Hecke Algebras and Representation Theory

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June 2019

Abstract

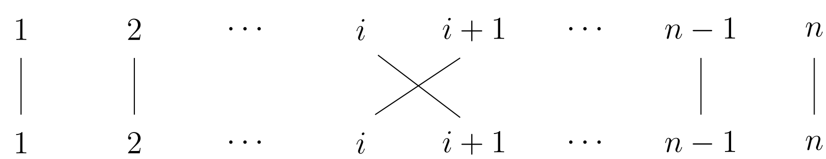
These notes were taken from a series of lectures given by Monica Vazirani at the ICMS Summer School and Workshop: Geometric representation theory and low-dimensional topology.

Lecture I

Recollections on Representation Theory of $\mathfrak{S}_n$

For the duration of the notes the main focus will be on Type A Hecke Algebras, Affine Hecke Algebras (AHA), and Double Affine Hecke Algebras (DAHA) and their respective representation theories. To draw the relevant comparisons between these we need to recall some notations from the representation theory of the symmetric group $\mathfrak{S}_n$(or S_n).

Let $s_i = (i\ i+1)$ be the usual transposition in $\mathfrak{S}_n$. These can be represented diagrammatically as

$s_i =$ 

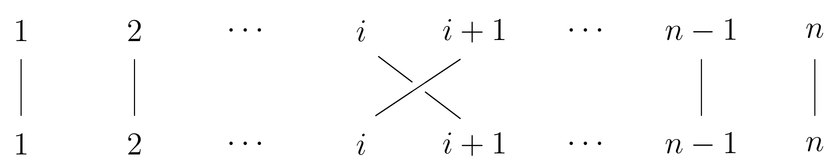
Each s_i satisfies the following braid relations

$$\text{Braid} = \begin{cases} s_i s_j = s_j s_i & |i-j| > 1\,, \\ s_i s_j s_i = s_j s_i s_j & |i-j| \leq 1\,. \end{cases}$$

These relations are called braid relations because they are the defining relations for the standard braid group on n strands:

$$\mathrm{Br}_n = \langle g_1, g_2, \ldots, g_{n-1} \mid \text{Braid} \rangle$$

Here we changed notation to g_i for the group element represented by the braid

$g_i =$ 

noting that the crossing now has a gap to indicate which strand is over the other. These relations along with the Quadratic relations $s_i^2 = 1$ for all i gives a presentation

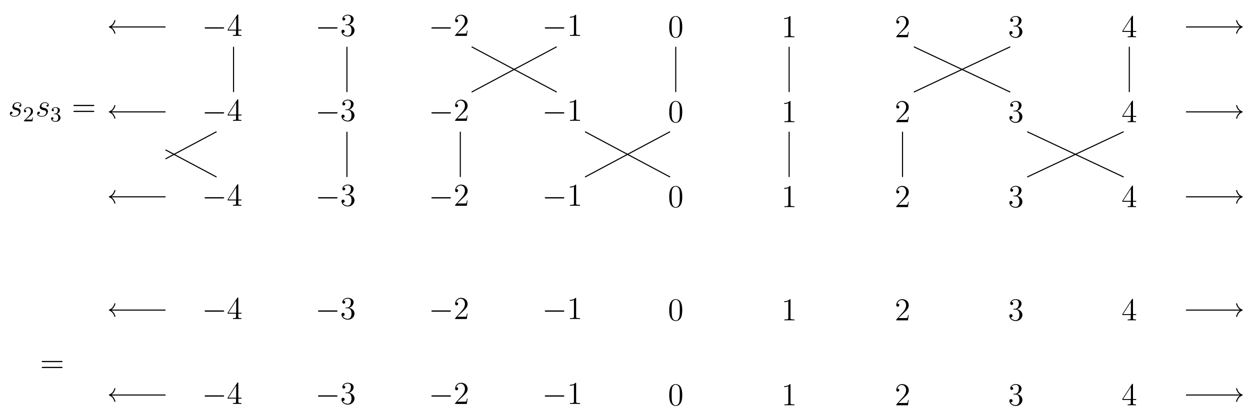
$$\mathfrak{S}_n = \langle s_1, s_2, \ldots, s_{n-1} \mid s_i^2 = 1, \text{Braid} \rangle$$

which is the usual Coxeter presentation of $\mathfrak{S}_n$. In the group algebra $\mathbb{C}\mathfrak{S}_n$ the Quadratic relations $s_i^2 = 1$ can be reformulated as the identity $(s_i - 1)(s_i + 1) = 0$. Moreover, there is a surjection $\mathrm{Br}_n \to \mathfrak{S}_n$.

Representations

There are many examples of sets that $\mathfrak{S}_n$ acts on.

- By definition it acts on $[n] = \{1, 2, \ldots, n\}$.
- It acts on $\mathbb{Z}$ n-periodically. For s_i it acts by transposing $i + kn \leftrightarrow i + 1 + kn$ for all $k \in \mathbb{Z}$. This can be visualized by either writing out a list of integers in order and placing the same crossings at each n-period (Figure 1)



  or writing down a fundamental domain and working there (Figure 2)

  or by placing the numbers of $[n]$ in order along the top and bottom circles of a hollow cylinder and drawing the lines on the wall as in (Figure 3)

  In the examples above we showed how to compose via stacking and chose to take $n = 4$ for the illustrations.

- It acts on $\mathbb{F}^n$ by permuting the coordinates. For example

$$s_2(a_1, a_2, a_3, \ldots, a_n) = (a_1, a_3, a_2, \ldots, a_n)\,.$$

Important to studying representations of $\mathfrak{S}_n$ are its subgroups, for which representations can be restricted or induced. The **parabolic** subgroups are as follows.

Definition. Let $J \subset [n-1]$. Define $w_J = \langle s_i \,|\, i \in J \rangle \leq \mathfrak{S}_n$. This is the **parabolic** subgroup, which is isomorphic to the Young subgroup $\mathfrak{S}_\beta$ associated to a **composition** β of n. A **composition** of n is a sequence of positive integers $\beta_1, \ldots, \beta_m$ such that $\sum_k \beta_k = n$, and a **weak composition** uses non-negative integers.

To determine β from J, which we assume to be in increasing order, is by the following procedure: taking the elements of $[n]$, first place bars at the beginning and end of the sequence, and second for every integer $m \in [n] \setminus J$ place a bar between m and $m + 1$; then β is the sequence determined by the number of elements between each bar. For example, with $J = \{2, 3, 6\}$ we have the schematic

$$|\,1\,|\,2\,3\,4\,|\,5\,|\,6\,7\,|$$

creating $\beta = (1312)$. Moreover, for this J, we have

$$w_J \cong \mathfrak{S}_{\{1\}} \times \mathfrak{S}_{\{2,3,4\}} \times \mathfrak{S}_{\{5\}} \times \mathfrak{S}_{\{6,7\}}$$

Where by $\mathfrak{S}_{\{2,3,4\}}$ we mean a copy of $\mathfrak{S}_3$ but it only acts on $\{2,3,4\}$.

Definition. We write $\mathbb{1}$ to indicate the one dimensional trivial representation. The algebraic object being represented will be clear from context.

For example, given w_J with $J = \{2,3,6\}$ the representation $\mathbb{1}$ is the vector space spanned by v and each $s_i \in w_J$ acts by identity $s_j(v) = v$.

Definition. Given a representation V of a group G and a subgroup $H \leq G$, the **restriction** of the representation of G to H is denoted $\operatorname{Res}_H^G V$, and the **induced representation** is the $\mathbb{F}[G]$ module

$$\operatorname{Ind}_H^G V = \mathbb{F}[G] \otimes_{\mathbb{F}[H]} V$$

We recall the fact that there is a natural isomorphism

$$\operatorname{Hom}_{\mathbb{F}[G]}(\operatorname{Ind}_H^G V, W) \cong \operatorname{Hom}_{\mathbb{F}[H]}(V, \operatorname{Res}_H^G W)$$

For our examples so far we have

$$\operatorname{Ind}_{w_J}^{\mathfrak{S}_n} \mathbb{1} \cong \mathbb{F}[\mathfrak{S}_n / w_j]$$

which is the permutation representation on the left cosets $\{\sigma w_j\}$. This representation has basis the set

$$w^J = \{\sigma w_J \,|\, \sigma \text{ has minimal length}\}$$

where the **length** is the number of generators in a reduced word for an element of $\mathfrak{S}_n$. Every coset has a *unique* such element. For example, with $\mathfrak{S}_4$ take $J = \{2,3\}$. Then $w_J = \langle s_2, s_3 \rangle$ so for any non-identity coset its representative must have an s_1 or an s_4. Thus here we have

$$w^J = \{w_J, s_1 w_J, s_2 s_1 w_J, s_3 s_2 s_1 w_J\}.$$

Theorem (Mackey)**.** Let $H, K \leq G$ and let V be a representation of K. Then

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G V \cong \bigoplus_{KgH} \operatorname{Ind}_{K \cap gHg^{-1}}^K \operatorname{Res}_{g^{-1}Kg \cap H}^H V^g$$

where V^g has underlying set V but now has the action $x.m = gxg^{-1}.m$.

For us we choose $V = \mathbb{1}$ and our parabolics to study the simpler objects of our representations:

$$\operatorname{Res}_{w_K}^{\mathfrak{S}_n} \operatorname{Ind}_{w_H}^{\mathfrak{S}_n} \mathbb{1} \cong \bigoplus_{w_K \sigma w_H} \operatorname{Ind}_{w_K \cap \sigma w_H \sigma^{-1}}^{w_K} \operatorname{Res}_{\sigma^{-1} w_K \sigma \cap w_H}^{w_H} \mathbb{1}^\sigma$$

Note that the sum can just as easily be indexed by minimal length double coset representatives, which again are unique.

Affine Forms of $\mathfrak{S}_n$.

We now want to present two extensions of $\mathfrak{S}_n$. First we will give their presentations and then utilize examples of their actions to motivate their presentations. These extensions are in a sequence

$$\mathfrak{S}_n \subset \hat{\mathfrak{S}}_n \subset \widetilde{\mathfrak{S}}_n$$

with presentations

$$\begin{aligned}\hat{\mathfrak{S}}_n &= \langle s_0, s_1, \ldots, s_{n-1} \,|\, \text{Braid},\ s_i^2 = 1\rangle \\ \widetilde{\mathfrak{S}}_n &= \langle \pi, s_0, s_1, \ldots, s_{n-1} \,|\, \text{Braid},\ s_i^2 = 1, \pi s_i \pi^{-1} = s_{i+1 \bmod n}\rangle\,.\end{aligned}$$

Here $\hat{\mathfrak{S}}_n$ is called the **affine symmetric group on n elements** and $\widetilde{\mathfrak{S}}_n$ is called the **extended affine symmetric group on n elements**.

We return to our original examples of actions of $\mathfrak{S}_n$ to see how the new elements of these new extensions act, which demonstrates useful relations between certain words in the group. We will just work with $\widetilde{\mathfrak{S}}_n$ for the remainder of this lecture.

- These again act on $\mathbb{Z}$ n-periodically. But now s_0 is the operation of transposing $kn \leftrightarrow 1+kn$ and corresponds to the following diagram (picture) . π acts by shifting $\pi(i) = i+1$, which is easy to imagine graphically. For each $0 \le i \le n$ define the bijection

  $$x_i : i + kn \mapsto i + (k+1)n\,.$$

  On the cylinder this element is represented as (picture). Based on our definitions note that

  $$x_1(i) = \pi s_{n-1} s_{n-2} \cdots s_1(i) = \begin{cases} i & i \not\equiv 1 \bmod n\,, \\ i+n & i \equiv 1 \bmod n\,. \end{cases}$$

  It is not difficult to reason that $s_i x_i s_i = x_{i+1}$ (just check it). From this it follows for each i that

  $$x_i = s_{i-1} \cdots s_2 s_1 \pi s_{n-1} \cdots s_i$$

  is an element of $\widetilde{\mathfrak{S}}_n$ and moreover it follows that

  $$\pi^n = x_1 x_2 \cdots x_n\,.$$

  An easy exercise is using the third relation for $\widetilde{\mathfrak{S}}_n$ to show that π^n is an element of the center $Z(\widetilde{\mathfrak{S}}_n)$, and seeing that the x_i generate an abelian subgroup.

  During this example we said that certain elements of $\widetilde{\mathfrak{S}}_n$ exist and satisfy certain relations, even though we were discussing their actions on the set $\mathbb{Z}$. Our conclusions do hold in the group because its action on $\mathbb{Z}$ is faithful. We will use these x_i later to construct convenient sets of generators, and subalgebras, for the affine Hecke algebras.

- They still act on $\mathbb{F}^n$. The s_i for $i > 0$ do the usual thing, but now s_0 and π act as

  $$\begin{aligned} s_0(a_1, a_2, \ldots, a_n) &= (a_n + 1, a_2, \ldots, a_{n-1}, a_1 - 1) \\ \pi(a_1, a_2, \ldots, a_n) &= (a_n + 1, a_1, \ldots, a_{n-1}) \end{aligned}$$

  We can give a geometric interpretation of s_0 for this representation. Given a point $x \in \mathbb{F}^n$ the point $s_i(x)$ corresponds to reflecting x across the hyperplane determined by $\ker(s_i - 1)$

(i.e. the points stabilized by s_i). For $n = 2$ we can visualize this via the following diagram: (picture)

In addition to this picture, we can reduce the dimension of these actions by considering the splitting $\mathbb{F}^n = V \oplus \mathbb{1}$ where V is the subrepresentation

$$V = \{x \in \mathbb{F}^n \mid \sum_i x_i = 0\}.$$

If we use our $n = 2$ example the reduction allows us to visualize s_1 as reflection about 0 and s_0 will be reflection about 1, which we can sketch as (picture).

We will end this lecture by remarking on the surjection $\widetilde{\mathfrak{S}}_n \twoheadrightarrow \mathfrak{S}_n$. It is possible to give the presentation

$$\widetilde{\mathfrak{S}}_n = \langle s_1, \ldots, s_{n-1}, x_1, \ldots, x_n \mid \text{Braid}, s_i^2 = 1, s_i x_i s_i = x_{i+1}, s_i x_j = x_j s_i \text{ when } |i-j| > 1\rangle$$

The map $\widetilde{\mathfrak{S}}_n \twoheadrightarrow \mathfrak{S}_n$ is defined by sending $s_i \mapsto s_i$ and $x_i \mapsto 1$. This can visualized by "filling in the cylinder" where now the strand that wrapped around the cylinder is allowed to pass through the interior and become straight.

Lecture 2

Our lecture began by recalling that the finite Hecke algebra $H_n^{fin}(t)$, $t \in \mathbb{F}^\times$ is generated by $T_1, ..., T_{n-1}$ with the braid relations, quadratic relations, and the condition $T_i - T_i^{-1} = t - t^{-1}$ for each $i \in [n-1]$. A basis for this algebra is indexed by the symmetric group, $\{T_\omega : \omega \in \mathfrak{S}_n\}$, where $\omega = s_{i_1} ... s_{i_\ell}$ (reduced decomposition) and $T_\omega = T_{i_1} T_{i_2} ... T_{i_\ell}$. Once we are given a definition of this type, one of the natural first questions to ask is when does this come in nature. One such example is the following:

Example. Let $G = \mathrm{GL}_n(\mathbb{F}_q)$, $B =$ Borel subgroup = upper-triangular invertible matrices in G, $\overline{W}$ = permutation matrices.

Its a fact that G can be written as the disjoint union over double-cosets as $G = \bigsqcup_{w \in \overline{W}} BwB$. So we can consider the algebra $\mathbb{C}[B \backslash G / B]$ as B bi-invariant functions. Lets actually see what this looks like for a specific value of n.

$\underline{n = 2}$: In this small dimensional example, we have $\mathrm{GL}_2(\mathbb{F}_q) = B \sqcup BsB$, where $s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ represents the non-trivial permutation in $\mathfrak{S}_2$. Furthermore, we can further decompose the non-trivial double coset as $BsB = \bigsqcup_{a \in \mathbb{F}_q} \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. This decomposition is not unique to the $n = 2$ case. For general n, we have

$$BwB = \bigsqcup_{a_i \in \mathbb{F}_q} B s_{i_1} x_{i_1}(a_1) s_{i_2} x_{i_2}(a_2) ... s_{i_\ell} x_{i_\ell}(a_\ell), \quad x_{i_j}(a_j) = I_n + a_j E_{i,i+1}$$

where $E_{i,i+1}$ is the permutation matrix for the permutation $(i\ i+1)$.

We can now check that $\mathbb{C}[B \backslash G / B] \cong e\mathbb{C}[g]e =_{\mathbb{C}[G]} (\mathrm{Ind}_B^G \mathbb{1}) \cong H_n(t = q^{1/2})$, where $e = \frac{1}{|B|} \sum_{b \in B} b$ is the idempotent of the function space and $q^{1/2} T_{s_i}$ = bump function on $B s_i B$ (the $q^{1/2}$ is needed for proper normalization). Taking the product to make T_w, we have (for the appropriate constant C) $C T_w$ = bump function on BwB.

Now we want to see different ways that we can present the extended affine Hecke Algebra of type A.

Presentation 1. Put on a cylinder. (ADD PICTURE)

Or more algebraically...

Presentation 2. Generators and relations (1): $\langle T_1, ..., T_{n-1}, T_0, \pi : \text{Braid, quadratic}, T_i - T_i^{-1} = t - t^{-1}, \pi T_i \pi^{-1} = T_{i+1}\rangle$. Here we note (and as is mentioned at the end of the last lecture notes) that π^n is central (which follows from using the only relation on π, $\pi^n T_i \pi^{-n} = \pi^{n-1} T_{i+1} \pi^{-(n-1)} = ... = T_{i+n} = T_i$)

Presentation 3. Generators and relations (2): An alternate presentation to the one above is $\langle T_1, ..., T_{n-1}, x_1^{\pm 1}, ..., x_n^{\pm 1} : \text{Braid, quadratic}, T_i x_i T_i = x_{i+1}, T_i x_k = x_k T_i \text{ when } k \neq i, i+1\rangle$, where the x_i's are polynomials, so they all commute with each other. One possible basis for this space is $\{T_w x^\beta : w \in \mathfrak{S}_n, \beta \in \mathbb{Z}^n\}$.

To go back and forth between (1) and (2), we have that $\pi = x_1 T_1 T_2 ... T_{n-1}$, which after some proper algebraic manipulations yields $\pi^n = x_1 x_2 ... x_n$.

We now want to apply this to a few examples. Such constructions can be applied in the case where

$$\begin{aligned} K &= \mathbb{Q}_p \text{ or } \mathbb{C}((v)) \text{ (or a finite extension of these)} \\ \mathcal{O} &= \mathbb{Z}_p \text{ or } \mathbb{C}[[v]] \\ \mathfrak{m} &= \langle p \rangle \text{ or } \langle v \rangle \end{aligned}$$

The projection $\mathcal{O} \to \mathcal{O}/\mathfrak{m}$ takes us into the field $\mathbb{F}_p$ or $\mathbb{C}$. Here we call the generator of the maximal ideal the uniformizer, denoted ω.

Example. Let $G = \mathrm{GL}_n(K) \supseteq \mathrm{GL}_n(\mathcal{O}) \supseteq I$ where I is the Iwahori subgroup containing matrices of the form $\begin{pmatrix} \mathcal{O}^\times & & & \mathcal{O} \\ & \mathcal{O}^\times & & \\ & & \ddots & \\ \mathfrak{m} & & & \mathcal{O}^\times \end{pmatrix}$ (here the upper-triangular portion can be anything in $\mathcal{O}$ and the lower-triangular can be anything in $\mathfrak{m}$, not just the corner entries). Utilizing the reduction map $G(\mathcal{O}) \xrightarrow{\psi} G(\mathcal{O}/\mathfrak{m}) \supseteq B$, we see that $I = \psi^{-1}(B)$. In the case of $K = \mathbb{Q}_p$, we have $\mathbb{C}[I \backslash G(K)/I] = AHA$. In a similar vein to above, we have a decomposition along a subgroup (in this case I), $G(K) = \sqcup_{w \in \tilde{W}} IwI$. Here we can represent the group elements as the following:

$$s_i \longleftrightarrow \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}, \quad s_0 \longleftrightarrow \left(\begin{array}{c|ccc|c} & & & & \omega^{-1} \\ \hline & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline \omega & & & & \end{array}\right), \quad \pi \longleftrightarrow \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ \omega & & & & 0 \end{pmatrix}.$$

In this setting, $G/B \leftrightarrow$ flags, $G/I \leftrightarrow$ affine flags of $\mathcal{O}$-lattices, and $G(K)/G(\mathcal{O}) \leftrightarrow$ affine Grassmannian. (Side bar: $BwB \leftrightarrow$ Schubert cells in the case of $\mathbb{C}$).

A relationship between the finite Hecke Algebra and Kazhdan-Lusztig Theory can be seen by looking at their respective bases:

$$\text{Finite Hecke} \to \{T_w : w \in \mathfrak{S}_n\} \ (\overline{T_w} = T_w^{-1}) \qquad \{C_w : w \in \mathfrak{S}_n\} \ (\overline{C_w} = C_w) \leftarrow \text{Kazhdan} - \text{Lusztig}$$

We now have all the necessary machinery to define the double affine Hecke Algebra.

Definition. Let $\mathbb{F}$ be a field and $q,t \in \mathbb{F}^\times$. The Double Affine Hecke Algebra (DAHA) $\mathbb{H}_n(q,t)$ of type A is the algebra with generators $T_1,...,T_{n-1},T_0,\pi,y_1^{\pm 1},...,y_n^{\pm 1}$ subject to the relations that $T_1,...,T_{n-1},T_0,\pi$ form an AHA (in t) $\cong H(X)$, $T_1,...,T_{n-1},y_1^{\pm 1},...,y_n^{\pm 1}$ form an AHA (in t) $\cong H(Y)$, with the additional relations $\pi y_i \pi^{-1} = y_{i+1}$, where we "set" $y_{n+1} = q^{-1}y_1$, $y_0 = qy_n$. The DAHA has basis $\{T_w y^\beta : w \in \tilde{\mathfrak{S}}_n, \beta \in \mathbb{Z}^n\}$.

Lecture 3

Recall that AHA$= \langle T_1,\dots,T_{n-1},T_0,\pi|$Braid, Quadratic $(T_i - t)(T_i + t^{-1}) = 0, \pi T_i \pi = T_{i+1}\rangle$. We can write π as $X_1T_1\dots T_{n-1}$. This has basis $\{T_\omega|\omega \in \tilde{\mathfrak{S}}_n\}$. This has another presentation AHA$(X) = \langle T_1,\dots,T_{n-1},X_1^{\pm 1},\dots,X_n^{\pm 1}|$Braid, Quadratic, X form a Laurent polynomial ring, $T_iX_iT_i = X_{i+1}, T_iX_j = X_jT_i$ for $i \neq j, j-1\rangle$. A basis for this is $\{T_\omega X^\beta|\omega \in \mathfrak{S}_n, \beta \in \mathbb{Z}^n\}$.

DAHA presentation 1:

$\langle T_1,\dots,T_{n-1},T_0,\pi,Y_1^{\pm 1},\dots,Y_n^{\pm 1}|$relations$\rangle$, where $T_1,\dots,T_{n-1},,T_0,\pi$ generate an AHA, and $T_1,\dots,T_{n-1},Y_1^{\pm 1},\dots,Y_n^{\pm 1}$ generate another AHA(Y). The other relations are $\pi Y_i \pi^{-1} = Y_{i+1}$ with convention $Y_{i+n} = q^{-1}Y_i, i \in \mathbb{Z}$. A basis is $\{Y^\gamma T_\omega|\gamma \in \mathbb{Z}^n, \omega \in \tilde{\mathfrak{S}}_n\}$.

Presentation 2: $\langle T_1,\dots,T_{n-1},Y_1^{\pm 1},\dots,Y_n^{\pm 1},X_1^{\pm 1},\dots,X_n^{\pm 1}|$relations$\rangle$, where $T_1,\dots,T_{n-1},Y_1^{\pm 1},\dots,Y_n^{\pm 1}$ generate another AHA(Y) and $T_1,\dots,T_{n-1},X_1^{\pm 1},\dots,X_n^{\pm 1}$ generate another AHA(X). A basis is $\{Y^\gamma T_\omega X^\beta|\gamma,\beta \in \mathbb{Z}^n, \omega \in \mathfrak{S}_n\}$. The relations are

1. $X_1\dots X_nY_i = q^{-1}Y_iX_1\dots X_n$
2. $Y_1\dots Y_nX_i = qX_iY_1\dots Y_n$
3. $X_1^{-1}Y_2^{-1}X_1Y_2 = T_1^2$

Macdonald Polynomials: Write $\mathbb{H}_n = \mathbb{H}_n(q,t) =$DAHA

Pol$= \mathrm{Ind}_{H(Y)}^{\mathbb{H}_n} \cong_{v.s.} \mathfrak{X} = \mathbb{F}[x_1^{\pm 1},\dots,x_n^{\pm 1}]$. We have that $(T_i - t)\mathbb{1} = 0$, $(Y_1 - 1)\mathbb{1} = 0$ and $\mathfrak{Y} = \mathbb{F}[y_1^{\pm 1},\dots,y_n^{\pm 1}]$ acts locally finitely on Pol.

For q,t generic, $\mathfrak{Y}$ acts semisimply with weight spaces of dimensions 1. Resulting weight basis (normalised) yield nonsymmetric MacDonald polys.

Note by 1), if f is a y weight vector, so is $x_1\dots x_nf$. What are symmetric macdonald polynomials? They are basis of Sym(X), triangular with respect to monomial basis and orthogonal with respect to $\langle -,-\rangle_{q,t}$. It's specialisations recover most of this interesting basis of Sym(X).

DAHA acts on Pol. It is an irred for generic q and t. Inside of both Sym(y) acts on Sym(x). Sym(y) weight basis are the symmetric macdonald polynomials.

Representation Theory

Of AHA, we use $\mathfrak{X}$, it is a large commutative subalgebra. For generic t, $\mathrm{Ind}_{H_J}^{H_n(x)}(1-\dim)$ tells the whole story: these span $K_0(\mathrm{Rep}H(X))$ ($H_J = \langle T_i, X_1^{\pm 1},\dots,X_n^{\pm 1}|j \in J\rangle$ parabolic).

$\mathrm{Ind}_{H_n^{\mathrm{fin}}}^{H_n(x)}\mathbb{1} \cong_{v.s} \mathbb{F}[x_1^{\pm 1},\dots x_n^{\pm 1}]$. $f \in \mathfrak{X}$, $T_if - f^{s_i}T_i = (t-t^{-1})\frac{f-f^{s_i}}{1-X_i/X_{i+1}}$. In Pol, $T_if = tf^{s_i} + (t-t^{-1})\frac{f-f^{s_i}}{1-X_i/X_{i+1}}$.

Let $e_+ =$ trivial idempotent $\in H_n^{\mathrm{fin}}(t)$. e_+Pol=Sym(X), $(T_i - t)e_+ = 0$.

Proposition. $Z(H_n(X)) = \mathrm{Sym}(X)$

Corollary. If M is a simple $H_n(X)$-module, then $\dim X \leq n!$

Lecture 4

Recall that we have the trivial idempotent $e_+ \in H_n^{\text{fin}}(t)$. As a beginning aside remark, the algebra $e_+\mathbb{H}(q,t)e_+$ will be called the spherical DAHA.
Now, recall that we defined $\text{Pol} = \text{Ind}_{H(Y)}^{\mathbb{H}}\mathbb{1}$. As a vector space, $\text{Pol} \cong \mathbb{F}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}]$. We can write a formula for Y_i acting on $\text{Pol}^+ \subseteq \text{Pol}$, since we know that $(T_i - t)\mathbb{1} = 0, (Y_i - t^{2(i-1)})\mathbb{1} = 0$. For this final lecture, we will focus on the representation Pol.

Proposition. For q, t generic, Pol is irreducible and has Y-weight basis that we can construct.

Let $F \in \mathcal{Y} =$ Laurent polynomials in Y. e have that $T_iF - F^{s_i}T_i = (t - t^{-1})\dfrac{F - F^{s_i}}{1 - \frac{Y_i}{Y_{i+1}}}$. Via presentation 1, a basis for Pol is $\{T_w \otimes \mathbb{1} \mid w \in \widetilde{\mathfrak{S}_n}/\mathfrak{S}_\mathfrak{n}\}$. The Y_i's act triangularly on this basis with respect to some order refining length. Thus, for $F \in \mathcal{Y}$, $FT_w \otimes \mathbb{1} = T_wF^w \otimes \mathbb{1} +$ (lower order terms) $\otimes \mathbb{1}$.
Let us look at the example when $n = 3$. (Y_1, Y_2, Y_3) acts on the weight vector $1 \otimes \mathbb{1}$ as $(1, t^2, t^4) = (b_1, b_2, b_3)$. The idea going forward is that $b_{i+1}/b_i = t^2$, then the vector $(b_1, \ldots, b_{i+1}, b_i, \ldots, b_n)$ is not a weight.
A couple tricks to remember as we are doing these calculations follow. Firstly, we have indexed Y_i by integers and we set $Y_k = q^{-1}Y_{k-3}$ for any k. Also, since by example, $Y_2\pi \otimes \mathbb{1} = \pi Y_1 \otimes \mathbb{1} = t1 \otimes \mathbb{1}$, we can deduce that $\pi(b_1, b_2, \ldots, b_n) = (qb_n, b_1, \ldots, b_{n-1})$. Some more examples include:

- $\pi \otimes \mathbb{1} \longleftrightarrow (qt^4, 1, t^2)$
- $T_0 \otimes \mathbb{1} \longleftrightarrow (qt^4, t^2, q^{-1})$
- $T_1T_0 \otimes \mathbb{1} \longleftrightarrow (t^2, qt^4, q^{-1})$
- $T_1T_2T_1T_0 \otimes \mathbb{1} \longleftrightarrow (q^{-1}, t^2, qt^4)$

Recall that $s_1s_2s_1s_0 \in \widetilde{\mathfrak{S}_2}$ is a translation by $(-1, 0, 1) \in (\mathbb{F}^\times)^n$. Note that all $(q^{\lambda_1}, q^{\lambda_2}t^2, q^{\lambda_3}t^4)$ and all $\mathfrak{S}_3$ orbits of these are distinct if $q^A \neq (t^2)^B$ for all $(0,0) \neq (A, B) \in \mathbb{Z}^2$ and q, t are not roots of unity. Thus, all $\mathcal{Y}$-weight spaces are 1 dimensional. Hence, there exists a $\mathcal{Y}$-weight basis of this representation. These basis elements will be the non-symmetric MacDonald polynomials, up to some rescaling. We would like to find this basis explicitly. To do so, we introduce the notion of <u>intertwiners</u>.

Proposition. There exist $\varphi_i \in \text{AHA}(Y) \subset \mathbb{H}, 0 \leq i \leq n-1$ such that

1. For $F \in \mathcal{Y}$, $F\varphi_i = \varphi_i F^{s_i}$.
2. The φ_i satisfy the braid relations. Thus, φ_w makes sense for all $w \in \widetilde{\mathfrak{S}_n}$. As an abuse of notation, set $\varphi_\pi = \pi$.

As it turns out, these intertwiners yield, up to rescaling, the nonsymmetric Macdonald polynomials by translating from presentation 1 into presentation 2. Let's compute a couple examples below.

- $\pi^3 \otimes \mathbb{1} = X_1X_2X_3 \otimes \mathbb{1} \rightsquigarrow X_1X_2X_3$ is a MacDonald polynomial.
- $\pi \otimes \mathbb{1} = X_1T_1T_2 \otimes \mathbb{1} = X_1 \otimes T_1T_2\mathbb{1} = t^2X_1 \otimes \mathbb{1} \rightsquigarrow X_1$ is a MacDonald polynomial.

Pick $\varphi_i = T_iY_i - Y_iT_i = T_i(Y_i - Y_{i+1}) + (t - t^{-1})Y_{i+1}$. As long as $Y_i - Y_{i+1}$ does not act as 0 on weight vector $\underline{u}$, then $\text{Span}\{\underline{u}, T_i\underline{u}\} = \text{Span}\{\underline{u}, \varphi_i\underline{u}\}$.

If we look at the quadratic relation on φ: $\varphi_i^2 = (tY_i - t^{-1}y_{i+1})(tY_{i+1} - t^{-1}Y_i) \in \mathcal{Y}^{s_i}$. Thus, φ_i acts invertibly on the weight vector $\underline{u}$ by the weight $(b_1, \ldots, b_n)$ as long as $b_{i+1}/b_i \neq t^{\pm 2}$. If we look at the Y-character of Pol, we see that Pol is irreducible.

Let M be a simple $\mathbb{H}$-Module which is Y-locally finite. Then if $(Y_i - Y_{i+1})$ does not act as 0 on $\underline{u}$ for all i, then $\varphi_i^2\underline{u} = 0 \Rightarrow \varphi_i\underline{u} = 0$.