Chapter Twentyfour.
Finite difference methods for the Black–Scholes PDE

Outline Solutions to odd-numbered exercises from the book:
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Mathematics, Stochastics and Computation,
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24.1 General equation of (23.9) for this F and $\mathbf{p}^i$ is

$$U_j^{i+1} = (1 - rk)U_j^i + \tfrac{1}{2}k\sigma^2(j^2)\left\{U_{j+1}^i - 2U_j^i + U_{j-1}^i\right\} + \tfrac{1}{2}krj\left\{U_{j+1}^i - U_{j-1}^i\right\},$$

which rearranges to

$$\frac{U_j^{i+1} - U_j^i}{k} - \tfrac{1}{2}\sigma^2(jh)^2\left\{\frac{U_{j+1}^i - 2U_j^i + U_{j-1}^i}{h^2}\right\} - rjh\left\{\frac{U_{j+1}^i - U_{j-1}^i}{2h}\right\} + rU_j^i = 0.$$

So (24.6) is reproduced. The boundary conditions are dealt with correctly, because

$$p_1^i = \tfrac{1}{2}k\sigma^2(1^2)U_0^i - \tfrac{1}{2}kr(1)U_0^i = \tfrac{1}{2}k(\sigma^2 - r)U_0^i$$

and

$$p_{N_x-1}^i = \tfrac{1}{2}k\sigma^2((N_x-1)^2)U_{N_x}^i + \tfrac{1}{2}kr(N_x-1)U_{N_x}^i = \tfrac{1}{2}k(N_x-1)(\sigma^2(N_x-1)+r)U_{N_x}^i.$$

General equation of (23.11) for this B is

$$U_j^{i+1} + rkU_j^{i+1} - \tfrac{1}{2}k\sigma^2(j^2)\left\{U_{j+1}^{i+1} - 2U_j^{i+1} + U_{j-1}^{i+1}\right\} - \tfrac{1}{2}krj\left\{U_{j+1}^{i+1} - U_{j-1}^{i+1}\right\} = U_j^i,$$

which rearranges to

$$\frac{U_j^{i+1} - U_j^i}{k} - \tfrac{1}{2}\sigma^2(jh)^2\left\{\frac{U_{j+1}^{i+1} - 2U_j^{i+1} + U_{j-1}^{i+1}}{h^2}\right\} - rjh\left\{\frac{U_{j+1}^{i+1} - U_{j-1}^{i+1}}{2h}\right\} = 0.$$

So (24.6) is reproduced. The boundary conditions are dealt with correctly, because

$$q_1^i = \tfrac{1}{2}k\sigma^2(1^2)U_0^{i+1} - \tfrac{1}{2}kr(1)U_0^{i+1} = \tfrac{1}{2}k(\sigma^2 - r)U_0^{i+1}$$

and

$$q_{N_x-1}^i = \tfrac{1}{2}k\sigma^2((N_x-1)^2)U_{N_x}^{i+1} + \tfrac{1}{2}kr(N_x-1)U_{N_x}^{i+1} = \tfrac{1}{2}k(N_x-1)(\sigma^2(N_x-1)+r)U_{N_x}^{i+1}.$$

24.3 The FTCS form (23.9) applies with

$$F = (1 - rk)I + \tfrac{1}{2}k\sigma^2 \widehat{D}_2 T_2 + \tfrac{1}{2}kr\widehat{D}_1 T_1,$$

where

$$\widehat{D}_1 = \begin{bmatrix} \frac{B+h}{h} & 0 & \dots & \dots & 0 \\ 0 & \frac{B+2h}{h} & 0 & \ddots & \vdots \\ \vdots & 0 & \frac{B+3h}{h} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{B+(N_x-1)h}{h} \end{bmatrix}$$

and $\widehat{D}_2 = \widehat{D}_1^2$, with

$$\mathbf{p}^i = \begin{bmatrix} \left(k\frac{1}{2}\sigma^2\frac{(B+h)^2}{h^2} - rk\frac{(B+h)}{2h}\right) V_0^i \\ 0 \\ \vdots \\ 0 \\ \left(k\frac{1}{2}\sigma^2\frac{(B+(N_x-1)h)^2}{h^2} - +rk\frac{(B+(N_x-1)h)}{2h}\right) V_{N_x}^i \end{bmatrix}.$$

Similarly, the BTCS form (23.11) applies with

$$B = (1 + rk)I - \tfrac{1}{2}k\sigma^2 \widehat{D}_2 T_2 - \tfrac{1}{2}kr\widehat{D}_1 T_1$$

and

$$\mathbf{q}^i = \begin{bmatrix} \left(k\frac{1}{2}\sigma^2\frac{(B+h)^2}{h^2} - rk\frac{(B+h)}{2h}\right) V_0^{i+1} \\ 0 \\ \vdots \\ 0 \\ \left(k\frac{1}{2}\sigma^2\frac{(B+(N_x-1)h)^2}{h^2} - +rk\frac{(B+(N_x-1)h)}{2h}\right) V_{N_x}^{i+1} \end{bmatrix}.$$

Crank-Nicolson is then given by (24.8).

24.5 Repeating the analysis that leads to (23.17), we simply need to re-define $\nu = k/h^2$ to $\nu = \frac{1}{2}\sigma^2 k/h^2$. So (23.17) becomes $\frac{1}{2}\sigma^2 k/h^2 \leq \frac{1}{2}$, that is, $\sigma^2 k \leq h^2$.