta^2 = \delta^2 E$.

Further,

$$E\delta^2 y_m = E(\delta^2 y_m) = E(y_{m+1} - 2y_m + y_{m-1}) = y_{m+2} - 2y_{m+1} + y_m.$$

So $\Delta^2 = E\delta^2$.

23.3 First row of (23.9):

$$U_1^{i+1} = (1-2\nu)U_1^i + \nu U_2^i + p_1^i = (1-2\nu)U_1^i + \nu U_2^i + \nu a(ik).$$

Generally,

$$U_j^{i+1} = (1-2\nu)U_j^i + \nu U_{j+1}^i + U_{j-1}^i.$$

Last row of (23.9):

$$U_{N_x-1}^{i+1} = (1-2\nu)U_{N_x-1}^i + \nu U_{N_x-2}^i + p_{N_x-1}^i = (1-2\nu)U_{N_x-1}^i + \nu U_{N_x-2}^i + \nu b(ik).$$

Hence, the formulation is correct.

23.5 Following the FTCS analysis, we have for BTCS

$$R_j^i = \left(\frac{\partial u}{\partial t} - \tfrac{1}{2}k\frac{\partial^2 u}{\partial t^2} + O(k^2) \right) - \left(\frac{\partial^2 u}{\partial x^2} - \tfrac{1}{12}h^2\frac{\partial^4 u}{\partial x^4} + O(h^4) \right).$$

Since u satisfies the PDE (23.2), we have

$$R_j^i = -\tfrac{1}{2}k\frac{\partial^2 u}{\partial t^2} - \tfrac{1}{12}h^2\frac{\partial^4 u}{\partial x^4} + O(k^2) + O(h^4).$$

23.7 Expanding the equation in the exercise gives

$$U_j^{i+1} - \tfrac{1}{2}\nu\left[U_{j+1}^{i+1} - 2U_j^{i+1} + U_{j-1}^{i+1}\right] = U_j^i + \tfrac{1}{2}\nu\left[U_{j+1}^i - 2U_j^i + U_{j-1}^i\right],$$

which rearranges to

$$(1+\nu)U_j^{i+1} = \tfrac{1}{2}\nu U_{j+1}^{i+1} + \tfrac{1}{2}\nu U_{j-1}^{i+1} + (1-\nu)U_j^i + \tfrac{1}{2}\nu U_{j+1}^i + \tfrac{1}{2}\nu U_{j-1}^i.$$

Multiplying by 2 gives (23.18).

23.9 General row of (23.19) gives

$$(1+\nu)U_j^{i+1} - \tfrac{1}{2}\nu U_{j-1}^{i+1} - \tfrac{1}{2}\nu U_{j+1}^{i+1} = (1-\nu)U_j^i + \tfrac{1}{2}\nu U_{j-1}^i + \tfrac{1}{2}\nu U_{j+1}^i,$$

which agrees with (23.18). The vector $\mathbf{r}^i$ has $r_1^i = \tfrac{1}{2}\nu[U_0^i + U_0^{i+1}]$ and $r_{N_x-1}^i = \tfrac{1}{2}\nu[U_{N_x}^i + U_{N_x}^{i+1}]$, as required.

23.11 General row of equation in exercise is

$$\tfrac{1}{2}U_j^{i+1}+\tfrac{1}{2}\left\{(1+2\nu)U_j^{i+1}-\nu U_{j-1}^{i+1}-\nu U_{j+1}^{i+1}\right\}=\tfrac{1}{2}U_j^{i}+\tfrac{1}{2}\left\{(1-2\nu)U_j^{i}+\nu U_{j-1}^{i}+\nu U_{j+1}^{i}\right\}.$$

This rearranges to

$$(1+\nu)U_j^{i+1}=\tfrac{1}{2}\nu U_{j+1}^{i+1}+\tfrac{1}{2}\nu U_{j-1}^{i+1}+(1-\nu)U_j^{i}+\tfrac{1}{2}\nu U_{j-1}^{i}+\tfrac{1}{2}\nu U_{j+1}^{i},$$

which is equivalent to (23.18). Also, $\frac{1}{2}(p^i+q^i)_1=r_1^i$ and $\frac{1}{2}(p^i+q^i)_{N_x-1}=r_{N_x-1}^i$, as required.