Notes on Gus Schrader's "The Cluster Approach to Character Varieties" talk

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1 Clusters

1.1 G-Local Systems

Let S be an oriented surface (usually with boundary) with a set $\{x_1, x_2, ..., x_n\} \subset \delta S$ of marked points. Let G be a simple complex Lie group (you can have PGL_n in mind)

Definition. A G-local system on S is a group homomorphism $\pi_1(S) \xrightarrow{\rho} G$

The set $\{\rho : \pi_1(S) \to G\}$ has a G-action given by conjugation. This action is not necessarily free, however.

1.2 Framed G-Local system(Fock and Gonchorov)

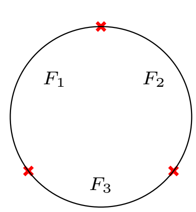
Definition. Let $B \subset G$ be a Borel subgroup, G/B be the flag variety, and P a principal G-bundle. Form the associated flag bundle $P \overset{G}{\times} G/B = Fl$. A framing is a flat section of $\mathrm{Fl}|_{\delta S \backslash \{x_1, x_2 ..., x_n\}}$

$\chi_{G,S}$ = Moduli Space of framed G-local systems

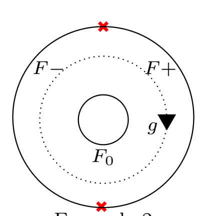
Example (1)**.** if $\delta S = S^1$ and has 3 marked points picture1 here so $\chi_{G,S} = \{F_1, F_2, F_3 \in G/B\}/G$ (G-action by conjugation)

Example. 2] if $\delta S = S^1 \sqcup S^1$ and has 2 marked on the same component picture1 here g, the monodromy, should preserve F_0 so $gF_0 = F_0$. Thus, $\chi_{G,S} = \{(g, F_-, F_0, F_+) \in G \times (G/B)^3 | gF_0 = F_0\}/G$

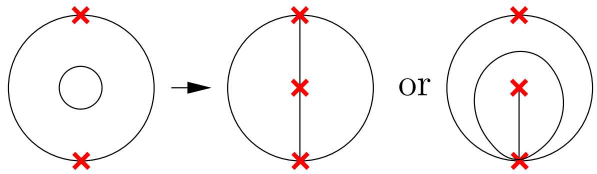
Theorem 1. (Fock-Gonchorov) $\chi_{G,S}$ is a rational variety.



Example 1



Example 2



Ideal Triangulations for example 2

1.3 Ideal Triangulations

Furthermore $\chi_{G,S}$ has a special collection of open embeddings $(\mathbb{C}^{\times})^d \hookrightarrow \chi_{G,S}$ where $d = dim\chi_{G,S}$. Let us first see how this construction goes in PGL_2.

- Shrink S^1 components of $\delta S \setminus x_1, x_2, ..., x_n$ into a puncture.

Definition. An Ideal Triangulation of S is a triangulation whose vertices are at punctures or marked points. (We allow "degenerate triangles" where two edges are glued together)

- For each ideal triangulation Δ, construct $\iota_\Delta : (\mathbb{C}^{\times})^d \to \chi_{G,S}$ by assigning the coordinate directions in $(\mathbb{C}^{\times})^d \leftrightarrow$ edges of the triangulation. We construct the local system from the coordinate directions
- retract $S \setminus \{x_1, ..., x_n\}$ onto the ribbon dual to Δ
- cut the ribbon up into:
  - squares for each edge of the dual graph of Δ (color the edges crossing Δ green, and the others red
  - hexagons around the vertices of the dual triangulation, red inside the ribbon (already drawn) and blue along the outside of the ribbon, oriented with the orientation of the surface
  - orient the green edges to match the rest of the ribbon (see figures 1,2 on the next page)
- then to each (oriented) blue segment, assign the uppertrianglar unipotent matrix E= $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$
- to each (oriented) red segment, assign a representative of the longest element of the Weyl group of G, the matrix w= $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
- to each (oriented) green segment crossing an edge assigned to the i-th toric coordinate direction x_i, assign the element of the torus, the matrix H(x_i) = $\begin{pmatrix} x_i & 0 \\ 0 & 1 \end{pmatrix}$
- we can consider paths along these colored segments as matrix multiplication, where going against the orientation uses the inverse matrix (note $w = w^{-1}$ in PSL_2)

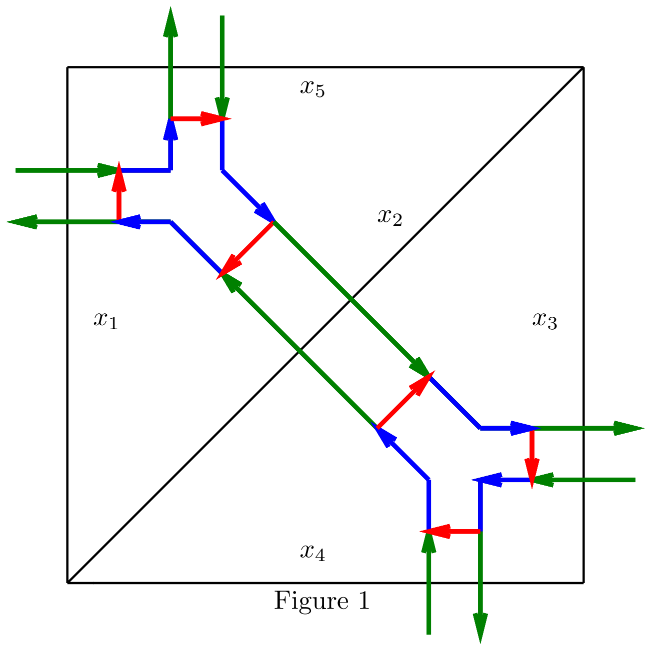


Figure 1

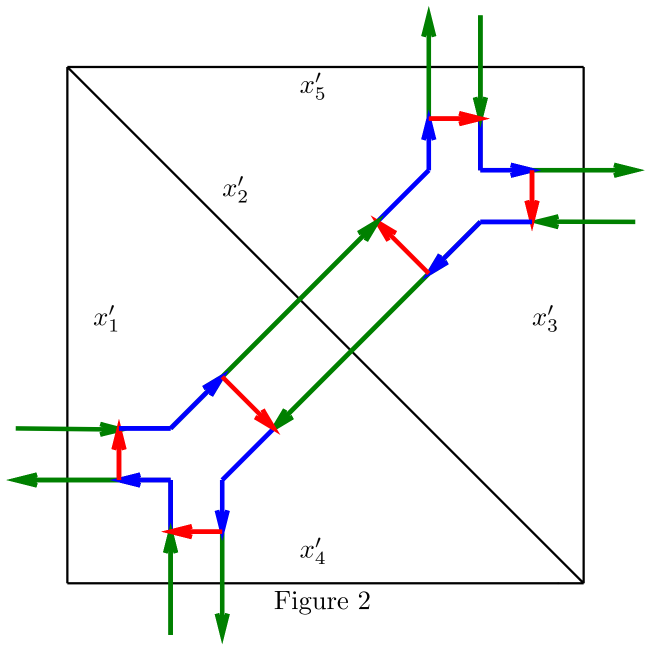
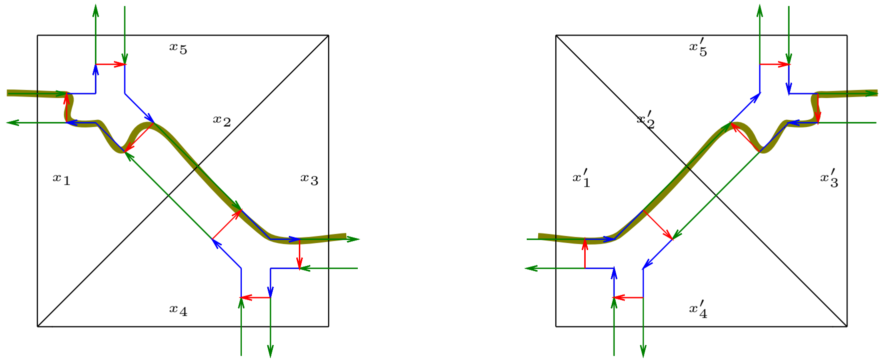


Figure 2



Exercise 1. Check that products around ribbon faces are trivial (i.e. it is path independent) Hint: remember we are working in PSL_2

Let's compare paths across two square with different triangulations

Path 1 gives us $H(x_1)wE^{-1}w^{-1}H(x_2)EH(x_3)$ Path 2 gives us $H(x'_1)EH(x'_2)w^{-1}Ew^{-1}H(x'_3)$ If you work out these calculations, you get that the coordinates of $\overrightarrow{x}$ and $\overrightarrow{x'}$ are related via

$$(x'_1, x'_2, x'_3) = (\frac{x_1x_2}{1+x_2}, \frac{1}{x_2}, \frac{x_2x_3}{1+x_2})$$
$$(x_1, x_2, x_3) = (x'_1(1+x'_2), \frac{1}{x'_2}, x'_3(1+x'_2))$$

Note: These Transition functions are subtraction-free bi-rational isomorphisms. Therefore positive points are sent to positive points, so we have a well defined set $\chi_{G,S}(\mathbb{R}_>0)$. In fact, for the case of G= PSL_2, it recovers the Teichmüller components in the $PSL_2(\mathbb{R})$ character variety.

1.4 Higher Rank

Consider an ideal triangulation and a reduced presentation for the longest word w_0 in W(G), the Weyl group.

Example. S=disk with 3 marked points on the boundary. $G = PGL_3$. $w_0 = s_1s_2s_1 = s_2s_1s_2$, so we have two special charts. For each simple reflection in the Weyl group, we have the corresponding sl_2 triple (E, F, H) in G. Embed E into G via that sl_2 triple. Embed H as the fundamental co-weight corresponding to the triple. And the spelling of w_0 tells us the order to multiply them together. We get $H_1H_2H_1$ for the first spelling and $H_2E_2H_1E_1H_2E_2$ for the second spelling.

$$H_1(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad H_2(x) = \begin{pmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

These toric coordinates also combinatorially encode the Goldman/Atiyah-Bott Poisson structure on $\chi_{G,S}$.

$$\{x_j, x_k\} = \epsilon_{j,k}x_jx_k \qquad \epsilon_{j,k} \in \mathbb{Z} \tag{1}$$

2 Quantization

2.1 Quantum Torus

$\hbar \in \mathbb{R}_{>0}$, $q = e^{\pi i \hbar^2}$ this promotes our toric coordinates to generators of the "quantum torus" where $X_1, ...X_d$ satisfying a skew-commutativity type relation

$$X_j X_k = q^{2\epsilon_{k,j}} X_k X_j \tag{2}$$

We can think of this in terms of the Heisenberg Algebra with generator $x_1, ..., x_d$ where

$$[x_j, x_k] = \frac{1}{2\pi i}\epsilon_{j,k} \tag{3}$$

then $X_k = e^{2\pi\hbar x_k}$
The Heisenberg algebra has the Hilbert space H_λ, where $\lambda \in (ker(\epsilon))^*$

Example. $\epsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $H = L^2(\mathbb{R})$

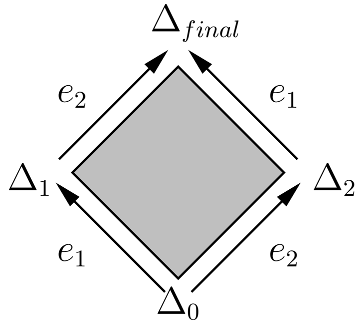
$$x_1 \mapsto \frac{1}{2\pi i}\frac{d}{dx}(= \hat{p} \text{ momentum operator})$$
$$x_2 \mapsto \_ \cdot x(= \hat{x} \text{ position operator})$$

2.2 Gluing Data

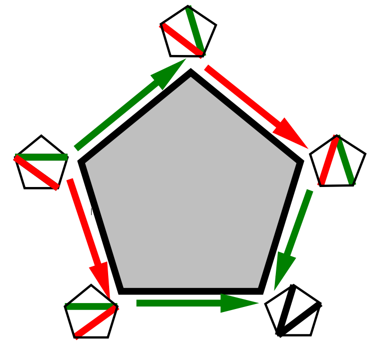
For each x_k, we can change the triangulated surface by doing a flip of the quadrilateral containing that edge as seen by comparing figures 1 and 2. Thus for each k, we have a unitary transformation $\phi(x_k) : H \to H$

Theorem 2. $[\hat{p}, \hat{x}] = \frac{1}{2\pi i} \Rightarrow \phi(\hat{p})\phi(\hat{x}) = \phi(\hat{x})\phi(\hat{p} + \hat{x})\phi(\hat{p})$

This theorem will help us show that we have a well-defined way of flipping between different triangulation.



Far Commuting



Pentagon Pentagon

Build a space, P, by having it's vertices correspond to ideal triangulations of your surface, modulo isotopy. Add edges between triangulations related by a single flip move.

Then we'll add two types of faces. First, add squares representing "far commuting" relations. In other words, if two edges (of the ideal triangulation) are not part of the same triangle, we can do the flips for both edges in either order because they're both a partial change of coordinates affecting different coordinates.

Finally, add pentagonal faces between the edges of the triangulations that only differ in a single pentagon as shown in the pentagon pentagon above (this corresponds to the right hand side of theorem 2). The colored arrows denote which edge is being flipped over.

Then, P is connected and simply-connected, meaning we can get between any two ideal triangulations via these flips, and that there is a well-defined unitary transformation for any ordered pair of triangulations. This gives us a representation of the mapping class group of our marked surface, $MCG(S)$, on H_S.