Chapter Eight. Black–Scholes PDE and Formulas

Outline Solutions to odd-numbered exercises from the book:
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8.1 Letting $Z_i = \mu^2 \delta t^2 + 2\mu\sigma\delta t^{\frac{3}{2}} Y_i + \sigma^2 \delta t Y_i^2$, we have

$$\mathbb{E}(Z_i) = \sigma^2 \delta t + \text{ higher powers of } \delta t,$$

and, using $\mathbb{E}(Y_i^3) = 0$ and $\mathbb{E}(Y_i^4) = 3$ from Exercise 3.7,

$$\mathbb{E}(Z_i^2) = 3\sigma^4 \delta t^2 + \text{ higher powers of } \delta t.$$

So

$$\mathbb{V}ar(Z_i) = \mathbb{E}(Z_i^2) - (\mathbb{E}(Z_i))^2 = 2\sigma^4 \delta t^2 + \text{ higher powers of } \delta t.$$

Hence, from the Central Limit Theorem, the LHS in (8.2) should look like

$$S(t)^2 N(L\sigma^2 \delta t, L 2\sigma^4 \delta t^2).$$

Since $L\delta t = \Delta t$, this becomes

$$S(t)^2 N(\sigma^2 \Delta t, 2\sigma^4 \Delta t \delta t).$$

8.3 Case 1: If $S < E$ then $\log(S/E) < 0$, so $d_1 \to -\infty$ and $d_2 \to -\infty$ as $t \to T^-$. Hence, in this case, $N(d_1)$ and $N(d_2)$ tend to zero as $t \to T^-$, so, in (8.19), $C(S,t)$ tends to zero.

Case 2: If $S = E$ then $d_1 \to 0$ and $d_2 \to 0$ as $t \to T^-$, so $N(d_1)$ and $N(d_2)$ tend to $\frac{1}{2}$. In (8.19) we get $C(S,t)$ tending to $\frac{1}{2}(S - E) = 0$.

Case 3: If $S > E$ then $\log(S/E) > 0$, so $d_1 \to \infty$ and $d_2 \to \infty$ as $t \to T^-$. Hence, in this case, $N(d_1)$ and $N(d_2)$ tend to one as $t \to T^-$, so, in (8.19), $C(S,t)$ tends to $S - E$.

Combining the three cases, we see that

$$\lim_{t \to T^-} C(S,t) = \max(S(T) - E, 0).$$

As $S \to 0^+$ we have $d_1 \to -\infty$ and $d_2 \to -\infty$, giving $C(S,t) \to 0$ in (8.19).

For large S we have d_1 and d_2 large, so that $N(d_1) \approx 1$ and $N(d_2) \approx 1$, so that $C(S,t) \approx S - Ee^{-r(T-t)} \approx S$ in (8.19).

8.5 Using the analysis in Exercise 8.3:

Case 1: If $S < E$ then $N(-d_1)$ and $N(-d_2)$ tend to one as $t \to T^-$, so, in (8.24), $P(S,t)$ tends to $E - S$.

Case 2: If $S = E$ then $N(-d_1)$ and $N(-d_2)$ tend to $\frac{1}{2}$. In (8.24) we get $P(S,t)$ tending to $\frac{1}{2}(E - S) = 0$.

Case 3: If $S > E$ then $N(-d_1)$ and $N(-d_2)$ tend to zero as $t \to T^-$, so, in (8.24), $P(S,t)$ tends to zero.

Combining the three cases, we see that

$$\lim_{t \to T^-} C(S,t) = \max(E - S(T), 0).$$

As $S \to 0^+$ we have $-d_1 \to \infty$ and $-d_2 \to \infty$, giving $P(S,t) \to Ee^{-r(T-t)}$ in (8.19).

For large S we have $-d_1$ and $-d_2$ very negative, so that $N(-d_1) \approx 0$ and $N(-d_2) \approx 0$, so that $P(S,t) \approx 0$ in (8.24).

8.7 As $E \to 0$ we have $\log(S/E) \to \infty$, and hence $d_1 \to \infty$ and $d_2 \to \infty$. This means that $N(d_1) \to 1$ and $N(d_2) \to 1$. So $SN(d_1) \to S$ and $Ee^{-r(T-t)}N(d_2) \to 0$. Overall, this gives $C(S,t) \to S$ in (8.19).

From a financial perspective, if the strike price $E = 0$ then there is a *guaranteed payoff* of $S(T)$. If the time-t option value were less than $S(t)$, then a speculator could buy the option and short-sell the asset. The option can then be bought (for price zero) at expiry and used to return the short-sold asset at expiry. This brings an immediate profit at time t and zero payoff at no risk at time T. Such an arbitrage opportunity cannot exist. Similarly, if the time-t option value were more than $S(t)$, then a speculator could sell the option and buy the asset. The speculator can fulfil the option by handing over the asset (for price zero) at expiry. This brings an immediate profit at time t and zero payoff at no risk at time T. Such an arbitrage opportunity cannot exist. Hence, to avoid an arbitrage opportunity, the time-t option value must be exactly $S(t)$.

Similarly, for the put, as $E \to 0$ we have $N(-d_1) \to 0$ and $N(-d_2) \to 0$, so that $P(S,t) \to 0$ in (8.24). From a financial perspective, it is clear that the option to "sell the asset for price $E = 0$ at expiry" will never be exercised; hence the option is worth nothing.

8.9 We have

$$\begin{aligned} \frac{\partial V}{\partial t} &= -(\sigma^2 - 2r)\frac{e^{(\sigma^2-2r)(T-t)}}{S} \\ \frac{\partial V}{\partial S} &= -\frac{e^{(\sigma^2-2r)(T-t)}}{S^2} \end{aligned}$$

$$\frac{\partial^2 V}{\partial S^2} = 2\frac{e^{(\sigma^2-2r)(T-t)}}{S^3}$$

Inserting this into LHS of the the Black–Scholes PDE (8.15) we find that

$$\begin{aligned}
\frac{\partial V}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV &= e^{(\sigma^2-2r)(T-t)} \left[\frac{-(\sigma^2-2r)}{S} + \tfrac{1}{2}\sigma^2 S^2 \frac{2}{S^3} \right. \\
&\quad \left. + rS\left(\frac{-1}{S^2}\right) - \frac{r}{S} \right] \\
&= e^{(\sigma^2-2r)(T-t)} \left[\frac{-\sigma^2}{S} + \frac{2r}{S} + \frac{\sigma^2}{S} - \frac{r}{S} - \frac{r}{S} \right] \\
&= 0
\end{aligned}$$

The practical implication is that this function $V(S,t)$ must be the value at time t and asset price S of the option with expiry-time payoff given by

$$V(S,T) = \frac{1}{S(T)}$$

(because the Black–Scholes PDE is valid for all European-style options).

8.11 Observe that the forward contract is equivalent to a European call option with zero strike price. By Exercise 8.7, the European call option with zero strike price has a time-zero value of $S(0)$. Discounting for interest, the value F, paid at time T, for the forward contract should be $S(0)e^{rT}$.