Chapter Three. Random Variables

Outline Solutions to odd-numbered exercises from the book:
An Introduction to Financial Option Valuation: Mathematics, Stochastics and Computation,
by Desmond J. Higham, Cambridge University Press, 2004
ISBN 0521 83884 3 (hardback)
ISBN 0521 54757 1 (paperback)

3.1 From (3.3) we have

$$\mathbb{P}(x_1 \leq X \leq x_2) = \int_{x_1}^{x_2} f(x)dx,$$

where the density function $f(x)$ is defined in (3.5). Since the integrand is the constant $1/(\beta-\alpha)$ over the range of integration, we get

$$\mathbb{P}(x_1 \leq X \leq x_2) = \frac{x_2 - x_1}{\beta - \alpha}.$$

3.3 We have

$$\begin{aligned}\mathbb{E}\left((X-\mathbb{E}(X))^2\right) = \mathbb{E}\left(X^2 - 2X\mathbb{E}(X) + (\mathbb{E}(X))^2\right) &= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + (\mathbb{E}(X))^2 \\ &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2.\end{aligned}$$

Then

$$\mathbb{V}ar(\alpha X) = \mathbb{E}(\alpha^2 X^2) - (\mathbb{E}(\alpha X))^2 = \alpha^2\mathbb{E}(X^2) - \alpha^2(\mathbb{E}(X))^2 = \alpha^2\mathbb{V}ar(X).$$

3.5 We have

$$\begin{aligned}\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}\,dx &= \frac{1}{\beta-\alpha}\left[\frac{x^3}{3}\right]_{\alpha}^{\beta} \\ &= \frac{1}{\beta-\alpha}\left(\frac{\beta^3-\alpha^3}{3}\right) \\ &= \frac{1}{\beta-\alpha}\frac{(\beta-\alpha)(\beta^2+\alpha\beta+\alpha^2)}{3} \\ &= \frac{\beta^2+\alpha\beta+\alpha^2}{3}\end{aligned}$$

and

$$\mathbb{V}ar(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = \frac{\beta^2+\alpha\beta+\alpha^2}{3} - (\frac{\alpha+\beta}{2})^2 = \frac{(\alpha+\beta)^2}{12}.$$

3.7 We have

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} \frac{xe^{-x^2/2}}{\sqrt{2\pi}}\, dx = 0,$$

because integrand is odd, and, integrating by parts,

$$\begin{aligned} \mathbb{E}(X^2) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x^2\, e^{-x^2/2} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\; xe^{-x^2/2} dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ \left[-xe^{-x^2/2} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} dx \right\} \\ &= 0 + 1 = 1. \end{aligned}$$

So $\mathbb{V}ar(X) := \mathbb{E}(X^2) - (\mathbb{E}(X))^2 = 1$.

Generally,

$$\mathbb{E}(X^p) = \int_{-\infty}^{\infty} \frac{x^p e^{-x^2/2}}{\sqrt{2\pi}}\, dx = 0 \quad \text{when } p \text{ is odd}$$

because integrand is odd. Letting $I_p = \mathbb{E}(X^p)$ for p even, we have

$$I_p = \int_{-\infty}^{\infty} x^{p-1} \frac{xe^{-x^2/2}}{\sqrt{2\pi}}\, dx = \left[-x^{p-1} \frac{e^{-x^2/2}}{\sqrt{2\pi}} \right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (p-1) x^{p-2} \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx$$

Hence, $I_p = 0 + (p-1) I_{p-2}$.

So $I_4 = 3I_2 = 3$.

Generally $I_p = (p-1)(p-3)(p-5)\ldots 1$,

$$\text{i.e.} \qquad \mathbb{E}(X^p) = (p-1)(p-3)(p-5)\ldots 1 \quad \text{for } p \text{ even}.$$

This expression is sometimes called the *skip factorial*.

3.9 From the symmetry of the bell-shaped curve,

$$\begin{aligned} N(\alpha) + N(-\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{s^2}{2}}\, ds + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\alpha} e^{-\frac{s^2}{2}}\, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{-\frac{s^2}{2}}\, ds + \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{s^2}{2}}\, ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\, ds \\ &= 1. \end{aligned}$$