Chapter Seven. Asset price model: part II

Outline Solutions to odd-numbered exercises from the book:
An Introduction to Financial Option Valuation:
Mathematics, Stochastics and Computation,
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7.1 We have, using $\mathbb{E}[Y_i] = 0$ and $\mathbb{E}[Y_i^2] = 1$,

$$\mathbb{E}\left[\left(\frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\right)^2\right] = \mathbb{E}\left[(\mu\delta t + \sigma\sqrt{\delta t}Y_i)^2\right] = \mu^2\delta t^2 + 2\mu\delta t\sigma\sqrt{\delta t}\mathbb{E}[Y_i] + \sigma^2\delta t\mathbb{E}\left[Y_i^2\right]$$
$$= \mu^2\delta t^2 + \sigma^2\delta t,$$

which has the form "$\sigma^2\delta t$ plus higher powers of δt".

Similarly, using $\mathbb{E}[Y_i^3] = 0$ and $\mathbb{E}[Y_i^4] = 3$ (Exercise 3.7), we have

$$\begin{aligned}\mathbb{E}\left[\left(\frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\right)^4\right] &= \mathbb{E}\left[(\mu\delta t + \sigma\sqrt{\delta t}Y_i)^4\right] \\ &= \mu^4\delta t^4 + 4\mu^3\delta t^3\sigma\sqrt{\delta t}\mathbb{E}[Y_i] + 6\mu^2\delta t^2\sigma^2\delta t\mathbb{E}[Y_i^2] \\ &\quad + 4\mu\delta t\sigma^3\delta t^{3/2}\mathbb{E}[Y_i^3] + \sigma^4\delta t^2\mathbb{E}[Y_i^4] \\ &= 3\sigma^4\delta t^2 + \text{ higher powers.}\end{aligned}$$

So

$$\begin{aligned}\mathbb{V}ar\left[\left(\frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\right)^2\right] &= \mathbb{E}\left[\left(\frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\right)^4\right] - \left(\mathbb{E}\left[\left(\frac{S(t_{i+1}) - S(t_i)}{S(t_i)}\right)^2\right]\right)^2 \\ &= 3\sigma^4\delta t^2 - \sigma^4\delta t^2 + \text{ higher powers} \\ &= 2\sigma^4\delta t^2 + \text{ higher powers.}\end{aligned}$$