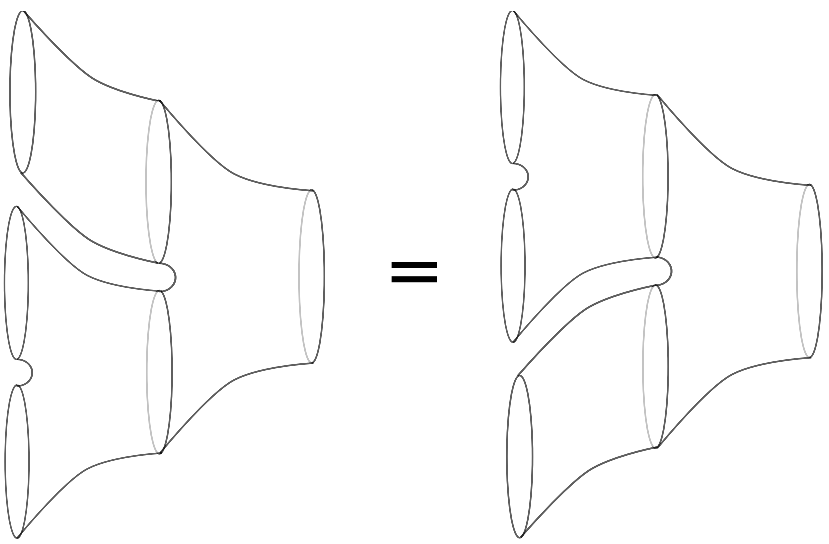
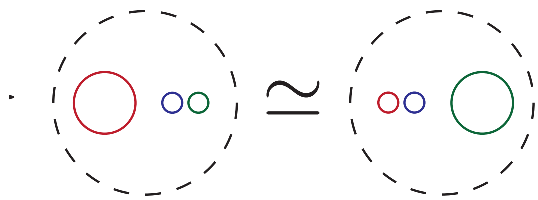
Lecture Notes of the Working Seminar at

The Hodge Institute

Topological Field Theories





Seminar organiser:
dr. D. Jordan

Winter 2016

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Preface: what are these notes?

These notes were compiled as a record of the Hodge Institute's Working Seminar on topological field theory, in Spring 2016. The aim of this seminar was to give an introduction to the language of topological field theory, and to provide an overview of the construction and classification of topological field theories in dimensions 1, 2, 3, and 4. Talks in the seminar were given by the participants in turn, and notes were taken by the post-graduate students, Jenny August, Matt Booth, Juliet Cooke, and Tim Weelinck. The seminar was led by David Jordan. Given the informal nature of the seminar, many proofs are omitted; when possible we have tried to provide references where complete details may be found.

Please feel free to email David Jordan, D.Jordan@ed.ac.uk, with any corrections, amendments, or suggestions! Additional references for unproved or incompletely proved claims are also very welcome, and we will add them in due time.

0.1 Layout

The organization of these notes is as follows. Chapter 1 gives a brief and informal overview of some elements of quantum field theory, emphasizing those most important in the formulation of topological field theory. In Chapter 2, we outline the well-known classification of 1-dimensional, 2-dimensional, and fully extended 2-dimensional field theories. This material is largely elementary but gives nevertheless some feel for the notion of a TQFT. In Chapter 3, we recall some basic definitions about quantum groups, focusing largely on the case of SL_2, and we recall the construction by Reshetikhin-Turaev of knot invariants from their representation theory. As such, this chapter is not directly related to TQFT's, however in Chapter 4, we will discuss how a (3,2,1)-TFT, a.k.a a once-extended 3D TQFT, may be constructed from the data of a modular tensor category. In the companion reference, Hodge Project: A construction of modular categories from quantum groups, we explain how a certain modification of the quantum group category, when the parameter q is a root of unity.

In Chapter 5, we return to the general theory needed to formally define cobordism categories: the language of higher categories and complete Segal spaces. This chapter ends with a precise statement of the cobordism hypothesis. Chapter 6 is an introduction to E_n-algebras and their factorization homology. These tools give a mechanism for defining and explicitly computing TQFT's, and are the most current mathematical topic covered in these notes. Finally, in Chapter 7, we give an overview of the paper [5], which proves that so-called *fusion categories*, i.e. tensor categories with a strong finiteness condition, are fully dualizable in the 3-category of tensor categories, hence define fully extended 3D TFT's. This leads to a "ground-up" construction of the Turaev-Viro fully extended 3-dimensional TFT.

Chapter 1

Origins of Topological Field Theory

Speaker: David Jordan
Notes: Juliet Cooke
Date: 15-1-2016

1.1 Quantum Field Theory

Let us briefly and very informally survey the basic set-up of quantum field theory – QFT – a central technique in mathematical physics, which served as the motivating origins for the development of topological field theory. This lecture shouldn't be taken too seriously, it is just an impressionistic overview.

In quantum field theory, one starts with a base manifold, typically a 4-manifold M^4 equipped with a metric with signature $(+ \; + \; + \; -)$ ('space time'), or a 3-manifold M^3 equipped with a Riemannian metric, or a surface $\Sigma = M^2$ with a conformal structure. On this manifold M we study a space of 'fields' $\mathcal{F}$. This means we have some bundle,

$$\begin{array}{c} \xi \\ \downarrow \\ M \end{array}$$

and we wish to study the vector space of sections $\mathcal{F} = \Gamma(M, \xi)$.[1] In physical situations there is a “Lagrangian"

$$\mathcal{L} = f\left(\varphi, \frac{\partial\varphi}{\partial x_i}, \frac{\partial^2\varphi}{\partial x_i \partial x_j}, \ldots\right), \; f \in \mathbb{C}[x_1, x_1, \ldots],$$

i.e. a linear functional on $\mathcal{F}$, valued in $\mathbb{C}$. The main object of study is the so-called 'partition function'

$$Z = \int_{\varphi \in \Gamma(M,\xi)} e^{\pi i \int_M \mathcal{L}(\varphi)} \, d\varphi,$$

which integrates the complex exponential of the action $\int_M \mathcal{L}(\varphi)$ of each field, over the space of all fields.

The partition function serves as the **measure** against which one computes correlations of fields ψ_1, ψ_2, i.e. the likelihood of observing ψ_1 given ψ_2 and vice versa. These integrals are ill-posed in general, not only divergent, but undefined because $d\varphi$ is meant to be a

[1] Often $\mathcal{F}$ arises as the associated jet bundle of a principal bundle for a gauge group G.

translation invariant measure on the infinite dimensional vector space of fields, and by Riesz's lemma, such a measure necessarily assigns either 0 or ∞ to every open set.

QFT addresses these divergence issues in various ways to still get meaningful physical quantities out, often by expanding partition functions formally in the neighborhood of the critical points of $\mathcal{L}$, which correspond to the classical physical states. We will not discuss this very interesting topic in this seminar, but rather we will look at a mathematical formalism for analyzing certain very simple QFT's, which bypasses these issues entirely.

Note that in any case, the partition function Z depends, a priori, on the extra data we have fixed on M, such as the metric. However, in certain field theories, especially 'super symmetric' ones, this dependence becomes trivial, and the field theory is said to be **topological**.

Example 1.1.1. An example of a topological quantum field theory is based on the Chern-Simons functional on a 3-manifold M^3 with Riemannian metric. We have a trivial G-bundle over M where G is a simply connected Lie group. We have

$$\mathcal{F} = \Omega^1(M, \mathfrak{g}).$$

Let a field (i.e. connection) be given $\nabla = A_i(m)dx_i$, then in local coordinates the Lagrangian is given as

$$\mathcal{L}(\nabla) = tr(F \wedge A - \frac{1}{3} A \wedge A \wedge A) \in \Omega^3(M^3)$$

where $F = dA + A \wedge A$ is the curvature of the connection, and the trace map is taken in some representation of $\mathfrak{g}$

$$\Omega^3(M, \mathfrak{g}^{\otimes 3}) \xrightarrow{tr} \Omega^3(M^3).$$

We will discuss a treatment of the Chern-Simons TQFT later in the seminar.

The aim of this course is to present an alternative framework for working with topoligical field theories, which bypasses from the state the technical difficulties involved in setting up traditional QFT.

1.2 Topological Quantum Field Theories

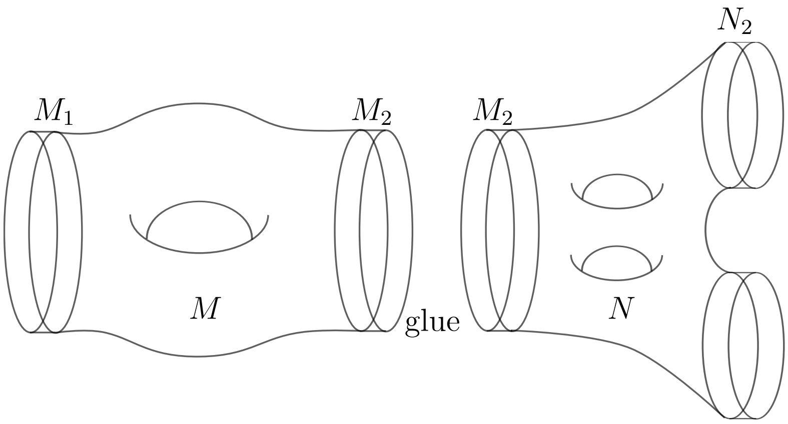
In the 1980s Atiyah and Segal realised that the formal properties of the (ill-defined) partition function may nevertheless lead to interesting **invariants** of manifolds; this led them to propose an axiomatisation of topological quantum field theories, a.k.a. TQFTs. In order to motivate the Atiyah-Segal axioms (stated at the end of the lecture), let us highlight a few mathematical structures that would follow from the QFT formalism, were it well-defined. These are compatibility with **boundary conditions**, **cobordisms** and **disjoint unions**.

i) Given submanifolds $M_1, M_2 \subseteq \partial M$ and fields supported in some collar neighbourhoods $M_1 \times I \subseteq M$, $M_2 \times I \subseteq M$ we can heuristically define

$$Z(\varphi_1, \varphi_2) = \int_{\substack{\varphi \in \Gamma(M.\Omega^* \xi) \\ \varphi|_{M_1 \times I} = \varphi \\ \varphi|_{M_2 \times I} = \varphi_2}} e^{i\pi \int_M \mathcal{L}(\varphi)} \, d\varphi$$

That is, we compute not the entire partition function, but rather its contributions from fields on M which restrict to φ_1 and φ_2 on each boundary component. This 'defines' a map:

$$Z(M) : \mathcal{F}(M_1 \times I) \to \mathcal{F}(M_2 \times I).$$



ii) These compose under gluing of oriented **cobordisms**.

$$Z(N) \circ Z(M) = Z(M \text{ glued to } N)$$

A cobordism from M_1 to M_2 (both $(n-1)$-manifolds) is a n-manifold M with boundary $\partial M = M_1 \sqcup M_2 = M_{in} \sqcup M_{out}$. A manifold is orientable if $\Lambda^*(TM)$ is trivial and an orientation is decomposition of this line into $\pm$.

iii) Let M_1, M_2 be manifolds of dimension $n-1$. We have:

$$Z(M_1 \sqcup M_2) = Z(M_1) \otimes Z(M_2)$$

1.3 Cobordism category

In order to capture the structures above, Atiyah and Segal studied the **cobordism category** $Cob^1(n)$. The cobordism category has

Objects: $(n-1)$-dimensional, compact, **oriented**, closed manifolds.

Morphisms: $Hom(M_1, M_2)$ consists of diffeomorphism classes of oriented cobordisms M^n with $\partial M = M_{in} \sqcup M_{out} = M_1 \sqcup M_2$.

Composition is given by gluing of cobordisms, as above. Quotienting by diffeomorphism is required for associativity. Orientation is needed to determine the 'source' and 'target' of cobordisms: the boundary has an orientation and those components which agree with the global orientation are incoming, and disagree are outgoing.

Atiyah and Segal defined an n-dimensional TQFTs as a symmetric monoidal functor

$$Z : Cob^1(n)^{\sqcup} \to Vect_k^{\otimes}.$$

The mondoidal structure on $Cob^1(n)^{\sqcup}$ is given by disjoint union $\sqcup$ and on $Vect_k^{\otimes}$ given by the usual tensor product on vector spaces.

This definition is already very interesting but we will extend it in various ways. The first way is to replace $Vect_k^{\otimes}$ with a general symmetric monoidal category $\mathcal{C}^{\otimes}$. So we define an n-dimensional TFT as a symmetric monoidal functor

$$Z : Cob^1(n)^{\sqcup} \to \mathcal{C}^{\otimes}.$$

We can also extend the definition to higher categories. Given $k = 1, 2, \ldots, \infty$, will introduce a higher cobordism category $Cob^k(n)$, and various examples of higher symmetric monoidal categories $\mathcal{C}^{\otimes}$, and study functors,

$$Z : Cob^k(n)^{\sqcup} \to \mathcal{C}^{\otimes},$$

Example 1.3.1. We have the 2-category of k-algebras,

$$\mathcal{C}^{\otimes} = \mathrm{Alg}_k^2 = \begin{cases} k \text{ algebras as objects} \\ A - B \text{ bimodules as morphisms} \\ \text{bimodule maps as 2-morphisms} \end{cases}$$

and

Example 1.3.2. The 3-category of tensor categories (which we may regard as algebra objects in Vect_k.

$$\mathcal{C}^{\otimes} = \mathrm{Alg}^2_{\mathrm{Vect}_k} = \begin{cases} \text{tensor categories as objects} \\ \text{bimodule categories as 1-morphisms} \\ \text{bimodule functors as 2-morphisms} \\ \text{natural transformations as 3-morphisms} \end{cases},$$

Similarly there is a 4-category of braided tensor categories, which will be discussed in Chapter 6.

1.4 The Cobordism Hypothesis

We shall also encounter the **cobordism hypothesis**: roughly this says that we can find 'generators' and 'relations' presentations for each $Cob^k(n)$, and hence classify TQFTs in terms of special objects in $\mathcal{C}^{\otimes}$ which are called "(fully) dualizable" objects. The hypothesis hypothesis has been worked on my many people including Baez-Dolan (conjecture), Lurie (proof), Scheimbauer (Segal spaces) and Ayala-Francis Tannaka (factorisation homology).

In brief:

- $Cob^{\infty}(n)$ is generated by a single object, the n-disc. Hence a fully extended TFT is completely determined by $Z(\mathbb{R}^n) \in \mathcal{C}^{\otimes}$.
- $Z(\mathbb{R})$ must be fully dualizable, which asserts the existance of many adjoints and duals in $\mathcal{C}^{\otimes}$ and is essentialy a strong finiteness condition.

See Chapter 5 for a more complete statement.

Chapter 2

TQFTs in 1 and 2 Dimensions

SPEAKER: JULIET COOKE
NOTES: JULIET COOKE
DATE: 22-1-2016

In this lecture we review the classification of TFT's in dimensions 1 and 2. We will largely follow [17].

2.1 The Category $\mathbf{Cob}^1(n)$

Definition 2.1.1. A $(n, n-1)$-dimensional TQFT is a symmetric monoidal functor

$$Z : Cob^1(n)^{\sqcup} \to \mathcal{C}^{\otimes}$$

we shall assume for simplicity that $\mathcal{C}^{\otimes} = \text{Vect}_k^{\otimes}$, the category of vector spaces over a field k equipped with the standard tensor product.

The superscript 1 of $Cob^1(n)$ tells us we are are dealing with a standard category; higher values are used to denote higher categories which will be considered later in this talk and in later talks. As result to describe $Cob^1(n)^{\sqcup}$ we need to know its objects and morphisms.

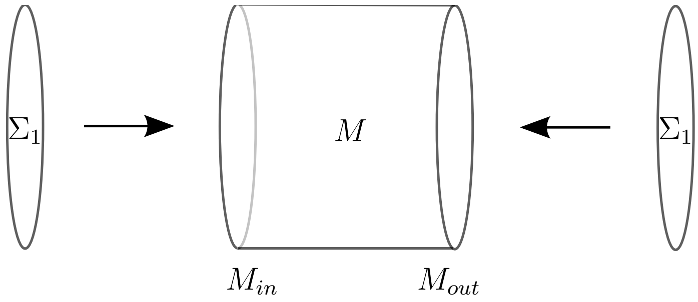
- The objects of $Cob^1(n)$ are $(n-1)$-dimensional manifolds Σ which are closed, compact and oriented.
- The morphisms of $Cob^1(n)$ are oriented cobordisms $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ up to equivalence of cobordisms.

Definition 2.1.2. An oriented cobordism from Σ_0 to Σ_1 is an oriented n-manifold M together with maps

$$\Sigma_0 \to M \leftarrow \Sigma_1$$

such that Σ_0 maps diffeomorphically onto M_{in} [1] and Σ_1 maps diffeomorphically onto M_{out}. We denote such a cobordism as $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$.

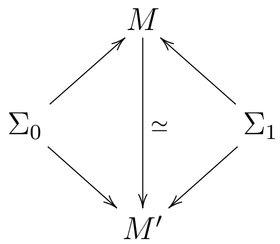
[1] The in and out boundaries of M are determined by how the choice of orientation M corresponds to the choice of orientation on parts of its boundary. We shall not go into detail as to how this is exactly done however the point is to add a directionality to our cobordism morphisms.



Two compatible cobordisms $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ and $\Sigma_1 \overset{M'}{\Longrightarrow} \Sigma_2$ can be composed by gluing together the n-manifolds along their common boundary Σ_1 [2] to give a new cobordism $\Sigma_0 \overset{M' \circ M}{\Longrightarrow} \Sigma_2$.

This composition operation is not strictly associative. However it is associate up to diffeomorphism relative to the boundaries Σ_0 and Σ_1 which leads to a good definition of morphisms by considering equivalence classes of cobordisms.

Definition 2.1.3. Two cobordisms $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ and $\Sigma_0 \overset{M}{\Longrightarrow} \Sigma_1$ are equivalent cobordisms if they have the same boundaries Σ_0, Σ_1 and the n-manifold M is diffeomorphic to M'. This can be expressed by saying the following diagram commutes:



Finally, $Cob^1(n)$ is a monoidal category with operation $\sqcup$, the disjoint union.

2.2 (1,0)-TQFTs

A $(1,0)$-dimensional TQFT is a symmetric monoidal functor

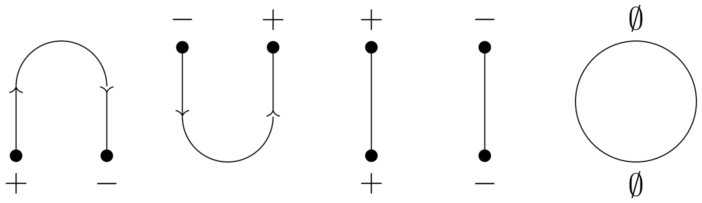
$$Z : Cob^1(1)^{\sqcup} \to \mathrm{Vect}_k^{\otimes}.$$

We shall now aim to classify such TQFTs. Unlike in higher dimensions where the aim would be to use TQFTs to understand the category $\mathrm{Cob}^1(n)$, we understand $\mathrm{Cob}^1(1)$ as this just requires knowledge of 0 and 1 dimensional manifolds and we can use this to understand the TQFT. The objects of $\mathrm{Cob}^1(1)$ are disjoint unions of oriented points

$$\left\{\overset{+}{\bullet}, \overset{-}{\bullet}\right\}.$$

The morphisms of $\mathrm{Cob}^1(1)$ are disjoint unions of

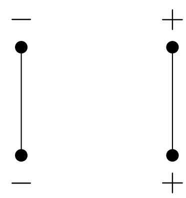
[2]Technically it is only a common boundary up to diffeomorphism. In order to compose we should also choose a small collar around the boundary in M and M'. Different choices of collar will give diffeomorphic cobordisms, so this choice will not matter given our final definition.



As $Z : Cob^1(1)^{\sqcup} \to \mathrm{Vect}_k^{\otimes}$ is monoidal, we have for disjoint unions $Z(A \sqcup B) = Z(A) \otimes Z(B)$ where either both A, B are 0-manifolds or they are both cobordisms of 0-manifolds. As a result to define Z it suffices to specify values on the two points and five cobordisms given above.

$$Z\left(\overset{+}{\bullet}\right) = V_+ \in \mathrm{Vect}_k^{\otimes}$$
$$Z\left(\overset{-}{\bullet}\right) = V_- \in \mathrm{Vect}_k^{\otimes}$$
$$Z\left(\underset{+\sqcup-}{\bigcap}\right) : V_+ \otimes V_- \to k$$
$$Z\left(\overset{-\sqcup+}{\bigcup}\right) : k \to V_- \otimes V_+$$
$$Z\left(\overset{-}{\underset{-}{|}}\right) : V_- \to V_-$$
$$Z\left(\overset{+}{\underset{+}{|}}\right) : V_+ \to V_+$$
$$Z\left(\bigcirc\right) : k \to k$$

The first thing to notice is that



are the identity cobordisms hence must be sent to identity maps in $\mathrm{Vect}_k^{\otimes}$ by the functor Z. Hence

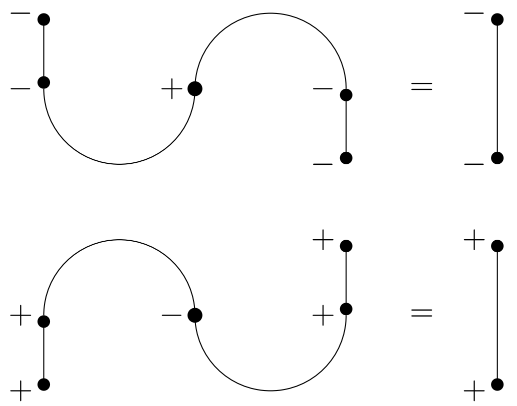
$$Z\left(\overset{-}{\underset{-}{|}}\right) = Id_{V_-}$$
$$Z\left(\overset{+}{\underset{+}{|}}\right) = Id_{V_+}$$

The map $Z\left(\underset{+\sqcup-}{\bigcap}\right) : V_+ \otimes V_- \to k$ gives a map $V_- \to Hom(V_+, k)$ and if this map is bijective then V_- can be identified with the dual vector space of V_+ and thus it would be sufficient to specify $\left(\overset{+}{\bullet}\right) = V_+$ only. This is in fact the case and follows from Zorro's Lemma.

Proposition 2.2.1 (Zorro's Lemma [3]). *For any pair of basis vectors $v_i \in V_-$ and $e_j \in V_+$,*

$$Z\left(\underset{+\sqcup-}{\bigcap}\right)(v_i, e_j) = \delta_{ij}.$$

Proof.



The map $Z\left(\overset{-\sqcup+}{\bigcup}\right) : k \to V_- \otimes V_+$ must have the form $k \mapsto \sum_{i=1}^{n} v_i \otimes e_j$ for some finite number of basis vectors $v_i \in V_-$ and $e_j \in V_+$. Consider a vector $e_j \in V_+$ under the action of the maps of the LHS of the lower diagram:

$$e_j = e_j \otimes k \mapsto e_j \otimes \sum_{i=1}^{n} v_i \otimes e_i \mapsto \sum_{i=1}^{n} e_i \left(Z\left(\underset{+\sqcup-}{\bigcap}\right)(v_i, e_j) \right).$$

The result must be the same as when we act on e_j by the map on the RHS which is just the identity map:

$$e_j \mapsto e_j.$$

Hence we have that

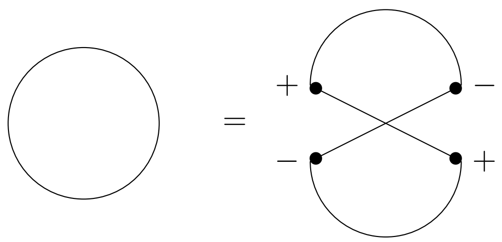
$$\sum_{i=1}^{n} e_i \underbrace{\left(Z\left(\underset{+\sqcup-}{\bigcap}\right)(v_i, e_j) \right)}_{\in k} = e_j$$

which implies that

$$Z\left(\underset{+\sqcup-}{\bigcap}\right)(v_i, e_j) = \delta_{ij}$$

as required. □

We have shown that $V_- = (V_+)^*$. Finally we claim that V_+ is a finite dimensional vector space:



[3]Zorro's Lemma also shows that $Z(\bigcup)$ and $Z(\bigcap)$ are adjoint, see chapter 5.

Hence

$$(\bigcirc) : 1 \mapsto \sum_{i=1}^{n} e^i \otimes e_i \mapsto \sum_{i=1}^{n} e^i(e_j) = \dim V_+,$$

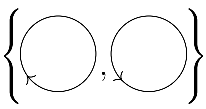
so that dim V_+ must be finite.

2.3 (2, 1)- TQFTs

We shall now consider a (2,1)-TQFT which is a symmetric monoidal functor:

$$Z : Cob^1(2) \to \text{Vect}_k^{\otimes}.$$

Again we know explicitly what $Cob^1(2)$ as we have a classification of surfaces. The objects of $Cob^1(2)$ are 1-dimensional closed compact oriented manifolds thus must be disjoint unions of



For the cobordisms we have the basic cobordisms given in Figure 2.1.

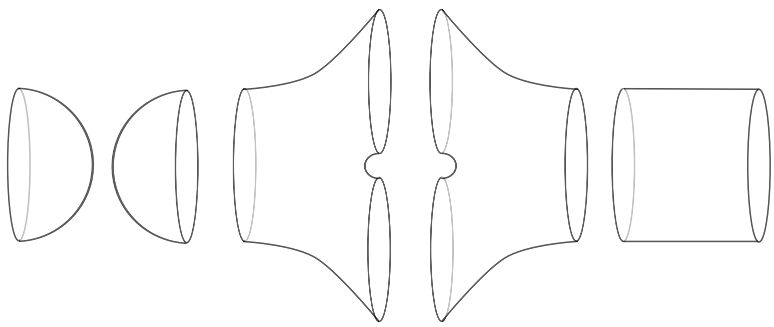


Figure 2.1: Basic Cobordisms

Using the classification of surfaces we can decompose any cobordism into a composition of these basis cobordism, such a decomoposition is some called the 'standard form'. We shall illustrate how this is done with an example. Suppose we have a cobordism $S^1 \sqcup S^1 \sqcup S^1 \overset{M}{\Longrightarrow} S^1 \sqcup S^1 \sqcup S^1 \sqcup S^1$ such that M has genus 2 then the standard form is given in Figure 2.2.

The first two of the basic cobordisms are used for constructing cobordism which are closed or have a single boundary component.

Firstly, we do not have to specify both orientations of the circle in our TQFT. If we denote $Z(\circlearrowleft) = A \in \text{Vect}_k^{\otimes}$ then $Z(\circlearrowright) = A^*$, the vector dual of A and A is finite dimensional. This is analgous to the $(1, 0)$-TQFT case and the proof uses a higher dimensional version of Zorro's lemma.

Definition 2.3.1. A (unital, associative) k-algebra is a k-vector space A together with two k-linear maps

$$\mu : A \otimes A \to A \text{ (multiplication)}, \; \eta : k \to A \text{ (unit map)},$$

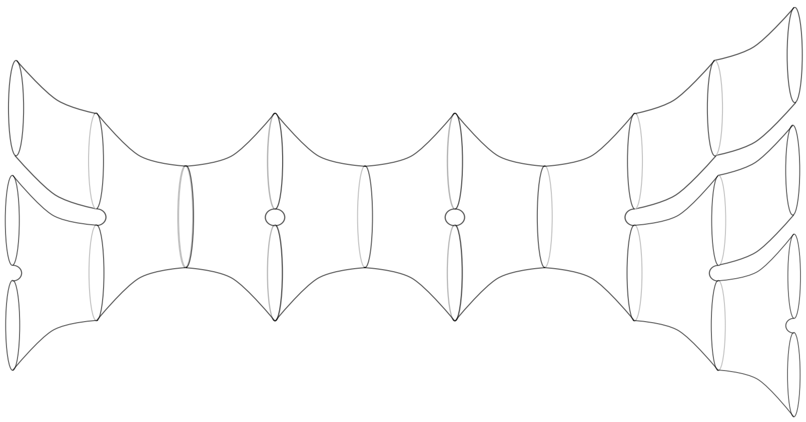


Figure 2.2: Example of a cobordism in 'standard form'

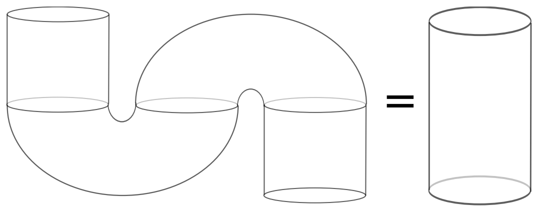


Figure 2.3: The diagram for Zorro's Lemma in 2-dimensions

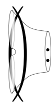
satisfying the associativity and the unit axiom:

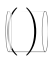
$$(\mu \otimes Id_A)\mu = (Id_A \otimes \mu)\mu$$
$$(\eta \otimes Id_A)\mu = Id_A = (Id_A \otimes \eta)\mu.$$

In other words, a k-algebra is precisely a monoid in the monoidal category $\mathrm{Vect}_k^{\otimes}$.

$A = Z(\circlearrowleft)$ is a k-algbera with maps defined:

$\eta = Z$  $: k \to A$ (unit)

$\mu = Z$  $: A \otimes A \to A$ (multiplication)

$Id_A = Z$  $: A \to A$ (identity)

These maps satisfy associativity and the unit axiom:

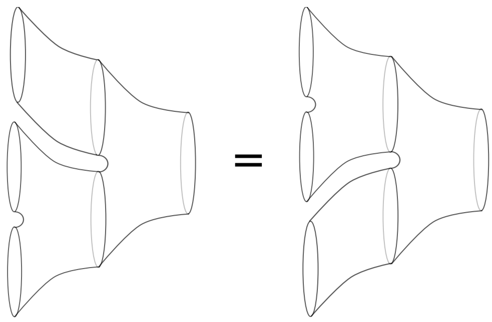


Figure 2.4: Associativity

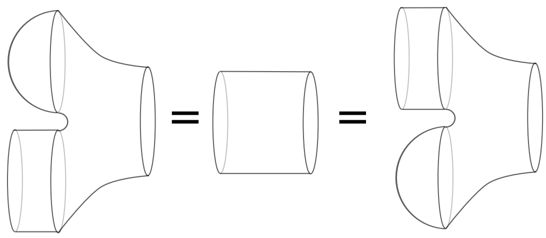


Figure 2.5: Unit axiom

Furthermore, A is a commutative algebra:

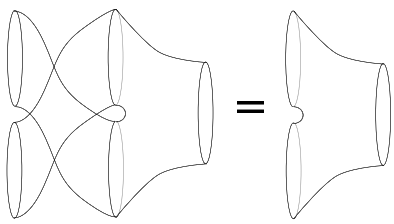


Figure 2.6: Commutativity

To conclude our classification of $(1, 0)$, we shall show that A is a Frobenius algebra.

Definition 2.3.2. A Frobenius algebra is a k-algbera A equipped with an associate non-degenerate pairing $\beta : A \otimes A \to k$ called the Frobenius form.

The Frobenius form on A is defined as:

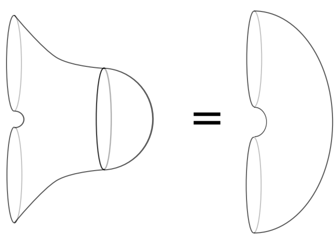


Figure 2.7: Frobenuis Form

The conditions of associativity and non-degeneracy are encoded in the following two figures.

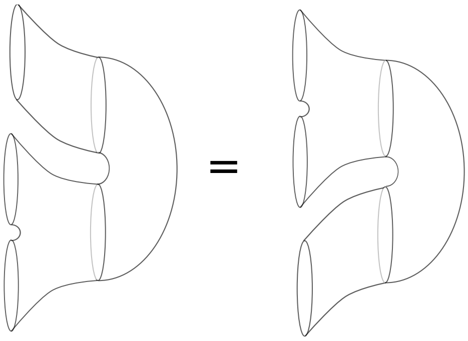


Figure 2.8: Associativity Condition

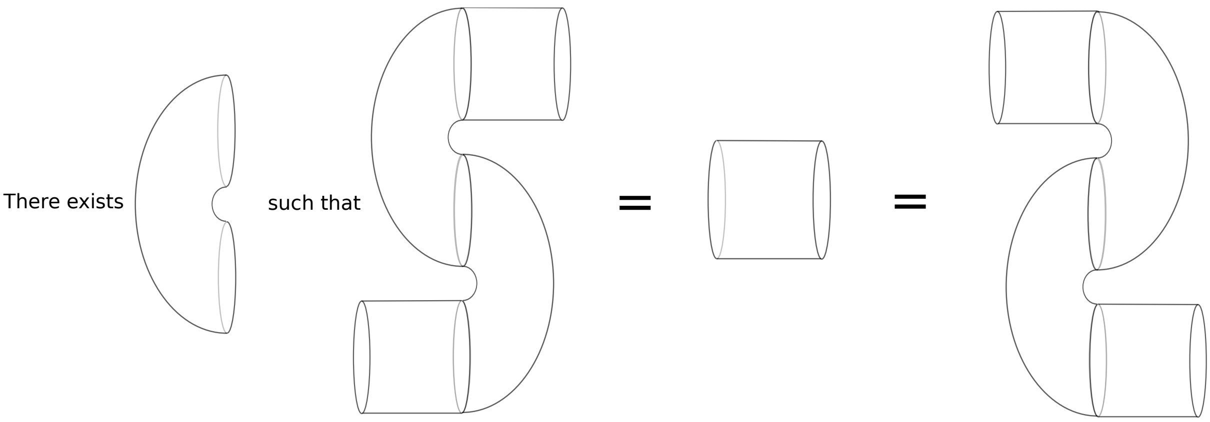


Figure 2.9: Non-degeneracy Condition

Another example of a Frobenius algebra is a matrix algebra defined over a field k with Frobenius form $\sigma(a,b) = tr(a \cdot b)$. To summarise: we have seen that for a $(2,1)$-TQFT the vector space $Z\left(\overset{+}{\bullet}\right)$ naturally carries the structure of a Frobenius algebra. Using the classification of surfaces to describe the morphisms and relations between them in $Cob^1(2)$ one can show:

Proposition 2.3.3. *(2,1)-TQFTs are completely determined by their value on* $Z\left(\overset{+}{\bullet}\right) = A \in Vect_k^{\otimes}$ *where A is a finite dimensional, commutative Frobenius algebra.* [4]

2.4 Fully extended (2, 1, 0) TQFTs

The definition of a TQFT can be extended to be a functor from higher cobordism categories $Cob^k(n)$ for $k > 1$. How this is done generally will be covered in chapter 5 but in this talk we shall consider an example where $k = 2 = n$.

Definition 2.4.1. A $(2,1,0)$-TQFT is a symmetric monoidal functor between weak 2-categories:

$$Z : Cob^2(2) \to \mathrm{Alg}_k^2.$$

We shall begin by defining the 2-category Alg_k^2.

- The objects of Alg_k^2 are k-algebras (k is a field).

[4] This is reversible; every finite dimensional, commutative Frobenius algebra is a TQFT.

- The 1-morphisms of Alg_k^2 are Bimodules.

  Definition 2.4.2. An $A-B$ bimodule ${}_AM_B$ is an abelian groups such that

  - M is a left A module and M is a right B module,
  - $(am)b = a(mb)$ for all $(a, m, b) \in A \times {}_AM_B \times B$.

  The composition of 1-morphisms is defined for compatible bimodules ${}_AM_B$ and ${}_BN_C$ to be

  $$ {}_AM_B \otimes_B {}_BN_C = \frac{{}_AM_B \otimes {}_BN_C}{\langle mb \otimes n - m \otimes bn | m \in {}_AM_B, b \in B, n \in {}_BN_C \rangle} $$

  where the unadorned $\otimes$ is the usual tensor product on vector spaces. This composition is only associative up to isomorphisms of bimodules.

- The 2-morphisms of Alg_k^2 are bimodule morphisms.

We now need to define the 2-category $Cob^2(2)$. We shall actually be considering the **framed**[5] $Cob^2(2)$ category this ensures that gluing works properly.

- The objects of $Cob^2(2)$ are 2-framed points $\left\{ \overset{+}{\bullet}, \overset{-}{\bullet} \right\}$.
- The 1-morphisms of $Cob^2(2)$ are compact 2-framed 1-manifolds.
- The 2-morphisms of $Cob^2(2)$ are diffeomorphism classes of compact 2-framed 2-manifolds with corners. [6]

Proposition 2.4.3. *Fully extended* $(2, 1, 0)$*-TQFTs*

$$ Z : Cob^2(2) \to Alg_k^2 $$

are completely determined by their value $B = Z\left(\overset{+}{\bullet} \right) \in Alg_k^2$*; which must be a finite dimensional, semi-simple, Frobenuis algebra and* $Z(\circlearrowleft)^*$ *is the centre of this Frobenius algebra.*

We will not have time to give a proof of this result however I shall give a sketch of how the proof works.

If $Z(\overset{+}{\bullet}) = A \in \text{Alg}_k^2$ then $Z(\overset{-}{\bullet}) = A^{op}$. As gluing a straight line segment to a 1-manifold has no effect on the topology $\underset{-}{\overset{+}{|}}$ must map to the identity map in Alg_k^2 which is the bimodule ${}_AA_A$:

$$ Z\left(\underset{-}{\overset{+}{|}} \right) = {}_AA_A. $$

[5] A 2-framing of a smooth l-manifold M, $l \leq 2$, is an isomorphism of the vector bundle $\mathbb{R}^{2-l} \oplus TM$ with the trivial bundle $M \times \mathbb{R}^2$.

[6] The source and target is determined by the choice of isomorphism $\mathbb{R}^{2-k} \oplus TM|_B \cong \mathbb{R}^{2-k+1} \oplus TB$ from M onto part of its boundary B.

Also

$$Z\left(\underset{+\sqcup-}{\bigcap}\right) = {}_{A\otimes A^{op}}A$$
$$Z\left(\overset{-\sqcup+}{\bigcup}\right) = A_{A\otimes A^{op}}$$

so gluing them together gives

$$Z(S^1) = A \otimes_{A\otimes A^{op}} A \cong \frac{A}{[A, A]}$$

as a vector space. We can define a trace on $\frac{A}{[A,A]}$:

$$tr' = Z\left(\bigcirc\right) : \tfrac{A}{[A,A]} \to k.$$

Thus we can define a trace on A by factoring through $\frac{A}{[A,A]}$:

$$tr : A \to \frac{A}{[A, A]} \xrightarrow{tr'} k.$$

Then the pairing $A\otimes A \to k$, defined by $(a, b) \mapsto tr(a\cdot b)$ for all $a, b \in A$, is a Frobenius form and thus A is a Frobenius algebra. To see that $\left(\frac{A}{[A,A]}\right)^*$ is the centre you prove that the map

$$z \mapsto \Big(tr(z \cdot \_) : A \to k\Big)$$

is an isomorphism $Z(A) \cong (\frac{A}{[A,A]})^*$ ($Z(A)$ is the centre of A) using the associativity property of the Frobenius form. Finally to show the Frobenius algebra A is finite dimensional is equivalent to showing it is separable.

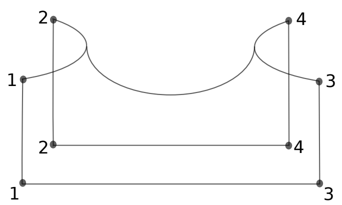


Figure 2.10: Saddle

To do this we use the map

$$Z(\text{saddle})\,{}_1A_2 \otimes {}_3A_4 \to {}_1A_3 \otimes {}_2A_4.$$

We define the element e to split the multiplication map $m : A \otimes A \to A$ into $A \otimes A = A' \oplus A''$ by

$$e := Z(\text{saddle})(1 \otimes 1) \in A \otimes A.$$ [7]

[7] e describes how the points (copies of A) are permuted.

Chapter 3

Quantum Groups and Link Invariants

SPEAKER: JENNY AUGUST
NOTES: JENNY AUGUST
DATES: 29-1-16 & 5-2-16

3.1 Introduction

This lecture gives a first example of quantum groups and shows one of their applications; knot invariants. It should be noted that this lecture will not make use of the definition of a topological field theory and it may look out of place in the seminar but we will see how it is connected in the following lecture. Unless otherwise stated, the reference for all this material is Kassel's *Quantum Groups* [9].

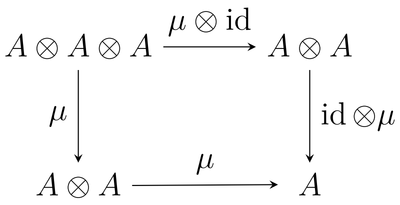
3.2 Quantum Groups

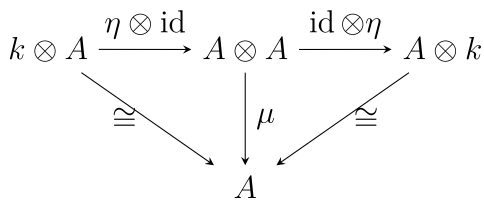
Unfortunately, there is no one definition for quantum groups and the term instead refers to various classes of objects, usually noncommutative algebras with some sort of additional structure. One such class consists of deformations of universal enveloping algebras of lie algebras and the specific example we will consider is a deformation of the universal enveloping algebra of $\mathfrak{sl}_2$. This is an example of a Hopf Algebra and so we begin by defining those.

3.2.1 Hopf Algebras

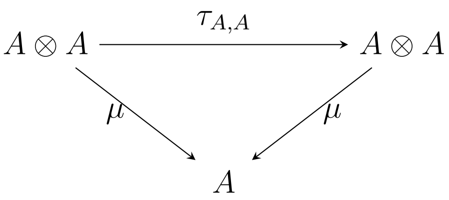
A Hopf Algebra is, in particular, an algebra and so we start with the definition of an algebra.

Definition 3.2.1. An associative algebra over a field k is given by a triple (A, μ, η) where A is a k-vector space and the k-linear maps $\mu : A \otimes A \to A$ and $\eta : k \to A$ are such that the following diagrams commute.



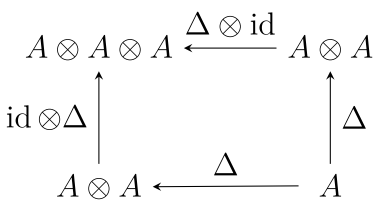


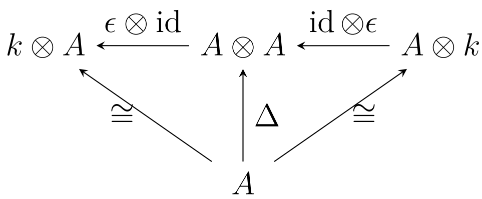
The first diagram gives the associativity of the algebra and second diagram shows the algebra is unital. Moreover, if we wish the algebra A to be commutative we ask that the following diagram also commutes, where $\tau_{A,A}(a \otimes b) = b \otimes a$.



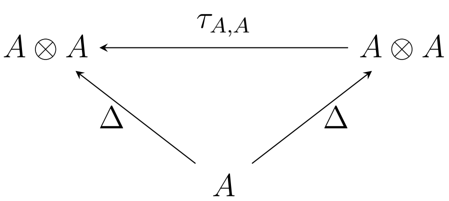
From this definition of algebra, it is very easy to define a coalgebra simply by reversing all the arrows. However, to be explicit we give the following definition.

Definition 3.2.2. A coassociative coalgebra over a field k is given by a triple (A, Δ, ϵ) where A is a k-vector space and the k-linear maps $\Delta : A \to A \otimes A$ and $\epsilon : A \to k$ are such that the following diagrams commute.





The two diagrams give the coassociativity of the coalgebra and the fact the coalgebra is counital. Moreover, if we wish the coalgebra A to be cocommutative we ask that the following diagram also commutes.



We call the maps Δ and ϵ the coproduct and counit of the coalgebra respectively. Given any two algebras, or coalgebras, we can define the notion of a morphism between them.

Definition 3.2.3. i) An algebra morphism between two algebras (A, μ_A, η_A) and (B, μ_B, η_B) is a linear map $f : A \to B$ such that $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

ii) A coalgebra morphism between two coalgebras $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ is a linear map $f : A \to B$ such that $(f \otimes f) \circ \Delta_A = \Delta_B \circ f$ and $\epsilon_B \circ f = \epsilon_A$.

This allows us to define a bialgebra.

Definition 3.2.4. A bialgebra is a quintuple $(A, \mu, \eta, \Delta, \epsilon)$ such that (A, μ, η) is an algebra and (A, Δ, ϵ) is a coalgebra with the additional requirement that Δ and ϵ are both morphisms of algebras.

Note that we could have equivalently asked that μ and η were coalgebra morphisms. A common example of a bialgebra is the group algebra of a finite group where $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for all $g \in G$. A Hopf Algebra is a bialgebra with some additional structure.

Definition 3.2.5. i) Let $(A, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. A linear map $S : A \to A$ is called an antipode for A if

$$\mu \circ (S \otimes \mathrm{id}_A) \circ \Delta = \mu \circ (\mathrm{id}_A \otimes S) \circ \Delta = \eta \circ \epsilon$$

ii) A Hopf algebra is a bialgebra with an antipode.

Note that not all bialgebras have an antipode so Hopf Algebras are a special class of bialgebras. Moreover, the antipode will be an antihomorphism i.e. $S(ab) = S(b)S(a)$ for all $a, b \in A$.

Now we turn our attention to the deformation of the universal enveloping algebra of $\mathfrak{sl}_2$, which will be the focus of this lecture. Recall that $\mathfrak{sl}_2$ is the 3-dimensional lie algebra generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with relations

$$[E, F] = H, \quad [H, E] = 2E \quad \text{and} \quad [H, F] = -2F.$$

We are interested in the representation theory of $\mathfrak{sl}_2$ i.e. the $\mathfrak{sl}_2$-modules. However, lie algebras are not associative and so we can not use the array of tools developed for studying the representation theory of associative unital algebras. Therefore, when studying a lie algebra $\mathfrak{g}$, we often choose to study an associative unital algebra which in some sense has "the same" representation theory as our original lie algebra. Such an algebra can be chosen in a universal way and is called the universal enveloping algebra of $\mathfrak{g}$, denoted $U(\mathfrak{g})$.

Note that $U(\mathfrak{sl}_2)$ is a cocommuative bialgebra with coproduct

$$\Delta(x) = 1 \otimes x + x \otimes 1 \quad \forall x \in \mathfrak{sl}_2.$$

However, for reasons which will become clear later, we choose to study the deformed algebra $U_q(\mathfrak{sl}_2)$ which can be defined as follows. Pick $q \in \mathbb{C}^\times$ such that q is not a root of unity. Then $U_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$-algebra generated by E, F, K, K^{-1}, with relations

$$KEK^{-1} = q^2 E, \quad KFK^{-1} = q^{-2} F \quad \text{and} \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

To motivate these relations, note that the generators act on the irreducible two dimensional representation of $U_q(\mathfrak{sl}_2)$ as

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad K = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

and that these matrices satisfy the relations above. In fact, we can think of K as q^H and if we take a certain limit as $q \to 1$ we get the original $U(\mathfrak{sl}_2)$ back.

The Hopf Algebra structure on $U_q(\mathfrak{sl}_2)$ is given by

$$\begin{aligned} \Delta(E) = 1 \otimes E + E \otimes K; \quad & \Delta(F) = K^{-1} \otimes F + F \otimes 1; \\ \Delta(K) = K \otimes K; \quad & \Delta(K^{-1}) = K^{-1} \otimes K^{-1}; \\ \epsilon(E) = \epsilon(F) = 0, \quad & \epsilon(K) = \epsilon(K^{-1}) = 1 \\ S(E) = -EK^{-1},\ S(F) = -KF, \quad & S(K) = K^{-1},\ S(K^{-1}) = K. \end{aligned}$$

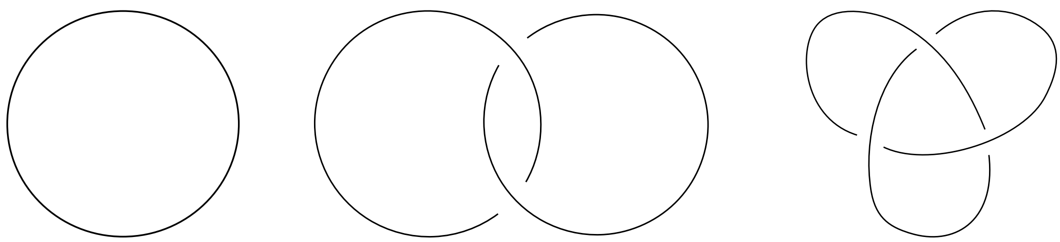
In particular, note that $U_q(\mathfrak{sl}_2)$ is not cocommutative which, although it seems like we are making things more complicated, is precisely what allows us to obtain knot invariants.

3.3 Knot Theory Basics

Knot theory is essentially studying what we get when we take a piece of string, tangle it up and then tie the ends together. We are actually going to look at a generalisation of knots, called links, in which more than one piece of string is allowed. A more mathematical definition is as follows.

Definition 3.3.1. A link is a collection of finitely many circles smoothly embedded in $\mathbb{R}^3$. A knot is a link consisting of a single circle.

Examples are shown in Figure 3.1. Note that (a) and (c) are examples of knots where as (b) is only a link.



(a) Unknot (b) Two linked circles (c) Trefoil

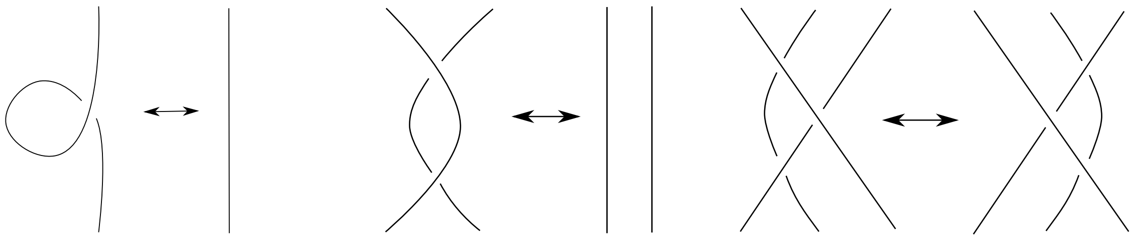
Figure 3.1: Examples of links.

The fundamental question of knot theory is to ask when two knots are "the same". Intuitively, two links are the same if you can get from one to the other without cutting the string anywhere.

Definition 3.3.2. Two links are considered isotopic if there exists an isotopy of $\mathbb{R}^3$ which maps one link to the other.

Figure 3.1 shows that we can draw links in the plane by keeping track of whether crossings are over-crossings or under-crossings. However, there are multiple ways of drawing the same link so we would like to know when two such diagrams represent the same link.

Proposition 3.3.3. *Two links are isotopic if and only if you can change from one to the other using the following* Reidemeister moves*:*



Despite this very useful characterisation, it can still be very difficult to tell whether or not two diagrams represent the same link. As a tool to help prove that two links are different, mathematicians have developed various link invariants.

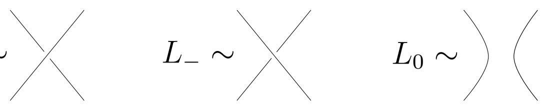
Definition 3.3.4. A link invariant assigns to each link an object such that, if two links are isotopic, they are assigned the same object.

These are useful because, if two different objects are assigned to two links, we know they can not be isotopic. A simple example is to assign to each link the number of circles which make up the link. It's clear that this is a link invariant but unfortunately it's not a very useful one. For example, it can't tell the difference between the unknot and the trefoil in Figure 3.1. Therefore, mathematicians looked for more sophisticated invariants and, since the 1920's, they have been assigning polynomials as link invariants. One such example is the Jones Polynomial.

Example 3.3.5. Let L be a link. We define the Jones Polynomial, $P_L(t)$, of L inductively.

- If L is the unknot then $P_L(t) = 1$.
- If L_+, L_- and L_0 are three links, identical except at a single crossing point where

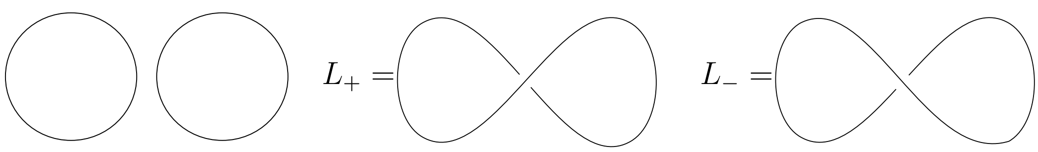
  $L_+ \sim$ $L_- \sim$ $L_0 \sim$

  then $t^{-1}P_{L_+}(t) - tP_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})P_{L_0}(t)$. This is called the Skein relation of the Jones Polynomial.

As an example, we find the Jones Polynomial of the link consisting of two unlinked circles. We define L_0 to be our desired link and the others as follows:

$L_0 =$ $L_+ =$ $L_- =$



We can see that both L_+ and L_- are isotopic to the unknot and so we get

$$(t^{\frac{1}{2}} - t^{-\frac{1}{2}})P_{L_0}(t) = t^{-1} - t \quad \text{and so} \quad P_{L_0}(t) = \frac{t^{-1} - t}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})}.$$

Since we can always put a link in terms of simpler links using this inductive method, we can calculate the Jones Polynomial of any link. To show it is a link invariant, you just need to show that it doesn't change when we alter a link by any of the Reidemeister moves.

The goal of the rest of the lecture is to construct link invariants, including the Jones Polynomial, using the category $U_q(\mathfrak{sl}_2)$-mod. The idea is that we will relate morphisms in $U_q(\mathfrak{sl}_2)$-mod to tangles. Tangles are a generalisation of links where we don't require the two ends of the string to be tied together.

Definition 3.3.6. A tangle is a smooth embedding of arcs and circles into $\mathbb{R}^2 \times I$ where the endpoints of the arcs lie on $\mathbb{R}^2 \times \partial I$.

Note that a link can be viewed as a tangle with only circles embedded. As with links, tangles can be drawn in the plane and two diagrams represent the same tangle if and only if they are related by the Reidemeister moves.

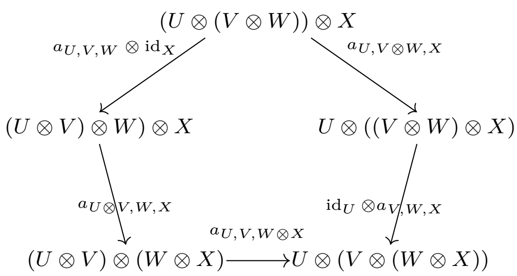
We are going to relate to each tangle a morphism in $U_q(\mathfrak{sl}_2)$-mod and, in particular, to each link, a morphism $\mathbb{C}(q) \to \mathbb{C}(q)$. This map can be thought of as an element of $\mathbb{C}(q)$ and will be the link invariant. The reason this is going to work is because $U_q(\mathfrak{sl}_2)$ is a Hopf Algebra which ensures $U_q(\mathfrak{sl}_2)$-mod is a balanced, rigid, braided tensor category which we discuss in more detail now.

3.4 Tensor Categories

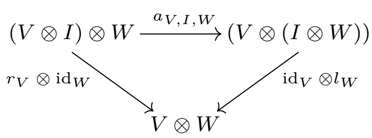
Definition 3.4.1. A tensor category is a category $\mathcal{C}$ with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and:

- A natural isomorphism $a : \otimes(\otimes \times \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes)$ called the associativity constraint;
- An object $I \in \mathcal{C}$ called the unit and natural isomorphisms $l : \otimes(I \times \mathrm{id}) \to \mathrm{id}$ and $r : \otimes(\mathrm{id} \times I) \to \mathrm{id}$ called the left and right unit constraints respectively,

such that the Pentagon and Triangle Axioms hold i.e. the two diagrams in Figure 3.3 commute for all objects $U, V, W, X \in \mathcal{C}$.



(a) The Pentagon Axiom



(b) The Triangle Axiom

Figure 3.3: The Pentagon and Triangle Axiom for Definition 3.4.1.

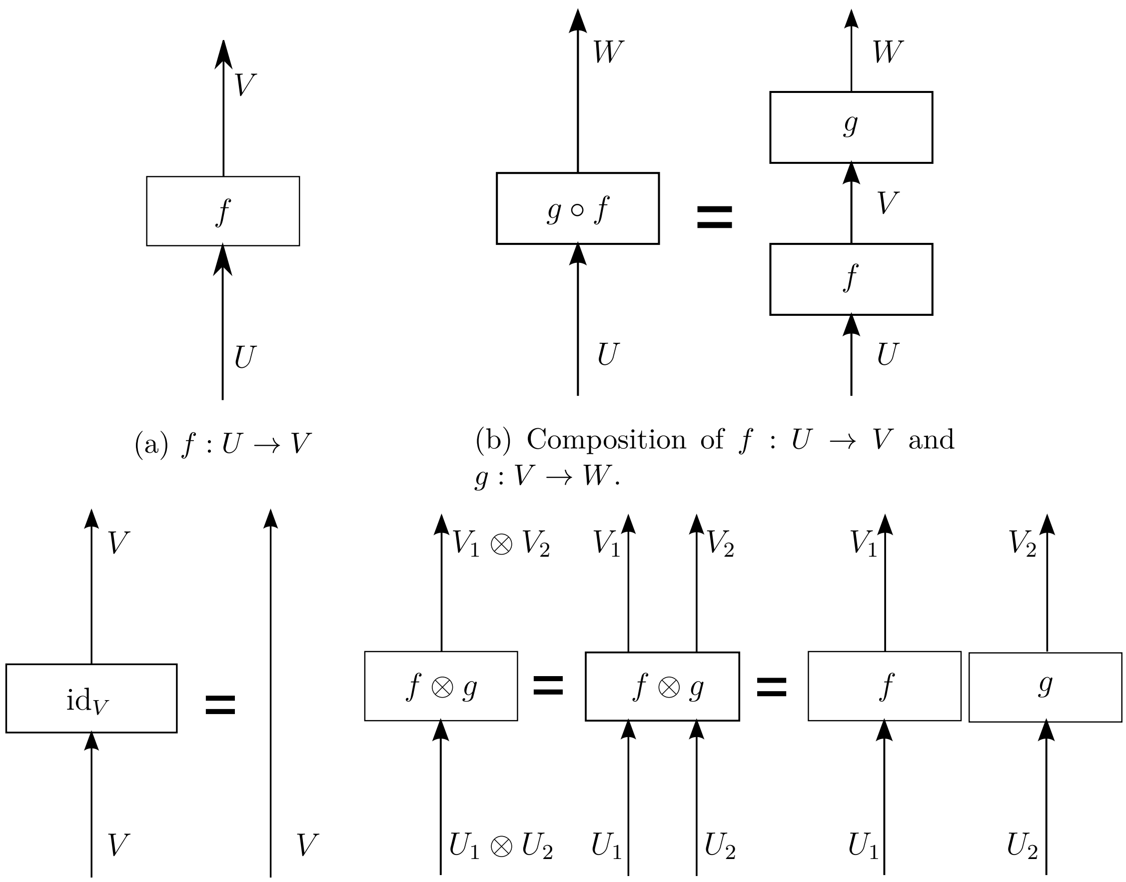
Example 3.4.2. i) The obvious example of a tensor category is the category of k-vector spaces, denoted $\mathrm{Vect}(k)$, with the usual tensor product. Here, the unit is given by k.

ii) Over a k-algebra A, every A-module is a k-vector space and so A-mod is a subcategory of Vect(k). Thus, it will inherit the tensor category structure from Vect(k) if, for any two A-modules V and W, we can give the vector space $V \otimes W$ an action of A. For a general k-algebra there is no canonical way to do this but if A has a coproduct $\Delta : A \to A \otimes A$ (such as $U_q(\mathfrak{sl}_2)$), we can define the action as

$$a \cdot (v \otimes w) = \Delta(a)(v \otimes w).$$

Thus $U_q(\mathfrak{sl}_2)$-mod is a tensor category.

As discussed, our link invariants are going to come from relating morphisms in $U_q(\mathfrak{sl}_2)$ to tangles. In Figure 3.4, we begin introducing how we might draw these morphisms to make this connection. We emphasise that the tensor product of two morphisms is simply the two morphisms drawn next to each other and composition by a morphism g means adding the picture related to g on top of the original morphism.



(a) $f : U \to V$

(b) Composition of $f : U \to V$ and $g : V \to W$.

(c) $\mathrm{id}_V : V \to V$

(d) Tensor product of $f : U_1 \to V_1$ and $g : U_2 \to V_2$.

Figure 3.4: Morphisms in a tensor category.

The definition of tensor category had a notion of associativity built into it but we would also like a notion of commutativity as this often makes structures easier to work with.

Definition 3.4.3. i) Define the flip functor, $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$, such that $\tau(V, W) = (V, W)$.

ii) A braided tensor category is a tensor category $(\mathcal{C}, \otimes)$ with a natural isomorphism

$$c : \otimes \to \otimes \circ \tau$$

called the commutativity constraint satisfying the Hexagon Axiom.

Note that this just means we have an isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ for every $V, W \in \mathcal{C}$ satisfying some compatibility relations.

Example 3.4.4. i) In $\mathrm{Vect}(k)$, we can take c such that $c_{V,W}(v \otimes w) = w \otimes v$. Notice that $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$ for all $V, W \in \mathcal{C}$ and so we say $\mathrm{Vect}(k)$ is a symmetric braided tensor category.

ii) In $U_q(\mathfrak{sl}_2)$-mod, the commutativity constraint from $\mathrm{Vect}(k)$ is not compatible with the action of $U_q(\mathfrak{sl}_2)$ and so we need to look for a different braiding.

Definition 3.4.5. In a bialgebra A, a universal R-matrix is an invertible element of $A \otimes A$ such that

$$\tau_{A,A} \circ \Delta(x) = R\Delta(x)R^{-1} \quad \forall x \in A.$$

If an R-matrix exists we can define a braiding in A-mod by

$$c_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w)).$$

where $\tau_{V,W}(v \otimes w) = w \otimes v$. If A is a cocommutative bialgebra, then you can take the R-matrix to be $1 \otimes 1$ and A-mod becomes a symmetric braided tensor category with c as in $\mathrm{Vect}(k)$. However, when A is not cocommutative, these R-matrices may not exist and can be very difficult to find even when they do. For $U_q(\mathfrak{sl}_2)$, you need to pass to the completion to find the R-matrix which has the expression

$$R = q^{\frac{H \otimes H}{2}} \exp_q(q - q^{-1})E \otimes F).$$

More information about this can be found [14]. Even though this R-matrix only exists in the completion, it is enough to give us a well defined braiding on $U_q(\mathfrak{sl}_2)$-mod. As an example, if we consider the 2 dimensional irreducible representation of $U_q(\mathfrak{sl}_2)$, $V = \mathbb{C}^2$ with standard basis $\{e_1, e_2\}$, then we get

$$c_{V,V} = q^2 \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

with respect to the basis $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$. Note that $c_{V,V}(e_1 \otimes e_1) = q^3 e_1 \otimes e_1$ and so it is clear that $c_{V,V}^2 \neq \mathrm{id}_{V \otimes V}$ and hence $U_q(\mathfrak{sl}_2)$-mod is not a symmetric tensor category.

While not being symmetric might make the category more complicated, it is exactly what we need to get useful link invariants. We represent the braiding isomorphisms in our category by the tangles shown in Figure 3.5 and, in particular, notice that if our category is symmetric, we see that we don't care whether or not a strand crosses above or below another. For example, this means we would be unable to tell the difference between two

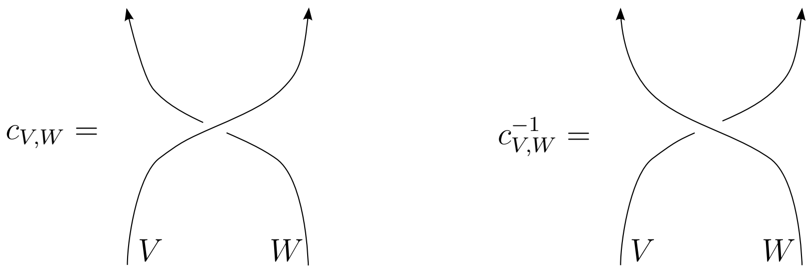


Figure 3.5: Representing the braiding isomorphisms as tangles.

linked circles and two unlinked circles. In particular, a knot invariant from a symmetric category would assign the same object to every knot and so be useless as a knot invariant. Thus, it is the non-cocommutativity of $U_q(\mathfrak{sl}_2)$ and so the lack of symmetry in $U_q(\mathfrak{sl}_2)$-mod which will give interesting knot invariants.

Recall that every vector space has a dual and this gives extra structure to the category $\mathrm{Vect}(k)$.

Definition 3.4.6. A braided tensor category is rigid if every object V has a dual object, denoted V^*, and morphisms

$$e_V : V^* \otimes V \to I$$
$$i_V : I \to V \otimes V^*$$

such that

$$(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \text{and} \quad (e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}.$$

Example 3.4.7. i) In $\mathrm{Vect}(k)$, we have $V^* = \mathrm{Hom}_k(V, k)$ and if V has basis $\{v_1, \ldots, v_n\}$, and the dual basis is $\{v^1, \ldots, v^n\}$ then we have

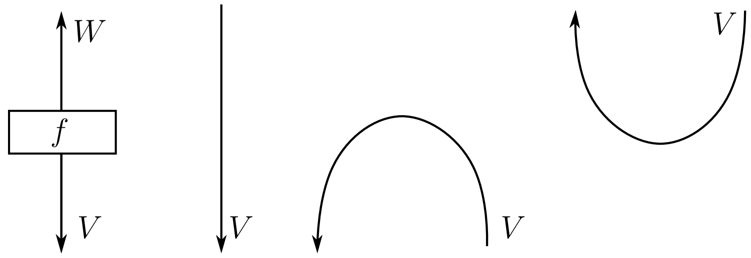
$$e_V(v^i \otimes v_j) = v^i(v_j)$$
$$i_V(1) = \sum_i v_i \otimes v^i$$

which are called evaluation and coevaluation respectively.

ii) As with the tensor category structure, the duals in $U_q(\mathfrak{sl}_2)$-mod and the corresponding maps are inherited from $\mathrm{Vect}(\mathbb{C})$, provided that we can define an action of $U_q(\mathfrak{sl}_2)$ on V^* such that e_V and i_V are both module homomorphisms. For this to hold, we need the antipode $S : U_q(\mathfrak{sl}_2) \to U_q(\mathfrak{sl}_2)$ which exists as $U_q(\mathfrak{sl}_2)$ is a Hopf Algebra. The action of $U_q(\mathfrak{sl}_2)$ on V^* is then given by

$$(a \cdot f)(v) = f(S(a) \cdot v) \quad \forall f \in V^*.$$

Now that we have these duals and these extra morphisms, we wish to know how to represent them as tangles. As we don't want to clutter up our tangles with lots of notation, if an arrow should be labelled by V^*, we instead label it by V but give the arrow the opposite orientation. This is shown in Figure 3.6. Also shown in the diagram

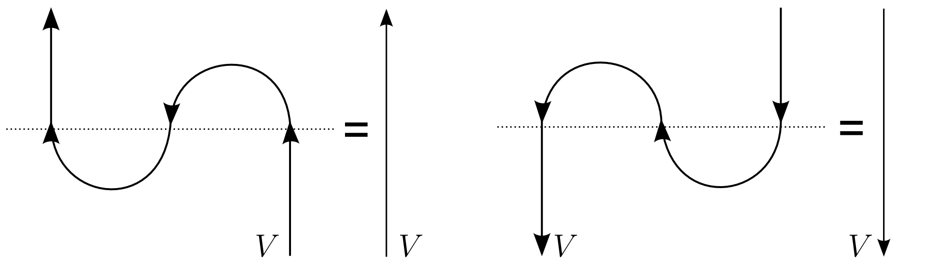


$f : V^* \to W$ $\quad \mathrm{id}_{V^*}$ $\quad e_V : V^* \otimes V \to k$ $\quad i_V : k \to V \otimes V^*$

Figure 3.6: Pictorial representations of the morphisms containing duals.

is how we draw e_V and i_V. We consider the unit as "the empty object" of the category and so if an arrow should be labelled by I, we just don't draw it. This is the key point, as drawing them like this is going to let us view links as morphisms from $\mathbb{C}(q)$ to $\mathbb{C}(q)$.

Before we go any further, we give an example of using the graphical representation of morphisms. For example, we can use them to express the conditions given in Definition 3.4.6 as shown in Figure 3.7. It is clear by the Reidemeister moves that the two tangles in each diagram are the same and so should represent the same morphism.



(a) $(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V$ $\quad$ (b) $(e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}$

Figure 3.7: Pictorial representations of the conditions in Definition 3.4.6.

To get a link invariant we are going to construct a functor from a category where tangles are the morphisms to the category $U_q(\mathfrak{sl}_2)$-mod. For this to be well defined we need two things:

i) A way of associating to any tangle a morphism in $U_q(\mathfrak{sl}_2)$-mod.

ii) If two tangles are isotopic we need that the two morphisms they represent are the same.

Unfortunately neither of these are satisfied at the moment. First, note that strands of tangles will have to be labelled by an object of $U_q(\mathfrak{sl}_2)$-mod for us to have any hope of knowing which morphism to send the tangle to. This problem will be solved by requiring that the category whose morphisms are tangles is in fact a category of "coloured" tangles. This precisely means that each strand of each tangle is coloured with an object of $U_q(\mathfrak{sl}_2)$-mod. However, even with this addition, we can't satisfy either of the above conditions as the following example shows.

Example 3.4.8. At this point, we have no way of associating a morphism to the top cap of the tangle in Figure 3.8. This would need to be a morphism $V \otimes V^* \to I$ which we don't even know exists at the moment. However, again we are saved by the extra structure that $U_q(\mathfrak{sl}_2)$-mod has.

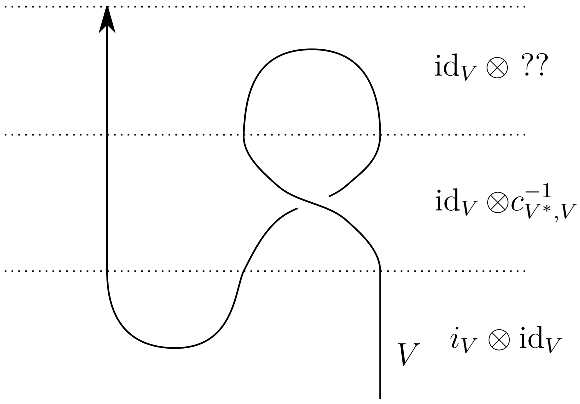


Figure 3.8: While we can associate morphisms to most of this tangle, we don't currently know what to assign to the top cap. Moreover, by Reidemister moves, this tangle is isotopic to the straight line corresponding to id_V and so the morphism it represents should be id_V.

Definition 3.4.9. i) The transpose of $f : U \to V$ is $f^* : V^* \to U^*$ where

$$f^* = (e_v \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes f \otimes \mathrm{id}_{U^*})(\mathrm{id}_{V^*} \otimes i_U).$$

ii) A braided tensor category $(\mathcal{C}, \otimes)$ is balanced if there exists a natural isomorphism

$$\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$$

such that

$$\theta_{V \otimes W} = (\theta_V \otimes \theta_W) c_{W,V} c_{V,W} \quad \text{and} \quad \theta_{V^*} = (\theta_V)^*.$$

iii) A ribbon category is a rigid, balanced, braided tensor category.

Example 3.4.10. i) Any symmetric braided tensor category is balanced with $\theta_V = \mathrm{id}_V$. In particular $\mathrm{Vect}(k)$ is a ribbon category.

ii) For $U_q(\mathfrak{sl}_2)$-mod, we again have to appeal to the extra struture that $U_q(\mathfrak{sl}_2)$ has. In particular, $U_q(\mathfrak{sl}_2)$ is a Ribbon Hopf Algebra which means there exists a special element, called the ribbon element which allows us to define the twist isomorphisms.

Why do we call a braided tensor category with a twist "balanced"? Consider Definition 3.4.6, which required each object of the category to have a dual object. In fact what we asked for there was for each object to have a right dual. We could instead have asked for left duals.

Definition 3.4.11. For an object V in a tensor categorey $\mathcal{C}$, the left dual of V is an object ${}^*V$ with morphisms

$$i'_V : I \to^* V \otimes V$$
$$e'_V : V \otimes^* V \to I.$$

such that

$$(\mathrm{id}_V \otimes e_V)(i_V \otimes \mathrm{id}_V) = \mathrm{id}_V \quad \text{and} \quad (e_V \otimes \mathrm{id}_{V^*})(\mathrm{id}_{V^*} \otimes i_V) = \mathrm{id}_{V^*}.$$

In a balanced category, we can take ${}^*V = V^*$ (hence the name balanced) and define

$$i'_V = (\mathrm{id}_{V^*} \otimes \theta_V) c_{V,V^*} \circ i_V$$
$$e'_V = e_V \circ c_{V,V^*}(\theta_V \otimes \mathrm{id}_{V^*}).$$

This solves one problem as we can now assign a morphism to the cap in Figure 3.8. However, this creates another problem as there is no reason for the morphism represented by Figure 3.8 to be the same as the identity morphism. In fact, they are not the same and we can see this if we draw the morphism i'_V out in full as shown in Figure 3.9. This tangle has a box labelled θ_V where as there is no such box on the identity map. However, we don't want to have to use the box to distinguish between the two tangles as boxes never appear on links and we need i'_V and e'_v to just be a cap and cup respectively. Therefore, we need a way to encode into our category of tangles that these two tangles

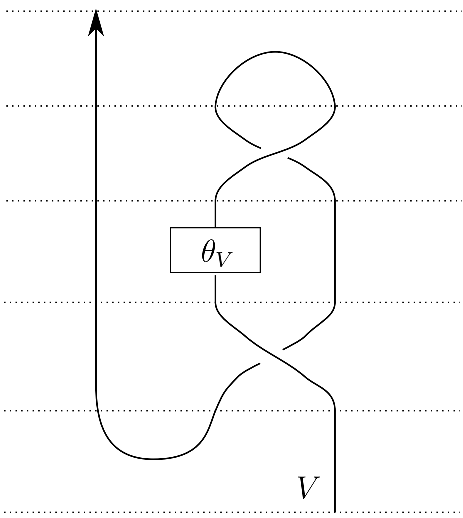


Figure 3.9: Using Reidemeister moves, this is isotopic to the straight line labelled with θ_V, which is not the same as the unlabelled line representing the identity.

are different. This invloves changing the category slightly by thickening the strands of all tangles slightly into ribbons. Figure 3.10 shows that the morphism from Figure 3.8 is now represented by a ribbon with a twist and the identity is a ribbon with no twist. This category is called the category of coloured ribbon tangles and we define it a bit more concretely now.

Definition 3.4.12. Given a category $\mathcal{C}$, we define a category $\mathrm{Rib}_{\mathcal{C}}$ to have

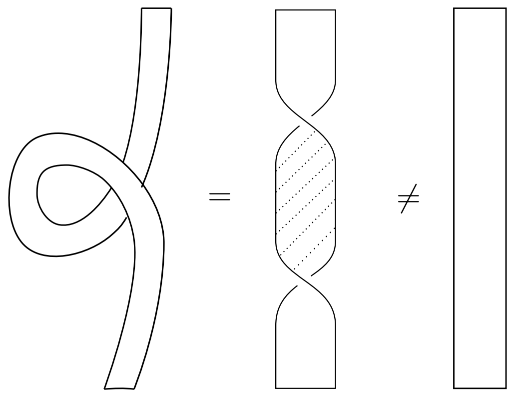


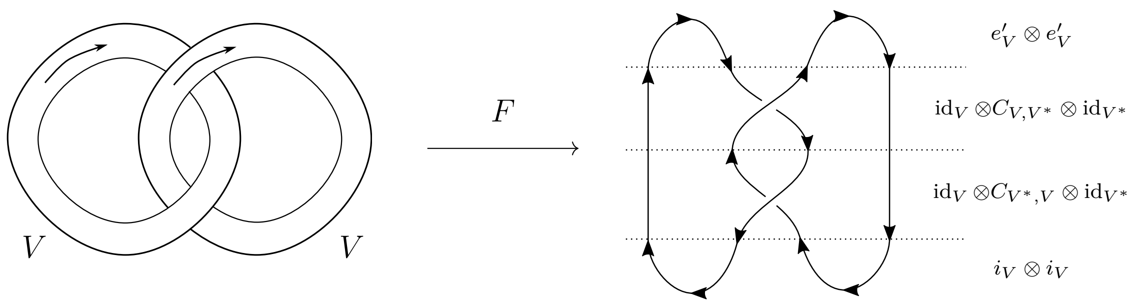
Figure 3.10: Thickening the strands into ribbons allows us to view a strand with a twist as different a strand with no twist.

- Objects are words made up of objects of $\mathcal{C}$, where each object in the word is assigned either an up or down arrow;
- Morphisms are ribbons connecting two words, such that both ends of a ribbon must be attached to the same object of $\mathcal{C}$ with the orientation consistent along the ribbon.

Composition is done by placing tangles on top of each other.

Now, taking $\mathcal{C} = U_q(\mathfrak{sl}_2) - \text{mod}$, we can define a functor $F : \text{Rib}_{\mathcal{C}} \to \mathcal{C}$. This maps objects and morphisms as in Figure 3.11.

Example 3.4.13. The following framed directed, coloured link maps as follows:



This morphism $(e'_V \otimes e'_V) \circ (\text{id}_V \otimes C_{V,V^*} \otimes \text{id}_{V^*}) \circ (\text{id}_V \otimes C_{V^*,V} \otimes \text{id}_{V^*}) \circ (i_V \otimes i_V)$ is a morphism from $\mathbb{C}(q) \to \mathbb{C}(q)$ and so can be thought of as an element of $\mathbb{C}(q)$. It is this element which we assign as the invariant to the link on the left hand side. However, we see there are several problems with our construction so far. The first is that this assignment required a choice: we chose to label both parts of the link by the representation V. If we had labelled them with a different choice, we would have got a different invariant. The second problem is that our invariant is only an invariant of framed, directed links rather than just links.

Unfortunately, we are never going to get rid of the choice involved. However, if we pick a represention V and let $\text{Rib}^V_{\mathcal{C}}$ be the full subcategory of $\text{Rib}_{\mathcal{C}}$ consisting of objects where every term is labelled by V, then we can view any framed, directed link, L as a morphism in this category, just by labelling every strand by V. Then, if F^V is the

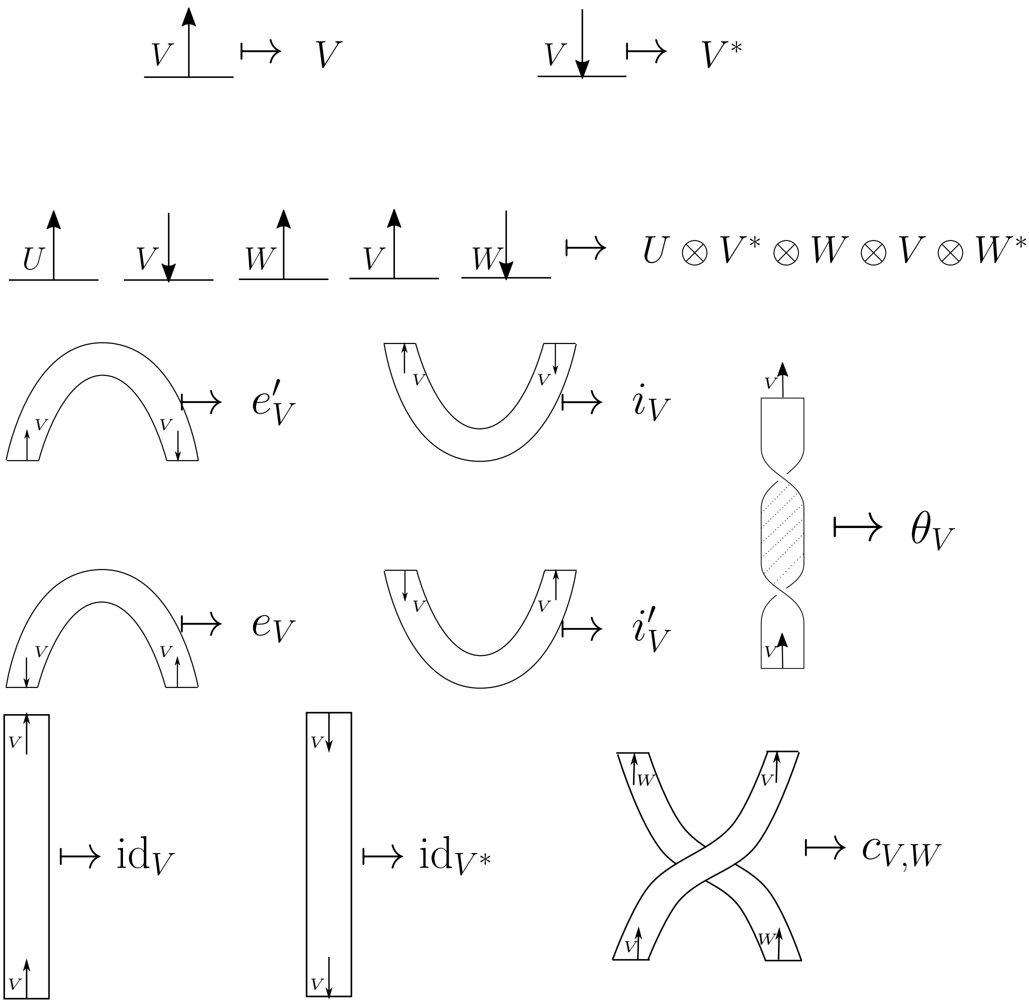


Figure 3.11: Shows where the functor $F : \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$ sends specific objects and morphisms.

restriction of F to this subcategory, $F^V(L)$ is the link invariant assigned to L. Moreover, if we choose $\mathcal{C} = U_q(\mathfrak{sl}_2)$-mod and $V = \mathbb{C}^2$, the irreducible representation of dimension two, then we get the added structure that $V^* \cong V$ and so the direction does not matter. Hence, by making this choice, we get an invariant of framed links.

Finally, we address how we can get a link invariant from this. One way to go about this is to assign a rule that, for each link, tells us how to frame it and then we can use our invariant of framed links. One such rule is called blackboard framing. Informally, this just involves taking the link diagram and thickening the lines as they are without adding any twists. Examples are given in Figure 3.12. However, a problem occurs because the two knots in Figure 3.12 are isotopic and so should be given the same link invariant but the blackboard framing takes them to two different framed links: if we unfold the second it would be a circle with a twist rather than just a circle. To solve this, we use the writhe of the link diagram which keeps track of the crossings in a link diagram.

Definition 3.4.14. In a link diagram L, we define positive and negative crossings as follows:

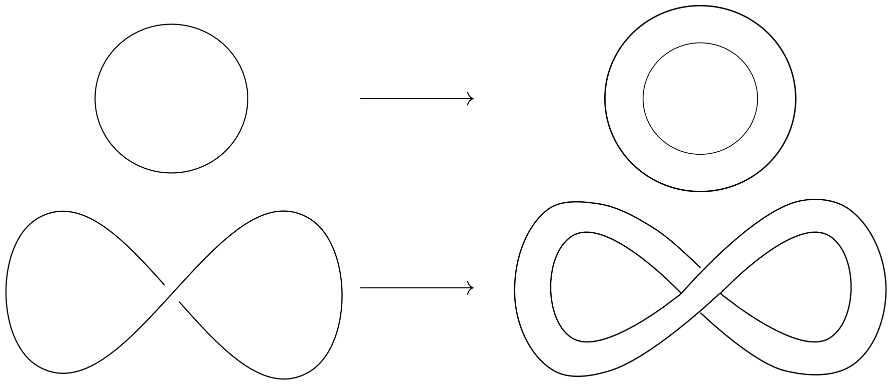
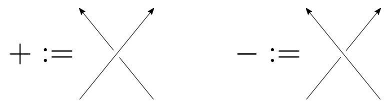


Figure 3.12: Examples of Blackboard Framing



Then the writhe of a link diagram L, denoted $\omega(L)$ is the total number of crossings minus the total number of negative crossings.

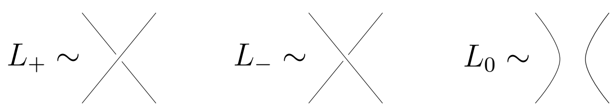
For example, the writhes of the two links in Figure 3.12 are 0 and 1 respectively. As you might expect, since the braiding θ_V keeps track of the twists in the correspondence we developed, we are going to use this to solve our problem with blackboard framing. Since V was chosen to be the irreducible representation of dimension 2, θ_V acts as a scalar and so we can think of it as an element of $\mathbb{C}(q)$. Then, if for a link L, we denote by L^b the blackboard framed link associated to L, the link invariant of L is

$$\theta_V^{\omega(L)} F^V(L^b).$$

Thus, we have constructed a knot invariant from the category $U_q(\mathfrak{sl}_2)$-mod using the large amount of structure it has. As a last remark, we note that the Jones Polynomial can be recovered from this construction. Recall that the Skein relation for the Jones polynomial was

$$t^{-1}P_{L_+}(t) - tP_{L_-}(t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})P_{L_0}(t)$$

where L_+, L_- and L_0 were identical link diagrams except at a single crossing point where



If we label each of the strands with our chosen representation V, then the three crossing points correspond to the maps $c_{V,V}, c_{V,V}^{-1}$ and $\mathrm{id}_{V\otimes V}$. However, we had a matrix for $c_{V,V}$ and the matrix for the identity is just the identity matrix. A quick calculation shows that

$$q^{-2}2c_{V,V} - q^2 c_{V,V}^{-1} = (q - q^{-1})\,\mathrm{id}_{V\otimes V}$$

and so taking $t = q^2$ we have precisely got the Skein relation back.

Chapter 4

Invariants of 3d manifolds, 3d TFTs and Quantum Groups

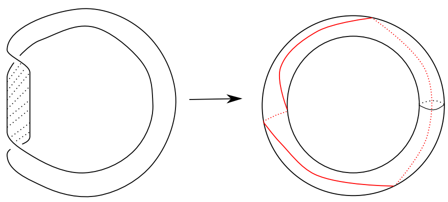
Speaker: Noah White
Notes: Jenny August
Dates: 5-2-16 & 12-2-16

4.1 Introduction

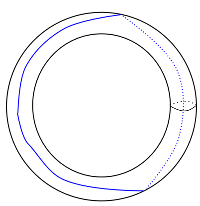
In the previous lectures, we classified 1- and 2-dimensional TFT's and saw how to obtain link invariants from a quantum group. These lectures seemed quite disjoint from one another and so in this lecture, we will see how the two topics are related. The key idea needed to link the two concepts is that 3-manifolds can be viewed as links and so we begin with this.

4.2 3-Manifolds From Links

Suppose K is a framed knot and let $T_1 \supseteq K$ be a small tubular neighbourhood with $\beta \in H_1(\partial T_1, \mathbb{Z})$ the cycle canonically given by the framing of the knot. For example,



where β is the cycle shown in red. Now, given another solid torus T and the cycle $\alpha \in H_1(\partial T, \mathbb{Z})$ given by



there is a canonical (up to isotopy) homeomorphism $f : \partial T_1 \to \partial T$ such that the induced map $f_* : H_1(\partial T_1, \mathbb{Z}) \to H_1(\partial T, \mathbb{Z})$ maps β to α.

We can extend this idea to links componentwise. For a link L in S^3, we denote by T_i some small tubular neighbourhood of the i-th component of L. For each of these T_i we get a corresponding map

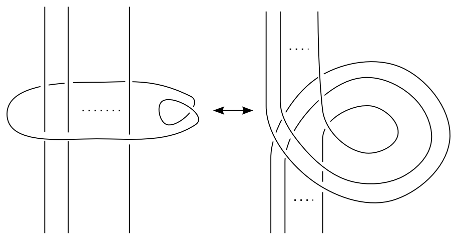
$$f_i : \partial T_i \to \partial T$$

as above. Then we define the *surgery of S^3 along L* to be the 3-manifold

$$M_L := \left(S^3 \setminus \cup_i T_i^{\text{int}}\right) \sqcup_{f_1} T \sqcup_{f_2} T \sqcup_{f_3} \cdots \sqcup_{f_N} T$$

where T_i^{int} denotes the interior of T_i. The key result is the following theorem, known as Kirby Calculus.

Theorem 4.2.1. *Every closed oriented compact 3-manifold can be obtained as M_L for some link L and $M_L \cong M_{L'}$ if and only if you can get from L to L' using the following move:*



The above move is know as the Kirby-Fenn-Rourke move and can contain any number of strands. This gives us a strategy for finding invariants of 3-manifolds; namely find a link invariant which is also invariant under the Kirby-Fenn-Rourke moves.

4.3 3D TFT's

Before, we go any further, we introduce how TFT's are connected. Recall from the first talk that a TFT is a symmetric monoidal functor

$$Z : Cob^m(n)^{\sqcup} \to \mathcal{C}^{\otimes}$$

where the objects of $Cob^m(n)^{\sqcup}$ are $(n-m)$-manifolds, the 1-morphisms are $(n-m+1)$-manifolds which are cobordisms between the objects, and so on until the m-morphisms are n-manifolds which are cobordisms between the $(m-1)$-morphisms.

In this case we are interested in TFT's from $Cob^2(3)$ into the category $\text{AbCat}_{\mathbb{C}}$, consisting of finite abelian categories over $\mathbb{C}$, functors and natural transformations. We choose $\text{AbCat}_{\mathbb{C}}$ because it is a 2-category like $Cob^2(3)$ and it has a monoidal structure given by the Deligne tensor product, denoted by $\boxtimes$, for which the category $\text{Vect}(\mathbb{C})$ is the identity.

So now we ask where a TFT would send closed manifolds. By definition, it must send a closed 1-manifold to a category. A closed 2-manifold can be thought of as an endomorphism of the unit (the empty manifold) of $Cob^2(3)$, similarly to how we viewed links as endomorphisms of the unit in $U_q(\mathfrak{sl}_2)$-mod when finding link invariants. Thus, any closed 2-manifold must be sent to an endofunctor of the identity category i.e. an endofunctor of Vect($\mathbb{C}$). Any such endofunctor can be shown to have the form

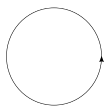
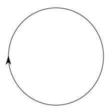
$$- \otimes V : \mathrm{Vect}(\mathbb{C}) \to \mathrm{Vect}(\mathbb{C})$$

for some vector space V and so can be identified with a vector space. Similarly, closed 3-manifolds must be sent to a natural isomorphism of the identity functor on Vect($\mathbb{C}$). The vector space representing this functor is $\mathbb{C}$ and so such a natural isomorphism can be identified with a map $\mathbb{C} \to \mathbb{C}$ which in turn can be represented by a single element of $\mathbb{C}$. This assignment of a number to a closed 3-manifold will be an invariant of 3-manifolds and so we can see that TFT's give us invariants of 3-manifolds.

So now we ask whether we can classify these 3d-TFT's. Recall from the first talk that we completely classified 2 dimensional TFT's. We showed that a TFT

$$Z : Cob^1(2)^{\sqcup} \to \mathrm{Vect}(k)^{\otimes}$$

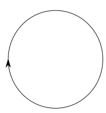
is completely determined by a corresponding finite dimensional commutative Frobenius algebra. To do this, we noted that the objects of $Cob^1(2)$ were disjoint unions of

 and 

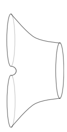
and so Z was completely determined on objects by where it sent one of these. Examining morphisms in $Cob^1(2)^{\sqcup}$ then told us what structure this image had to have. We can play exactly the same game for 3 dimensional TFT's. We want to study TFT's

$$Z : Cob^2(3) \to \mathrm{AbCat}_{\mathbb{C}}$$

where the objects of $Cob^2(3)$ are 1-manifolds, the 1-morphisms are 2-manifolds and the 2-morphisms are 3-manifolds. As before, this TFT is completely determined by the category $\mathcal{C}$ where it sends



and so now we need to look at morphisms involving this object in $Cob^2(3)$. As before, the cobordism



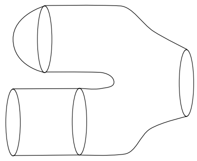
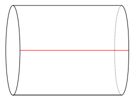
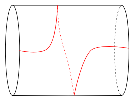


Figure 4.1: The image of this morphism of $Cob^2(3)$ is the functor $m\circ(F\boxtimes\mathrm{id}_{\mathcal{C}}) : \mathrm{Vect}(\mathbb{C})\boxtimes\mathcal{C}\to\mathcal{C}$. Since it is diffeomorphic to a cylinder $m\circ(F\boxtimes\mathrm{id}_{\mathcal{C}})$ must be isomorphic to $\mathrm{id}_{\mathcal{C}}$.

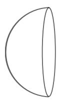
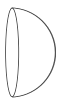
means that we must have a map $\mathcal{C}\boxtimes\mathcal{C}\to\mathcal{C}$ and by the same arguments as for the 2-dimensional case, this map must be associative, commutative and have a unit. However, in this case, each of these only has to hold up to natural isomorphism as $\mathrm{AbCat}_{\mathbb{C}}$ has a further level of morphisms than $\mathrm{Vect}(\mathbb{C})$. This precisely means that $\mathcal{C}$ must be a braided tensor category.
Now, we consider the 2-manifolds

 and 

where the second one is a twist of the first. These are diffeomorphic to each other and so must be sent to two isomorphic functors $F, G : \mathcal{C}\to\mathcal{C}$ which are the identity on objects. Thus, for each $V\in\mathcal{C}$, there are isomorphisms

$$\theta_V : V\to V$$

which are natural in V. This gives a balancing structure on $\mathcal{C}$ which, along with the fact that $\mathcal{C}$ must be rigid, means that $\mathcal{C}$ must be a ribbon category. The proof that $\mathcal{C}$ must be rigid is a bit involved and so we do not present it here but it can be found in [9]. Additionally, applying Zorro's lemma as we did in the first lecture, we see that $\mathcal{C}$ must have only finitely many simple objects. In fact we can go even further and show that $\mathcal{C}$ must be a semisimple category. To see this we need to examine the image of

 and 

under the TFT. The first, which we call the cap, must map to a functor $F : \mathrm{Vect}(\mathbb{C})\to\mathcal{C}$ and the second, which we call the cup, to a functor $G : \mathcal{C}\to\mathrm{Vect}(\mathbb{C})$ as the unit of $\mathrm{AbCat}_{\mathbb{C}}$ was $\mathrm{Vect}(\mathbb{C})$. However, Figure 4.1 shows that the functor $m\circ(F\boxtimes\mathrm{id}_{\mathcal{C}}) : \mathrm{Vect}(\mathbb{C})\boxtimes\mathcal{C}\to\mathcal{C}$ (where m is the product on $\mathcal{C}$) must be isomorphic to $\mathrm{id}_{\mathcal{C}}$ as both are the image of cylinders in $Cob^2(3)$. Therefore, we get isomorphisms $F(\mathbb{C})\otimes X\cong X$, for every $X\in\mathcal{C}$ which are natural in X. Similarly, by flipping Figure 4.1 upside down, we get isomorphisms $X\otimes F(\mathbb{C})\cong X$ and thus $F(\mathbb{C})$ must be the unit of $\mathcal{C}$.
Now, for dualisability reasons, which haven't been discussed yet in the seminar but will be discussed later on, the image of the cup must be right adjoint to the image of the cap. Therefore, we get isomorphisms

$$GX\cong\mathrm{Hom}_{\mathrm{Vect}(\mathbb{C})}(\mathbb{C}, GX)\cong\mathrm{Hom}_{\mathcal{C}}(F(\mathbb{C}), X)\cong\mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, X)$$

which are natural in X. Therefore, by Yoneda's Lemma, we have $G = \mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, -)$.
So now, we want to use this information to see that the category $\mathcal{C}$ must be semisimple. Note that an abelian category is semisimple if and only if all of its objects are projective which precisely means that $\mathrm{Hom}(X, -)$ is exact for all objects. However, since our category is a ribbon category, for any $X \in \mathcal{C}$, we have

$$\mathrm{Hom}_{\mathcal{C}}(X, -) \cong \mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, X^* \otimes -)$$

and, since tensoring in a tensor category is exact, we see that $\mathrm{Hom}_{\mathcal{C}}(X, -)$ will be exact as soon as $\mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, -)$ is exact. However, again for dualisaibilty reasons, the image of the cup must have left and right adjoints and thus $\mathrm{Hom}_{\mathcal{C}}(1_{\mathcal{C}}, -)$ must have left and right adjoints and must therefore be exact. Therefore, $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is exact for any X and so $\mathcal{C}$ is semisimple.

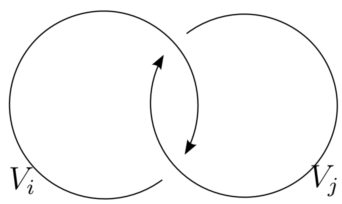
With a bit more work we get that any 3d TFT into $\mathrm{AbCat}_{\mathbb{C}}$ must be a modular tensor category which we define now.

4.4 Modular Tensor Categories

Recall that a ribbon category is a balanced, rigid, braided tensor category.

Definition 4.4.1. A modular tensor category is a ribbon category $\mathcal{C}$ such that:

- $\mathcal{C}$ is semisimple;
- $\mathcal{C}$ has finitely many simple objects $\{V_i\}_{i \in I}$;
- the matrix $\tilde{s} = (\tilde{s}_{ij})$ is invertible where

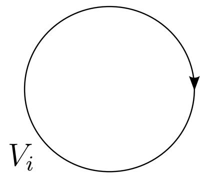
$\tilde{s}_{ij} =$ 

are viewed as elements of $\mathbb{C}$.

The first question we might ask about such a category is why is it called a modular category. The reason for this is that is gives representations of the the modular group $PSL(2, \mathbb{Z})$ i.e. the group $SL(2, \mathbb{Z})$ quotiented out by its centre. To see this we introduce some notation.

Definition 4.4.2. Suppose $\mathcal{C}$ is a modular tensor category. Let

- $\theta_i := \theta_{V_i} \in \mathrm{End}(V_i) = \mathbb{C}$ where θ is the balancing natural isomorphism in $\mathcal{C}$;
- $d_i := \dim(V_i)$ where

$\dim(V_i) =$ 

is viewed as an element of $\mathbb{C}$;

- $p^{\pm} := \sum_{i\in I} \theta_i^{\pm 1} d_i^2$;
- $D := \sqrt{p^+p^-}$;
- $s := \frac{1}{\sqrt{p^+p^-}}\tilde{s}$;
- $t = (t_{ij})$ where $t_{ij} = \delta_{ij}\theta_i$.

Then, with a few calculations, you find

$$(st)^3 = \sqrt{\frac{p^+}{p^-}}s^2 \quad \text{and} \quad s^4 = 1.$$

If we now recall that $SL(2,\mathbb{Z})$ has a common presentation with generators and relations

$$S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, (ST)^3 = S^2, S^4 = 1$$

which are remarkably similar to the relations above. They are only out by a scalar factor and thus, the equations from the modular tensor category actually give a presentation of $PSL(2,\mathbb{Z})$, also known as the modular group. Thus we call these categories modular tensor categories.

However, returning to our key discussion we want to give an invariant of 3-manifolds. It turns out that, with a TFT and hence a modular tensor category we can define the following invariant.

Definition 4.4.3. The Reshetkhin-Turaev invariant of a 3-manifold M_L is

$$\tau(M_L) = D^{-N-1}\left(\frac{p^+}{p^-}\right)^{\frac{\sigma(L)}{2}} RT_V(L)$$

where

- N is the number of components in the link;
- $\sigma(L)$ is the wreath number of L (this is just a topological invariant of the link L);
- V is the sum of all simple modules in the modular tensor category $\mathcal{C}$, each with weight d_i;
- $RT(L)$ is the framed link invariant defined in the last lecture.

So now we know that given a 3d TFT, we get a modular tensor category which in turn gives an invariant of 3-manifolds. We now wish to give an example of a modular tensor category and for this, we return to quantum groups.

4.5 Quantum Groups

In the previous lecture, we defined $U_q(\mathfrak{sl}_2)$ where q was chosen not be be a root of unity. This had the advantage that $U_q(\mathfrak{sl}_2)$-mod had similar structure to $\mathfrak{sl}_2$-mod in the sense that both had a unique irreducible representation of each dimension. However, we are now going to consider the case where q is a root of unity. We define $U_q(\mathfrak{sl}_2)$ in the same way i.e. it is the $\mathbb{C}(q)$ algebra generated by $E, F, K^{\pm}$ with relations

$$KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F \quad \text{and} \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

However, if we first think of q as a formal symbol rather than a root of unity, we want to think of assigning q to a root of unity as a specialisation of the algebra. The first will be a $\mathbb{C}(q)$-algebra but the specialisation will only be a $\mathbb{C}$-algebra. If we are specialising q to be the root of unity ε, then

$$\frac{1}{q - \varepsilon} \in U_q(\mathfrak{sl}_2)$$

when considered as a $\mathbb{C}(q)$-algebra but which would have to be sent to 1/0 in when considered as a $\mathbb{C}$-alegbra which clearly isn't possible. Thus, we need a different way to think about specialising which involves integral forms.

Definition 4.5.1. Let U be a $\mathbb{C}(q)$-algebra. An integral form for U is a $\mathbb{C}[q, q^{-1}]$-subalgebra $U_{\mathbb{Z}} \subset U$ such that

$$U_{\mathbb{Z}} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}(q) \cong U.$$

An integral form $U_q(\mathfrak{sl}_2)_{\mathbb{Z}}$ is useful because, as it is only a $\mathbb{C}[q, q^{-1}]$-algebra, it no longer contains $\frac{1}{q-\varepsilon}$ for $\varepsilon \in \mathbb{C}$ but it still contains much of the infomation of $U_q(\mathfrak{sl}_2)$. We can now specialise $U_q(\mathfrak{sl}_2)$ by actually specialising the integral form of $U_q(\mathfrak{sl}_2)$.

Definition 4.5.2. The specialisation of an integral form $U_{\mathbb{Z}}$ of a $\mathbb{C}(q)$-algebra U is the fibre U_ε over ε i.e.

$$U_\varepsilon = U_{\mathbb{Z}} \otimes_{\mathbb{C}[q,q^{-1}]} \mathbb{C}_\varepsilon$$

where $\mathbb{C}_\varepsilon = \mathbb{C}$ with q acting as ε.

However, the problem with integral forms is that they involve a choice and different integral forms will give different specialisations as the following example shows.

Example 4.5.3. Consider the $\mathbb{C}(q)$-algebra

$$U = \frac{\mathbb{C}(q)\langle x, \partial_q \rangle}{[\partial_q, x] = q}$$

The most obvious integral form, which we call U_1, is generated by x and ∂_q. Then

$$U_1 \otimes \mathbb{C}_{q=0} \cong \mathbb{C}[u, v].$$

However, another integral form, which we call U_2 is generated by x and $q^{-1}\partial_q$ and then

$$U_2 \otimes \mathbb{C}_{q=0} \cong \mathbb{C}\langle u, v \rangle / [q^{-1}\partial_q, x] = 1.$$

Since

$$\begin{aligned}[q^{-1}\partial_q, x] \otimes 1 &= q^{-1}[\partial_q, x] \otimes 1 \\ &= 1 \otimes 1 \\ &\neq 0\end{aligned}$$

we see that $U_1 \otimes \mathbb{C}_{q=0} \ncong U_2 \otimes \mathbb{C}_{q=0}$ and so the choice of integral form does really matter.

Chapter 5

Higher Categories, Complete Segal Spaces and the Cobordism Hypothesis

Speaker: Matt Booth
Notes: Matt Booth
Dates: 16-2-2016 & 26-2-2016

These lectures will follow Lurie's treatment in [10].

A note on notation I try to denote usual categories with italic letters, and higher categories (2-categories, (∞, n)-categories, etc.) with calligraphic letters.

5.1 Extending $\mathbf{Cob}(n)$

5.1.1 Extending down

We've seen that the cobordism categories $\mathbf{Cob}(n)$ should really have more structure than just categories. In particular we should have, for every integer $k \leq n$, a k-category[1] $\mathbf{Cob}_k(n)$ with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{closed oriented } (n-k)\text{-manifolds} \\ \text{1-morphisms} &\longleftrightarrow \text{oriented cobordisms} \\ \text{2-morphisms} &\longleftrightarrow \text{cobordisms between cobordisms} \\ &\vdots \\ k\text{-morphisms} &\longleftrightarrow \text{(diffeomorphism classes of) } n\text{-manifolds with corners} \end{aligned}$$

We'd like to have a nice definition of k-category that includes $\mathbf{Cob}_k(n)$. Here's an obvious definition to make:

Definition 5.1.1. A **strict 1-category** is a category. A **strict k-category** is defined inductively as a category enriched over strict $(k-1)$-categories.

[1] In this section we'll treat higher categories at an informal level. Note that the concept of a "k-category" has not been defined!

This definition is not the correct one. In particular $\mathbf{Cob}_k(n)$ is not a strict k-category since composition is not strictly associative, only associative up to isomorphism. We could adjust the definition of $\mathbf{Cob}_k(n)$ so that composition does become strictly associative, but this quickly gets messy.

Moral 5.1.2. We're going to need a better notion of k-category, where composition need only be associative up to isomorphism.

5.1.2 Extending up

Let's suppose we have a good definition of what a k-category is. Then we can define a (k, n)-category to be a k-category where all of the i-morphisms are invertible for $n < i \leq k$.

Example 5.1.3. A $(1, 0)$-category should just be a groupoid.

It's also often useful to allow $k = \infty$; in fact we're going to define (∞, n)-categories later. This gives us a definition of (k, n)-categories simply by ignoring the morphisms above level k.

Example 5.1.4. An $(\infty, 0)$-category is an **∞-groupoid**. Given a topological space X, we should be able to form an ∞-groupoid $\pi_{\leq\infty}(X)$ called the **fundamental ∞-groupoid** of X, with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{points of } X \\ \text{1-morphisms} &\longleftrightarrow \text{paths between points} \\ \text{2-morphisms} &\longleftrightarrow \text{homotopies between paths} \\ \text{3-morphisms} &\longleftrightarrow \text{homotopies between homotopies} \\ &\vdots \end{aligned}$$

The fundamental groupoid of X remembers all of X up to weak homotopy equivalence. More formally, the fundamental groupoid construction is an equivalence between topological spaces (up to weak homotopy equivalence) and ∞-groupoids (up to equivalence). This assertion is known as the **homotopy hypothesis**[2]. This allows us to think of $(\infty, 0)$-categories as really being topological spaces. So as well as generalising category theory, higher category theory should also generalise topology.

Recall that in defining $\mathbf{Cob}(n)$, we defined a morphism $M \to N$ to be a diffeomorphism class of (oriented) cobordisms $M \to N$. Instead of considering two diffeomorphic cobordisms to be the same map, we could say that they differ by an invertible 2-morphism.

[2]This is not a theorem yet, since we don't have a definition of ∞-groupoid. We could either **define** an ∞-groupoid to be a topological space, or we could regard the homotopy hypothesis as being a **condition** that our models for higher categories need to satisfy.

Hence we should have an $(\infty, 1)$-category $\mathbf{Cob}^t(n)$ with

$$\begin{aligned}
\text{objects} &\longleftrightarrow \text{closed oriented } (n-1)\text{-manifolds} \\
\text{1-morphisms} &\longleftrightarrow \text{oriented cobordisms} \\
\text{2-morphisms} &\longleftrightarrow \text{diffeomorphisms between cobordisms} \\
\text{3-morphisms} &\longleftrightarrow \text{isotopies between diffeomorphisms} \\
&\vdots
\end{aligned}$$

Note that this definition allows us to keep track of the diffeomorphism groups of our cobordisms.

We can combine our two higher-categorical versions of $\mathbf{Cob}(n)$ into a single (∞, n)-category $\mathbf{Bord}_n$ with

$$\begin{aligned}
\text{objects} &\longleftrightarrow \text{0-manifolds} \\
\text{1-morphisms} &\longleftrightarrow \text{cobordisms between 0-manifolds} \\
\text{2-morphisms} &\longleftrightarrow \text{cobordisms between cobordisms} \\
&\vdots \\
n\text{-morphisms} &\longleftrightarrow n\text{-manifolds with corners} \\
(n+1)\text{-morphisms} &\longleftrightarrow \text{diffeomorphisms} \\
(n+2)\text{-morphisms} &\longleftrightarrow \text{isotopies between diffeomorphisms} \\
&\vdots
\end{aligned}$$

Moral 5.1.5. We're going to need a good definition of (∞, n)-categories. Note that the disjoint union operation on 0-manifolds should turn $\mathbf{Bord}_n$ into a symmetric monoidal (∞, n)-category.

5.1.3 Intuitive statement of the cobordism hypothesis

The cobordism hypothesis is stated in terms of **framed** cobordisms. This is a technical point and won't really concern us. Denote the (∞, n)-category of framed cobordisms by $\mathbf{Bord}_n^{\mathrm{fr}}$.

If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category then consider the category of $\mathcal{C}$-valued fully extended framed TFTs: we can identify this category with the category $\mathbf{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ of symmetric monoidal functors from $\mathbf{Bord}_n^{\mathrm{fr}}$ to $\mathcal{C}$.

The cobordism hypothesis more or less says that the evaluation functor $Z \mapsto Z(*)$ determines a bijection between isomorphism classes of $\mathcal{C}$-valued fully extended framed TFTs and isomorphism classes of objects in $\mathcal{C}$ satisfying suitable finiteness conditions.[3]

Remark 5.1.6. This specialises to a statement about $\mathbf{Cob}(n)$ by taking homotopy n-categories.

[3]By suitable finiteness conditions' we mean **full dualisability**, which we'll see a definition of in § 5.5. In $\mathbf{Vect}_k$ the fully dualisable objects are exactly the finite-dimensional vector spaces.

5.2 $(\infty, 1)$-categories as complete Segal spaces

We'll first define $(\infty, 1)$-categories and then soup up our definition in § 5.3 to get to (∞, n)-categories.

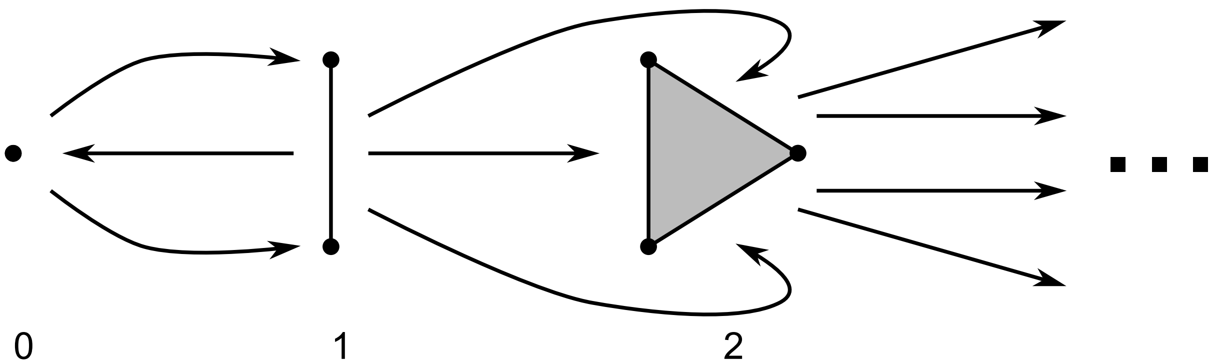
Intuitively, an $(\infty, 1)$-category should be a **topological category**; one where the hom-sets have the structure of topological spaces and all maps in sight are continuous.[4] Higher morphisms are homotopies, homotopies between homotopies, and so on. However, while intuitive, this definition is very difficult to work with.

There are several other models[5] but we're going to use **complete Segal spaces** as our models for $(\infty, 1)$-categories since they generalise easily to (∞, n)-categories.

5.2.1 Preliminary: simplicial sets

Definition 5.2.1. The **simplex category** Δ has objects $[n] = \{0, 1, \ldots, n\}$ for every natural number n and morphisms the weakly order-preserving maps.

It looks like



where we've omitted the maps from [2] to [1]. The maps going to the right are the **face maps** and the maps going to the left are the **degeneracy maps**.

Definition 5.2.2. A **simplicial object** in a category C is a functor $\Delta^{\mathrm{op}} \to C$. More concretely a simplicial object is a collection of objects X_n indexed by the nonnegative integers together with various face and degeneracy maps.

Definition 5.2.3. A **morphism** between two simplicial objects $F : \Delta^{\mathrm{op}} \to C$ and $G : \Delta^{\mathrm{op}} \to C$ is a natural transformation $F \to G$. Concretely, a morphism of simplicial objects is a collection of maps $X_n \to Y_n$ commuting with the face and degeneracy maps.

Proposition 5.2.4. *The collection of simplicial objects in a category C and their morphisms itself forms a category, which we denote sC.*

A simplicial object $X_\bullet$ looks like

$$X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \;\substack{\longrightarrow \\ \longrightarrow}\; \cdots$$

[4] One way to think about this is that an $(\infty, 1)$-category should be enriched in $(\infty, 0)$-categories, which are the same thing as topological spaces.

[5] A good account of these is given in [3].

We'll be interested in **simplicial sets**; simplicial objects in **Set**.[6] Later we'll be interested in simplicial topological spaces.

Example 5.2.5. Given a topological space X, we can define (functorially) a simplicial set $\mathrm{Sing}(X)$ that at level n is the set $\mathrm{Hom}(\Delta^n, X)$ of maps from the n-simplex $\Delta^n \subseteq \mathbb{R}^{n+1}$ to X. We also have a geometric realisation functor $|\cdot| : s\mathbf{Set} \to \mathbf{Top}$ and in fact $|\mathrm{Sing}(X)|$ is weakly homotopy equivalent to X. Simplicial sets are good combinatorial models of topological spaces.[7]

Example 5.2.6. Given a category C, the **nerve** is a simplicial set $N(C)$ which at level n consists of the strings $C_0 \xrightarrow{f_1} C_2 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n$ of n composable morphisms. It's easy to recover C up to isomorphism (not just equivalence) from its nerve $N(C)$.

We might wonder which simplicial sets are the nerves of categories.

Proposition 5.2.7 (the Nerve Theorem)**.** *A simplicial set X is isomorphic to the nerve of a category if and only if for all $m, n \geq 0$ the diagram*

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

induced by the maps

$$0 < 1 < \cdots < m \longleftarrow 0 < 1 < \cdots < m$$

$$\begin{array}{ccccc} m < m+1 < \cdots < m+n & \qquad [m+n] & \longleftarrow & [m] & \qquad m \\ \uparrow & \uparrow & & \uparrow & \uparrow \\ 0 < 1 < \cdots < n & \qquad [n] & \longleftarrow & [0] & \qquad 0 \end{array}$$

$$0 \longleftarrow 0$$

is Cartesian (i.e. a pullback diagram).

Whenever this diagram appears, we will fix the convention that the maps featuring are the maps described above.

5.2.2 Homotopy theory

Our philosophy is that $(\infty, 1)$-category theory should be category theory, but done in a homotopy-theoretic manner. This is because an $(\infty, 1)$-category is just a topological category where the higher morphisms are given by homotopies.

[6] A simplicial set is the same thing as a set-valued presheaf on Δ.

[7] Technically $s\mathbf{Set}$ and **Top** are Quillen equivalent (via these two functors), so they have the same homotopy theory.

Since Proposition 5.2.7 tells us that we can recover a category from its nerve, we'll try to code up the concept of a nerve in homotopy theory. We'll see that a Segal space is precisely this concept of 'homotopy nerve'. However we'll see that a Segal space alone won't quite be enough to recover an $(\infty, 1)$-category: we'll need some more conditions.

Definition 5.2.8. Let $X \xrightarrow{f} Z \xleftarrow{g} Y$ be a diagram of topological spaces[8] and continuous maps. The **homotopy fibre product** $X \times_Z^h Y$ of X and Y along f and g is the space $X \times_Z Z^{[0,1]} \times_Z Y$ whose points are triples (x, y, p) with $x \in X$, $y \in Y$ and $p : [0, 1] \to Z$ a path in Z from $f(x)$ to $g(y)$.

Remark 5.2.9. There is a canonical map from $X \times_Z Y$ to $X \times_Z^h Y$ given by $(x, y) \mapsto (x, y, p)$ where p is the constant path from $f(x) = g(y)$ to itself.

Example 5.2.10. Let X be a space and $p : * \to X$ be the inclusion of a basepoint. Then the homotopy fibre product of $* \xrightarrow{p} X \xleftarrow{p} *$ is the space ΩX of loops in X based at p. The usual fibre product is the one point space $*$.

Example 5.2.11. More generally, the homotopy fibre product of $* \xrightarrow{p} Y \xleftarrow{f} X$ is the homotopy fibre of f over the basepoint p in Y.

The usual fibre product of topological spaces does not respect homotopy equivalences. The homotopy fibre product is invariant under homotopy equivalence: if we replace f and g by homotopic maps then the weak homotopy type of $X \times_Z^h Y$ does not change.

Remark 5.2.12. Another nice property of the homotopy fibre product is that we have a long exact sequence of homotopy groups

$$\cdots \to \pi_n(X \times_Z^h Y) \to \pi_n(X) \times \pi_n(Y) \to \pi_n(Z) \to \cdots \to \pi_0(X) \times \pi_0(Y)$$

Proposition 5.2.13. *The homotopy fibre product $X \times_Z^h Y$ comes with two canonical projection maps to X and Y making the diagram*

$$\begin{array}{ccc} X \times_Z^h Y & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

commute up to canonical homotopy. Moreover if the square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

*is homotopy commutative then there is a unique up to homotopy map $W \to X \times_Z^h Y$ making the two triangles obtained strictly commutative. For this reason we often call $X \times_Z^h Y$ the **homotopy pullback** of X and Y along f and g.*

[8]For technical reasons we need to work with a 'convenient category' of spaces. For example we can use CGH (compactly generated Hausdorff) spaces as in [18] or CGWH (compactly generated weak Hausdorff) spaces as in [19]. For concreteness we may suppose that all topological spaces are CGWH.

Definition 5.2.14. A homotopy commutative square

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

is **homotopy Cartesian** (or just **h-Cartesian**) if there is a weak homotopy equivalence $W \to X \times_Z^h Y$ such that the triangles obtained are strictly commutative.

Neither

$$\text{h-Cartesian} \implies \text{Cartesian}$$

nor

$$\text{Cartesian} \implies \text{h-Cartesian}$$

is true in general! If our maps are sufficiently nice (e.g. if $X \to Z$ is a fibration) then a Cartesian square is homotopy Cartesian. In this situation we can compute the homotopy fibre product by computing the usual fibre product.

5.2.3 Segal spaces

Definition 5.2.15 (Rezk)**.** A simplicial topological space $X_\bullet$ is a **Segal space** if for all $m, n \geq 0$ the diagram

$$\begin{array}{ccc} X_{m+n} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_0 \end{array}$$

is h-Cartesian. We can equivalently specify that for all n the **Segal maps**

$$X_n \to \underbrace{X_1 \times_{X_0}^h X_1 \times_{X_0}^h \cdots \times_{X_0}^h X_1}_{n}$$

are weak homotopy equivalences.

Remark 5.2.16. This is not a universally accepted definition. Some authors, for example [4], specify in addition that $X_\bullet$ should be **Reedy fibrant**, a 'niceness' condition on simplicial spaces that ensures that the homotopy pullback $X_m \times_{X_0}^h X_n$ is the usual pullback $X_m \times_{X_0} X_n$. In this case it's enough to demand that $X_{m+n} \to X_m \times_{X_0} X_n$ is a weak homotopy equivalence. Reedy fibrancy is a technical condition that can always be satisfied. For more on model category theory and the definition of Reedy fibrancy, the reader can consult e.g. Appendix A.2 of [11].

What do Segal spaces have to do with $(\infty, 1)$-categories? Let's suppose for the moment that we already have a good theory of $(\infty, 1)$-categories. Just like a 1-category has an underlying groupoid, obtained by throwing away all of the noninvertible morphisms, an $(\infty, 1)$-category should have an underlying ∞-groupoid obtained in the same way:

Idea 5.2.17. Let $\mathcal{C}$ be any $(\infty,1)$-category. We can loosely define the **underlying ∞-groupoid** of $\mathcal{C}$, which I will denote by $\pi_{\leq\infty}(\mathcal{C})$, to be the ∞-groupoid with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{objects of } \mathcal{C} \\ \text{1-morphisms} &\longleftrightarrow \text{invertible 1-morphisms in } \mathcal{C} \\ \text{2-morphisms} &\longleftrightarrow \text{2-morphisms between invertible 1-morphisms of } \mathcal{C} \\ &\vdots \end{aligned}$$

Since we can identify ∞-groupoids with topological spaces, we may think of $\pi_{\leq\infty}(\mathcal{C})$ as a topological space $B_0\mathcal{C}$ which we refer to as the **classifying space for objects of $\mathcal{C}$**. Note that by definition the fundamental ∞-groupoid of $B_0\mathcal{C}$ is the underlying ∞-groupoid of $\mathcal{C}$.

Clearly $B_0\mathcal{C}$ should not in general encode all of the information about $\mathcal{C}$. For example it doesn't know about noninvertible morphisms or how composition works. We can extend the above definition to get classifying spaces for n-morphisms of $\mathcal{C}$ (since we can think of an object as a 0-morphism), and hopefully this collection of classifying spaces should allow us to recover $\mathcal{C}$.

Idea 5.2.18. Let $[n]$ be the 1-category associated to the ordered set $\{0, 1, 2\ldots, n\}$. Let $\mathcal{C}$ be an $(\infty,1)$-category. We can think of an n-morphism in $\mathcal{C}$ as a functor $[n] \to \mathcal{C}$. The collection $\mathrm{Fun}([n],\mathcal{C})$ of functors $[n] \to \mathcal{C}$ itself naturally has the structure of an $(\infty,1)$-category, so it has an underlying ∞-groupoid $\pi_{\leq\infty}(\mathrm{Fun}([n],\mathcal{C}))$. Let $B_n\mathcal{C}$ be the topological space associated to this ∞-groupoid. We call $B_n\mathcal{C}$ the **classifying space for n-morphisms in $\mathcal{C}$**. Again, by definition the fundamental ∞-groupoid of $B_n\mathcal{C}$ is the underlying ∞-groupoid of $\mathrm{Fun}([n],\mathcal{C})$.

What kind of object should the collection $B_\bullet\mathcal{C}$ be? Moreover, to what extent does it determine $\mathcal{C}$? The answer to the first question is that $B_\bullet\mathcal{C}$ should be a simplicial space, and moreover a Segal space. The Segal conditions formalise the idea that giving a chain

$$A_0 \to A_1 \to \cdots \to A_{n+m}$$

of composable morphisms should be equivalent to giving two chains

$$A_0 \to \cdots \to A_n \qquad\qquad A_n \to \cdots \to A_{n+m}$$

and moreover that it should not matter where we break the chains. To answer the second question, we can try to define an 'inverse' to the operation $\mathcal{C} \to B_\bullet\mathcal{C}$ and see if we need to add any extra data to a general Segal space in order to extract an ∞-category.

Idea 5.2.19. Given a Segal space $X_\bullet$ we should be able to construct an $(\infty,1)$-category $\mathcal{C}(X_\bullet)$ which has

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{points of } X_0 \\ \text{Mapping spaces } \mathrm{Map}(x,y) &\longleftrightarrow \{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\} \\ \text{composition law} &\longleftrightarrow \text{given by } X_2 \\ \text{higher associativity information} &\longleftrightarrow \text{given by } X_3, X_4 \ldots \end{aligned}$$

Observe that the connected components of the space $\mathrm{Map}(x, y)$ should be precisely the homotopy classes of 1-morphisms in $\mathcal{C}(X_\bullet)$. With this in mind, we can construct a 1-category from a Segal space:

Definition 5.2.20. The **homotopy category** $hX_\bullet$ of a Segal space $X_\bullet$ is the category whose objects are the points of X_0, and whose homsets are

$$\begin{aligned}\mathrm{Hom}_{hX_\bullet}(x, y) &:= \pi_0(\mathrm{Map}(x, y)) \\ &= \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\})\end{aligned}$$

Remark 5.2.21. The homotopy category of $X_\bullet$ records some of the basic information about $\mathcal{C}(X_\bullet)$ - it knows what the objects should be, for example - but it forgets all of the homotopical information by identifying all homotopic maps. It can be thought of as a 1-categorical 'flattening' of the $(\infty, 1)$-category $\mathcal{C}(X_\bullet)$.

5.2.4 Completeness

If we start with a general Segal space $X_\bullet$, how does it compare to the Segal space $Y_\bullet := B_\bullet(\mathcal{C}(X_\bullet))$? The fundamental groupoid of Y_0 is the classifying space for 0-morphisms of $\mathcal{C}(X_\bullet)$. This receives a map from the fundamental groupoid of X_0 but this map is not necessarily an equivalence, since there may be invertible 1-morphisms in $\mathcal{C}(X_\bullet)$ which do not come from paths in X_0. We'd like to impose an extra condition on our Segal spaces which ensures that every invertible 1-morphism in $\mathcal{C}(X_\bullet)$ comes from an essentially unique path in X_0.

Definition 5.2.22. Let p_i be the map from $[0]$ to $[1]$ given by $0 \mapsto i$. For any Segal space $X_\bullet$ write $p_i^* : X_1 \to X_0$ for the map induced by p_i. If $f \in X_1$ then write $x := p_0^*(f)$ and $y := p_1^*(f)$ so that we can think of f as a path from x to y. The map $\{f\} \to \{x\} \times_{X_0} X_1 \times_{x_0} \{y\} \to \{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\}$ determines an element $[f]$ of $\mathrm{Hom}_{hX_\bullet}(x, y) = \pi_0(\{x\} \times^h_{X_0} X_1 \times^h_{x_0} \{y\})$. Say that f is **invertible** if $[f]$ is an isomorphism.

Example 5.2.23. If $X_\bullet$ is a Segal space let $\delta : X_0 \to X_1$ be the map induced by the unique map $[1] \to [0]$. Then for every $x \in X_0$, the map $[\delta(x)]$ is the identity map id_x in the homotopy category. So $\delta(x)$ is invertible for every x.

Definition 5.2.24. If $Z \subseteq X_1$ is the subspace of invertible elements of a Segal space $X_\bullet$, then say that $X_\bullet$ is **complete** if $\delta : X_0 \to Z$ is a weak homotopy equivalence.

So a complete Segal space is one where every isomorphism in $\mathcal{C}(X_\bullet)$arises from an essentially unique path in X_0. In fact, if $X_\bullet$ is complete then we should have an equivalence $X_\bullet \cong B_\bullet(\mathcal{C}(X_\bullet))$.

Remark 5.2.25. If one is more careful and starts with a rigorous axiomatisation of $(\infty, 1)$-categories then the above assertions and intuitive ideas can be turned into theorems. This was done by Toën in [21].

We've seen that the well-defined theory of complete Segal spaces should correspond to the as-yet-undefined theory of $(\infty, 1)$-categories. With this in mind, we make the following rather bold definition:

Definition 5.2.26. An $(\infty, 1)$**-category** is a complete Segal space.

Proposition 5.2.27 (Rezk)**.** *Any Segal space $X_\bullet$ has a completion; i.e. admits a homotopy universal morphism*[9] $X_\bullet \to Y_\bullet$ *where $Y_\bullet$ is complete. In general $Y_\bullet$ is unique up to homotopy and we refer to it as the* ***completion*** *of $X_\bullet$, denoted $\hat{X}_\bullet$. The map $X_\bullet \to \hat{X}_\bullet$ is functorial.*

Remark 5.2.28. Complete Segal spaces are the fibrant objects of a suitable model structure on the category of simplicial spaces, just as quasicategories are the fibrant objects of the Joyal model structure on the category of simplicial sets.

5.3 (∞, n)-categories as n-fold complete Segal spaces

Now we have a definition of $(\infty, 1)$-category as a certain functor $\Delta^{\mathrm{op}} \to \mathbf{Top}$, we're going to generalise this and define an (∞, n)-category as a certain functor $(\Delta^{\mathrm{op}})^{\times n} \to \mathbf{Top}$.

Definition 5.3.1. An ***n*-fold simplicial object** in a category C is a functor

$$\underbrace{\Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}}}_{n} \to C$$

Example 5.3.2. A 0-fold simplicial object is an object. A 1-fold simplicial object is just a simplicial object in the usual sense.

In general an n-fold simplicial object in a category C is a collection $X_{i_1 \cdots i_n}$ of objects of C indexed by n-tuples of nonnegative integers $\underline{i} = (i_1, \ldots, i_n)$ along with a collection of face and degeneracy maps. We'll always use an underbar to denote multiindices in this manner. We think of n-fold simplicial objects as having n 'directions' in which to compose.

Definition 5.3.3. An ***n*-fold simplicial space** is an n-fold simplicial object in the category **Top**.

Definition 5.3.4. A map $X \to Y$ of n-fold simplicial spaces is a **weak homotopy equivalence** if all of the maps $X_{\underline{i}} \to Y_{\underline{i}}$ are weak homotopy equivalences.

Definition 5.3.5. A diagram

$$\begin{array}{ccc} W & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X & \longrightarrow & Z \end{array}$$

of n-fold simplicial spaces is **homotopy Cartesian** if for all multiindices $\underline{i}$ the square

$$\begin{array}{ccc} W_{\underline{i}} & \longrightarrow & Y_{\underline{i}} \\ \downarrow & & \downarrow \\ X_{\underline{i}} & \longrightarrow & Z_{\underline{i}} \end{array}$$

is homotopy Cartesian.

Definition 5.3.6. An n-fold simplicial space X is **essentially constant** if it's weakly homotopy equivalent to a constant n-fold simplicial space.

[9] A morphism of Segal spaces is a morphism of the underlying simplicial spaces.

Via currying, whenever $n > 0$ we can always think of an n-fold simplicial object in C as a simplicial object in the category of $(n-1)$-fold simplicial objects in C. This idea will form the basis of our inductive definition of an n-fold complete Segal space.

Definition 5.3.7. For $n > 0$ an n-fold simplicial space X, thought of as a simplicial object in the category of $(n-1)$-fold simplicial spaces, is said to be an **n-fold Segal space** if the following conditions are met:

i) Every X_k is an $(n-1)$-fold Segal space.

ii) For all m and l the diagram

$$\begin{array}{ccc} X_{m+l} & \longrightarrow & X_m \\ \downarrow & & \downarrow \\ X_l & \longrightarrow & X_0 \end{array}$$

is a homotopy Cartesian square of $(n-1)$-fold simplicial spaces.

iii) X_0 is an essentially constant $(n-1)$-fold simplicial space.

Moreover, we say that an n-fold Segal space is **complete** if

iv) Each X_k is a complete $(n-1)$-fold Segal space.

v) The Segal space $Y_\bullet = X_{\bullet,0,0,\ldots,0}$ is complete.

Definition 5.3.8. An (∞, n)**-category** is a complete n-fold Segal space.

Proposition 5.3.9. *Any n-fold Segal space has a completion.*

Loosely, an n-fold complete Segal space is a 'fattened' or 'spread out' version of an (∞, n)-category. Some illuminating diagrams are given in §2.2.1 of [4].

5.4 The (∞, n)-category $\mathbf{Bord}_n$

In this section we'll code up our ideas about $\mathbf{Bord}_n$ to define an n-fold simplicial space $\mathbf{PBord}_n$. We'll indicate how this is an n-fold Segal space that in general is not complete. Then we can define the (∞, n)-category $\mathbf{Bord}_n$ to be the completion $\widehat{\mathbf{PBord}}_n$ of $\mathbf{PBord}_n$. Our exposition will be fairly informal; for a more rigorous explanation see §2 of [4].

5.4.1 The level sets $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$

We want to think of $(\mathbf{PBord}_n)_{(k_1,\ldots,k_n)}$ as a collection of $k_1 k_2 \cdots k_n$ composed cobordisms, with k_i cobordisms in the i^{th} direction.

Idea 5.4.1. Cobordisms are easier to deal with when we consider them as submanifolds of some large $\mathbb{R}^m$. So we'll define sets of cobordisms living in $\mathbb{R}^m$ for varying m, and then take a limit over m. Whitney's embedding theorem will ensure that we get all of the cobordisms, since every l-dimensional manifold can be embedded in $\mathbb{R}^{2l}$.

Definition 5.4.2. Let V be a finite-dimensional real vector space and fix a multiindex $\underline{k} = (k_1, \ldots, k_n)$. Define $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ to be the set of tuples

$$(M, (t_0^i, \ldots, t_{k_i}^i)_{i=1\ldots n})$$

satisfying the following:

i) For each $1 \leq i \leq n$, $t_0^i \leq \cdots \leq t_{k_i}^i$ is an ordered tuple of $k_i + 1$ real numbers.

ii) M is a closed n-dimensional submanifold of $V \times \mathbb{R}^n$ and the composition $\pi : M \hookrightarrow V \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$ is proper[10].

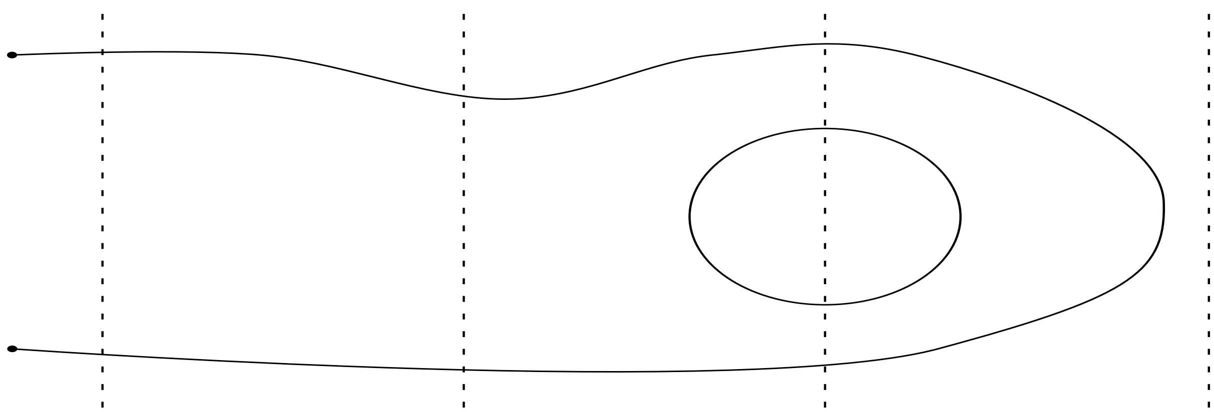
iii) For a subset S of $\{1, \ldots, n\}$ let $p_S : M \to \mathbb{R}^S$ denote the composition $M \xrightarrow{\pi} \mathbb{R}^n \twoheadrightarrow \mathbb{R}^S$. Then we require that for every $1 \leq i \leq n$ and every $0 \leq j \leq k_i$, that for all $x \in p_{\{i\}}^{-1}\left(t_j^i\right)$, the map $p_{\{i,\ldots,n\}}$ is submersive[11] at x.

Remark 5.4.3. What's the motivation behind this definition? If we want to think of M as being a collection of composed cobordisms, the numbers t_j^i record the 'cutting points' where we glue two cobordisms together. So the region of M between the hyperplanes corresponding to t_j^i and t_{j+1}^i should be the $(j+1)^{\text{st}}$ cobordism glued in the i^{th} direction.

Condition iii) says that in particular the set $p_{\{n\}}^{-1}\left(t_j^n\right)$ is an $(n-1)$-dimensional submanifold that we can think of as one of the $(n-1)$-cobordisms that we glue together to get M.

Furthermore the set $p_{\{n-1,n\}}^{-1}\left\{t_{j_{n-1}}^{n-1}, t_{j_n}^n\right\}$ is an $(n-2)$-dimensional manifold that is one of the $(n-2)$-cobordisms joined by an $(n-1)$-cobordism. Similarly, the preimage $p_{\{m,\ldots,n\}}^{-1}\left\{t_{j_m}^m, \ldots, t_{j_n}^n\right\}$ is an $(m-1)$-dimensional manifold that we can loosely think of as one of our $(m-1)$-morphisms.

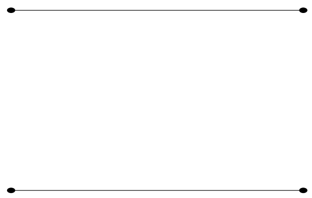
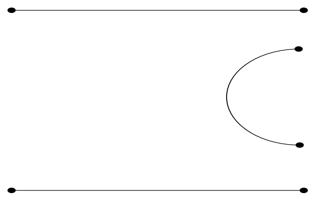
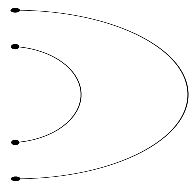
Example 5.4.4. Here is an element of $\mathbf{PBord}_1^{\mathbb{R}}$:



The cutting points indicated by the dotted lines allow us to view this as a composition of the three 1-cobordisms

[10]A map is **proper** if preimages of compact sets are compact.

[11]A map $f : M \to N$ is **submersive at** $m \in M$ if the differential $\mathrm{d}f_x : T_xM \to T_xN$ is surjective. A map is **submersive** if it's submersive at every point of its domain.


,

and


5.4.2 The topological spaces $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$

Fact 5.4.5 ([7], Chapter II). The set $\mathrm{Emb}(X, Y)$ of smooth embeddings of a smooth manifold X into a smooth manifold Y has a topology, the **Whitney C^∞ topology**.

Theorem 5.4.6 ([6])**.** *The space* $\mathrm{Sub}(V \times \mathbb{R}^n)$ *of closed n-dimensional submanifolds of $V \times \mathbb{R}^n$ can be identified with the space*

$$\bigsqcup_{[L]} \mathrm{Emb}(L, V \times \mathbb{R}^n)/\mathrm{Diff}(L)$$

where the disjoint union is taken over diffeomorphism classes of n-dimensional manifolds L. Moreover the topology on $\mathrm{Sub}(V \times \mathbb{R}^n)$ *has neighbourhood basis at $M \subseteq V \times \mathbb{R}^n$ the sets*

$$\{N \subseteq V \times \mathbb{R}^n : N \cap K = f(M) \cap K \text{ for all } f \in W\}$$

where K is a compact subset of $V \times \mathbb{R}^n$ and $W \subseteq$ $\mathrm{Emb}(M, V \times \mathbb{R}^n)$ *is a neighbourhood (in the Whitney C^∞ topology) of the inclusion $M \hookrightarrow V \times \mathbb{R}^n$.*

Remark 5.4.7. The space $\mathrm{Sub}(V \times \mathbb{R}^n)$ is sometimes denoted by $\Psi(V \times \mathbb{R}^n)$.

Since we can view $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ as a subset of $\mathrm{Sub}(V \times \mathbb{R}^n) \times \mathbb{R}^k$, for some $k \in \mathbb{N}$ depending only on $\underline{k}$, we can give $\left(\mathbf{PBord}_n^V\right)_{\underline{k}}$ the subspace topology.

5.4.3 The n-fold simplicial space $\mathbf{PBord}_n$

Proposition 5.4.8. *There are face and degeneracy maps making the collection of spaces*

$$\left\{\left(\mathbf{PBord}_n^V\right)_{\underline{k}} : \underline{k} \in \mathbb{N}^n\right\}$$

into an n-fold simplicial space.

Call this n-fold simplicial space $\mathbf{PBord}_n^V$. Loosely, the face maps forget a number t_j^i whereas the degeneracy maps repeat a number t_j^i.

Now we can remove the dependence on the vector space V. Let $\mathbb{R}^\infty$ be the unique real vector space of countably infinite dimension. We define the n-fold simplicial space $\mathbf{PBord}_n$ to be the limit

$$\mathbf{PBord}_n := \varinjlim_{V \subseteq \mathbb{R}^\infty} \mathbf{PBord}_n^V$$

5.4.4 The n-fold Segal spaces $\mathbf{PBord}_n$ and $\mathbf{Bord}_n$

We need to verify that the n-fold simplicial space $\mathbf{PBord}_n$ is in fact an n-fold Segal space. The important point is to prove that the Segal map

$$\begin{array}{c}(\mathbf{PBord}_n)_{k_1,\dots,k_i+k_i',\dots,k_n}\\ \downarrow \\ (\mathbf{PBord}_n)_{k_1,\dots,k_i,\dots,k_n} \underset{(\mathbf{PBord}_n)_{k_1,\dots,0,\dots,k_n}}{\times^h} (\mathbf{PBord}_n)_{k_1,\dots,k_i',\dots,k_n}\end{array}$$

is a weak homotopy equivalence.

Fact 5.4.9. $\mathbf{PBord}_n$ is nice enough for the homotopy fibre product above to be weakly homotopy equivalent to the usual fibre product. More technically, $\mathbf{PBord}_n$ is Reedy fibrant - see 5.2.16 for further discussion.

Corollary 5.4.10. *To verify that the above Segal map is a weak homotopy equivalence, we may replace the homotopy fibre product with the genuine fibre product of topological spaces.*

Replacing the homotopy pullback with the usual pullback, we see that an element of

$$(\mathbf{PBord}_n)_{k_1,\dots,k_i,\dots,k_n} \underset{(\mathbf{PBord}_n)_{k_1,\dots,0,\dots,k_n}}{\times} (\mathbf{PBord}_n)_{k_1,\dots,k_i',\dots,k_n}$$

is a pair of submanifolds M and N of $V \times \mathbb{R}^n$ for some V, together with data allowing us to glue them on their intersection. Gluing them together gets us an element of $(\mathbf{PBord}_n)_{k_1,\dots,k_i+k_i',\dots,k_n}$. The Segal map is in fact a homeomorphism, not just a homotopy equivalence. We can now deduce the following:

Theorem 5.4.11. $\mathbf{PBord}_n$ *is an n-fold Segal space.*

The n-fold Segal space $\mathbf{PBord}_n$ is not in general complete. We define $\mathbf{Bord}_n := \widehat{\mathbf{PBord}}_n$ to be its completion. Then $\mathbf{Bord}_n$ is an (∞, n)-category.

Remark 5.4.12. The spaces $\mathbf{PBord}_1$ and $\mathbf{PBord}_2$ **are** complete. However, for $n \geq 6$, $\mathbf{PBord}_n$ is **not** complete; this is because not all invertible cobordisms $M \to N$ arise from diffeomorphisms $M \to N$. The **s-cobordism theorem** says that for $n \geq 6$, this statement is equivalent to the vanishing of an invariant of the cobordism known as the **Whitehead torsion**. It's known that for $n \geq 6$ that there are n-bordisms which have nontrivial Whitehead torsion, and hence that $\mathbf{PBord}_n$ is not complete.

5.4.5 Extra structure on $\mathbf{Bord}_n$

Most importantly, $\mathbf{Bord}_n$ is a **symmetric monoidal** (∞, n)**-category**, which means that it has a symmetric monoidal structure (given by the disjoint union) compatible with the (∞, n) structure.

We can also restrict to cobordisms with certain properties: for example there is an (∞, n)-category $\mathbf{Bord}_n^{\mathrm{fr}}$ of framed cobordisms, and an (∞, n)-category $\mathbf{Bord}_n^{\mathrm{or}}$ of oriented cobordisms. Both of these categories also carry a symmetric monoidal structure.

The (∞, n)-category $\mathbf{Bord}_n^{\text{fr}}$ of framed cobordisms will be our focus from now on, since the Cobordism Hypothesis is stated in terms of framed cobordisms.

Note on the constructions The construction of $\mathbf{Bord}_n$ outlined above is similar to Lurie's definition in [10]. Lurie's original definition contained an error, and this was corrected by Calaque and Scheimbauer in [4] (which consists mainly of material from [16]) who construct their spaces differently.

They first construct a Segal space of intervals in $\mathbb{R}^n$ and then lift this Segal space structure to $\mathbf{PBord}_n$. The definitions in [4] correspond roughly to the definitions here by taking our t_j^i to be points in their intervals.

5.5 Adjoints and dualisablity

Given a topological field theory $Z : \mathbf{Bord}_n^{\text{fr}} \to \mathcal{C}$, we'd like to classify the kind of objects of $\mathcal{C}$ that could be the image of the 0-manifold $*$ under Z. Such objects should satisfy some finiteness condition: for example when $n = 1$ and $\mathcal{C} = \mathbf{Vect}_k$ we saw that $Z(*)$ had to be finite-dimensional, and conversely that any finite-dimensional vector space can be obtained as the image of $*$ under some TFT.

The correct analogue of finite-dimensionality in the ∞-categorical setting is **full dualisability**, and to define this is the goal of the current section.

It turns out that requiring dualisability for objects is not enough: we'll also need a notion of dualisability for k-morphisms as well. In the 2-category **Cat** we already have a reasonable notion of dualisability for 1-morphisms: a left dual (if it exists) for a functor $F : C \to D$ should be its left adjoint $G : D \to C$. We extend this definition to general 2-categories and then to general (∞, n)-categories. Adjoints and duals are very closely related in higher categories.

5.5.1 Duals for objects

Recall the following 1-categorical definition:

Definition 5.5.1. Let C be a monoidal category. Let V be an object of C. A **right dual** for V is the data of an object $V^\vee$ and maps

$$\begin{aligned} \text{ev} : V \otimes V^\vee \to 1 \qquad & \text{the **evaluation map**} \\ \text{coev} : 1 \to V^\vee \otimes V \qquad & \text{the **coevaluation map**} \end{aligned}$$

such that the triangles[12]

$$\begin{array}{ccc} V & & \\ {\scriptstyle \text{id}_V \otimes \text{coev}} \downarrow & \searrow^{\text{id}_V} & \\ V \otimes V^\vee \otimes V & \xrightarrow[\text{ev} \otimes \text{id}_V]{} & V \end{array} \qquad \begin{array}{ccc} V^\vee & & \\ {\scriptstyle \text{coev} \otimes \text{id}_{V^\vee}} \downarrow & \searrow^{\text{id}_{V^\vee}} & \\ V^\vee \otimes V \otimes V^\vee & \xrightarrow[\text{id}_{V^\vee} \otimes \text{ev}]{} & V^\vee \end{array} \qquad (1)$$

[12]These triangles are technically pentagons; here we have ignored the associators and the isomorphisms $X \otimes 1 \xrightarrow{\sim} X \xleftarrow{\sim} 1 \otimes X$.

commute. We also say in this situation that V is a **left dual** of $V^\vee$.

Remark 5.5.2. If C is symmetric monoidal, then the notions of right dual and left dual coincide and we refer simply to the **dual**.

Example 5.5.3. If C is the symmetric monoidal category $\mathbf{Vect}_k$ (with monoidal structure given by the usual tensor product over k) then a vector space V has a dual if and only if it is finite-dimensional. More specifically, we can always define a space $V^* = \mathrm{Hom}(V, k)$ and an evaluation map $V \otimes_k V^* \to k$, but we can only define a compatible coevaluation map if V is finite-dimensional.

Proposition 5.5.4. *Left and right duals (if they exist) are unique up to unique isomorphism.*

We can easily extend the definition of a dualisable object to higher categories, by taking the homotopy category.

Definition 5.5.5. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Say that an object X of $\mathcal{C}$ is **dualisable** if it admits a dual when considered as an object of the homotopy category $h\mathcal{C}$.

If Z is an oriented or framed topological field theory with target $\mathcal{C}$, then any object X of $\mathcal{C}$ with $X = Z(*)$ must be dualisable since we can obtain $X^\vee$ by evaluating Z on a point with the opposite orientation to that of $*$. In general the condition that X be dualisable is not strong enough for such a TFT to exist. However, for $n = 1$ it turns out that dualisability is sufficient, so this problem will only manifest itself in higher dimensions.

In general we should require that the morphisms in $\mathcal{C}$ should also have duals, which leads us to the notion of adjoints.

5.5.2 Adjoints in 2-categories

Recall the unit-counit definition of an adjunction:

Definition 5.5.6. Let C, D be two categories and $F : C \to D$ and $G : D \to C$ two functors. An **adjunction** between F and G consists of two natural transformations

$$u : \mathrm{id}_C \Rightarrow GF \qquad \text{the } \textbf{unit}$$
$$v : FG \Rightarrow \mathrm{id}_D \qquad \text{the } \textbf{counit}$$

such that the following two triangles[13] commute:

$$\begin{array}{ccc} F \circ \mathrm{id}_C & & \\ {\scriptstyle \mathrm{id}_F \diamond u}\downarrow & \searrow^{\mathrm{id}_F} & \\ F \circ G \circ F & \xrightarrow[v \diamond \mathrm{id}_F]{} & \mathrm{id}_D \circ F \end{array} \qquad \begin{array}{ccc} \mathrm{id}_C \circ G & & \\ {\scriptstyle u \diamond \mathrm{id}_G}\downarrow & \searrow^{\mathrm{id}_G} & \\ G \circ F \circ G & \xrightarrow[\mathrm{id}_G \diamond v]{} & G \circ \mathrm{id}_D \end{array} \qquad (2)$$

In this situation we say that F is a **left adjoint** of G and that G is a **right adjoint** of F.

[13] Once again, these triangles are really pentagons. If we think of **Cat** as a strict 2-category, then they are squares since we don't need any associators.

Note that the expression $\eta \diamond \theta$ means the horizontal composition of the natural transformations η and θ rather than the vertical composition.

Remark 5.5.7. Observe the formal similarity of the triangles of equation (2) to the ones of equation (1). This is a good indication that adjoints are 'higher duals'.

Proposition 5.5.8. *Adjoints, if they exist, are unique up to unique isomorphism.*

The category **Cat** is the prototypical example of a 2-category: the objects of **Cat** are all (small) categories, the 1-morphisms are functors, and the 2-morphisms are natural transformations. Definition 5.5.6 didn't really rely on any of the properties of **Cat**, and so we can immediately generalise it to any 2-category:

Definition 5.5.9. Let $\mathcal{C}$ be any 2-category. Let X, Y be objects of $\mathcal{C}$ and let $F : X \to Y$ and $G : Y \to X$ be two 1-morphisms. Say that a 2-morphism $u : \mathrm{id}_X \to G \circ F$ is the **unit of an adjunction between** F **and** G if there exists another 2-morphism $v : F \circ G \to \mathrm{id}_Y$ such that the following two triangles commute:

$$\begin{array}{ccc} F \circ \mathrm{id}_X & & \\ {\scriptstyle \mathrm{id}_F \diamond u} \downarrow & \searrow^{\mathrm{id}_F} & \\ F \circ G \circ F & \xrightarrow[v \diamond \mathrm{id}_F]{} & \mathrm{id}_Y \circ F \end{array} \qquad\qquad \begin{array}{ccc} \mathrm{id}_X \circ G & & \\ {\scriptstyle u \diamond \mathrm{id}_G} \downarrow & \searrow^{\mathrm{id}_G} & \\ G \circ F \circ G & \xrightarrow[\mathrm{id}_G \diamond v]{} & G \circ \mathrm{id}_Y \end{array}$$

In this case we say that v is the **counit**, and that F (resp. G) is **left** (resp. **right**) adjoint to G (resp. F).

Remark 5.5.10. If u is the unit of an adjunction, then a compatible counit v is uniquely determined, and vice versa. So it's enough to specify the existence of either u or v.

Example 5.5.11. A category with a single object is the same thing as a monoid. Similarly if $\mathcal{C}$ is a 2-category with a single object $*$ then the category $\mathrm{Hom}_{\mathcal{C}}(*, *)$ is a monoidal category.

Conversely if M is a monoidal category then we can build a 2-category $\mathcal{B}M$ with a single object *, hom-category $\mathrm{Hom}_{\mathcal{B}M}(*, *) \cong M$ and composition law for 1-morphisms given by the tensor product on M.

Then an object X of M is right dual to an object Y of M if and only if it is right adjoint to Y when both are considered as 1-morphisms of $\mathcal{B}M$. We often call $\mathcal{B}M$ the **delooping** of M.

Adjoints are closely related to invertibility:

Proposition 5.5.12. *Let $\mathcal{C}$ be a 2-category in which every 2-morphism is invertible. Let f be a 1-morphism of $\mathcal{C}$. Then the following are equivalent:*

i) f is invertible.

ii) f admits a left adjoint.

iii) f admits a right adjoint.

Definition 5.5.13. Say that a 2-category $\mathcal{C}$ **has adjoints for 1-morphisms** if every 1-morphism has both a left and a right adjoint.

5.5.3 Adjoints in higher categories

We'd like to generalise Definition 5.5.9 from 2-categories to higher categories.

Definition 5.5.14. Let $n \geq 2$ and let $\mathcal{C}$ be an (∞, n)-category. Let $h_2\mathcal{C}$ be the **homotopy 2-category** of $\mathcal{C}$, with

$$\begin{aligned} \text{objects} &\longleftrightarrow \text{objects of } \mathcal{C} \\ \text{1-morphisms} &\longleftrightarrow \text{1-morphisms of } \mathcal{C} \\ \text{2-morphisms} &\longleftrightarrow \text{isomorphism classes of 2-morphisms of } \mathcal{C} \end{aligned}$$

Remark 5.5.15. **Homotopy n-categories** are defined similarly.

Definition 5.5.16. Let $\mathcal{C}$ be an (∞, n)-category. Say that $\mathcal{C}$ **has adjoints for 1-morphisms** if $h_2\mathcal{C}$ has adjoints for 1-morphisms. More generally, for $1 < k < n$ say that $\mathcal{C}$ **has adjoints for k-morphisms** if for any two objects X, Y of $\mathcal{C}$ the $(\infty, n-1)$-category $\mathrm{Map}(X, Y)$ has adjoints for $(k-1)$-morphisms. Say that $\mathcal{C}$ **has adjoints** if it has adjoints for k-morphisms for all $0 < k < n$.

Remark 5.5.17. If every k-morphism in $\mathcal{C}$ is invertible then $\mathcal{C}$ has adjoints for k-morphisms. The converse is true provided that all $(k+1)$-morphisms are invertible - this is a consequence of Proposition 5.5.12.

Remark 5.5.18. The condition that $\mathcal{C}$ have adjoints depends on the choice of n. If we regard $\mathcal{C}$ as an $(\infty, n+1)$-category with all $(n+1)$-morphisms invertible then in general $\mathcal{C}$ does not have adjoints for n-morphisms unless it is an ∞-groupoid.

If $\mathcal{C}$ is monoidal then we can ask for a bit more:

Definition 5.5.19. Let $\mathcal{C}$ be a monoidal (∞, n)-category. Say that $\mathcal{C}$ **has duals** if the following two conditions are satisfied:

i) Every object X has both a left and a right dual when considered as an object of the homotopy category $h\mathcal{C}$.[14]

ii) $\mathcal{C}$ has adjoints.

Remark 5.5.20. We can generalise our earlier construction of Example 5.5.11. If $\mathcal{C}$ is a monoidal (∞, n)-category then it is possible to build an $(\infty, n+1)$-category $\mathcal{BC}$ (the **delooping** of $\mathcal{C}$) with a single object $*$, recovering $\mathcal{C}$ as the mapping object $\mathrm{Hom}_{\mathcal{BC}}(*, *)$. Then $\mathcal{C}$ has duals if and only if $\mathcal{BC}$ has adjoints.

5.5.4 Full dualisability

Given a symmetric monoidal (∞, n)-category we'd like to pick out the largest subcategory with duals.

Theorem 5.5.21. *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. Then there exists another symmetric monoidal (∞, n)-category $\mathcal{C}^{\mathrm{fd}}$ with duals, and a symmetric monoidal functor $i : \mathcal{C}^{\mathrm{fd}} \to \mathcal{C}$, universal among symmetric monoidal functors $j : \mathcal{D} \to \mathcal{C}$ where $\mathcal{D}$ has duals.*

[14] Note that $h\mathcal{C}$ inherits its monoidal structure from $\mathcal{C}$. If $\mathcal{C}$ is symmetric monoidal then this condition is the condition that every object be dualisable.

Remark 5.5.22. $\mathcal{C}^{\text{fd}}$ is determined up to equivalence by the above properties. In general we can obtain $\mathcal{C}^{\text{fd}}$ from $\mathcal{C}$ by repeatedly discarding morphisms that don't admit adjoints (and objects that don't admit duals).

Example 5.5.23. If $\mathcal{C}$ is a symmetric monoidal $(\infty, 1)$-category then $\mathcal{C}^{\text{fd}}$ is equivalent to the full subcategory of $\mathcal{C}$ spanned by the dualisable objects.

Definition 5.5.24. Say that an object X of $\mathcal{C}$ is **fully dualisable** if it belongs to the essential image[15] of the functor i.

Example 5.5.25. For each $n > 0$, the (∞, n)-category $\mathbf{Bord}_n^{\text{fr}}$ has duals. Every k-morphism can be identified with an oriented manifold M; the morphism $\bar{M}$ (M with the opposite orientation) is both a left and a right adjoint to M.

Example 5.5.26. If $\mathcal{C}$ is the $(\infty, 1)$-category $\mathbf{Vect}_k$, then an object of $\mathcal{C}$ is fully dualisable if and only if it is finite-dimensional.

This generalises to the following:

Proposition 5.5.27. *An object of a symmetric monoidal $(\infty, 1)$-category is fully dualisable if and only if it is dualisable.*

In general full dualisability is a much stronger condition than dualisability! In dimension 2, there are some simple criteria for testing whether or not an object is fully dualisable:

Proposition 5.5.28. *Let $\mathcal{C}$ be a symmetric monoidal $(\infty, 2)$-category. Let X be an object of $\mathcal{C}$. Then X is fully dualisable if and only if it admits a dual $X^\vee$ and the evaluation map $\mathrm{ev} : X \otimes X^\vee \to 1$ has both a left and a right adjoint.*

5.6 The Cobordism Hypothesis

In this short section we rigourously state the Cobordism Hypothesis. We begin with some bookkeeping.

5.6.1 Terminology

Definition 5.6.1. An (∞, n)**-functor** between two (∞, n)-categories $\mathcal{C}$ and $\mathcal{D}$ is a map of the underlying simplicial spaces (which is itself a natural transformation between the defining functors).

Theorem 5.6.2. *The collection $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of (∞, n)-functors between two (∞, n)-categories itself forms an (∞, n)-category.*

Remark 5.6.3. The collection of all (small) (∞, n)-categories naturally forms an $(\infty, n+1)$-category with mapping objects $\mathrm{Map}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.

Proposition 5.6.4. *We can also define **symmetric monoidal** (∞, n)**-functors** between symmetric monoidal (∞, n)-categories. The collection of symmetric monoidal (∞, n)-functors between two symmetric monoidal (∞, n)-categories $\mathcal{C}$ and $\mathcal{D}$ itself forms an (∞, n)-category, which we refer to as $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$.*

[15] Recall that the **essential image** of a functor $F : \mathcal{D} \to \mathcal{E}$ is the smallest isomorphism-closed subcategory of $\mathcal{E}$ containing the image of F.

Definition 5.6.5. A **fully extended framed n-dimensional topological field theory** is a symmetric monoidal (∞, n)-functor with source $\mathbf{Bord}_n^{\mathrm{fr}}$. The collection of all fully extended framed n-TFTs with target $\mathcal{C}$ is the (∞, n)-category $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$.

Definition 5.6.6. Given an (∞, n)-category $\mathcal{C}$, I will denote the underlying $(\infty, 0)$-category[16] of $\mathcal{C}$ by $\pi_{\leq\infty}(\mathcal{C})$. This notation is not standard.

5.6.2 A Precise Statement

Claim 5.6.7 (the Cobordism Hypothesis)**.** *If $\mathcal{C}$ is a symmetric monoidal (∞, n)-category then the evaluation functor $Z \mapsto Z(*)$ induces an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C}) \xrightarrow{\sim} \pi_{\leq\infty}(\mathcal{C}^{\mathrm{fd}})$$

In particular, the Cobordism Hypothesis states that $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid, and hence a topological space. We can think of it as a classifying space for fully dualisable objects in $\mathcal{C}$.

It is not too difficult to prove that $\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr}}, \mathcal{C})$ is an ∞-groupoid. The hard part of proving the Cobordism Hypothesis is proving that the induced functor is an equivalence. A sketch proof of this is given by Lurie in [10].

[16] a.k.a. ∞-groupoid

Chapter 6

Introduction to Factorization Homology

SPEAKER: TIM WEELINCK
NOTES: TIM WEELINCK
DATES: 4-3-2016 & 8-3-2016

6.1 E_n-algebras or Coefficients

Today our notion of $(\infty, 1)$-categories are topological categories i.e. categories enriched over topological spaces (compactly generated, weakly Hausdorff - to be precise). We will come across categories that are classical 1-categories (the category of vector spaces), 2-categories (the category of categories) or categories that come equipped with a model structure (the dg category of vector spaces). We will point out along the way how to obtain the associated topological category.

Definition 6.1.1. Let $\mathrm{Disk}_1^{\mathrm{fr}}$ denote the topological category with objects disjoint copies of the framed real line $\emptyset$, $\mathbb{R}$, $\mathbb{R}^{\sqcup 2}$, $\mathbb{R}^{\sqcup 3}$, etcetera and morphism spaces[1]

$$\mathrm{Map}(\mathbb{R}^{\sqcup i}, \mathbb{R}^{\sqcup j}) = \{\text{smooth open embeddings } \mathbb{R}^{\sqcup i} \hookrightarrow \mathbb{R}^{\sqcup j} \text{ respecting framing}\}.$$

The topology on the morphism sets is inherited from the compact-open topology on the space of continuous maps. We will make use of the following structure of the hom spaces.

i) For all i, j paths in $\mathrm{Map}(\mathbb{R}^{\sqcup i}, \mathbb{R}^{\sqcup j})$ are given by isotopies of embeddings.

ii) Any framing preserving embedding $\mathbb{R} \hookrightarrow \mathbb{R}$ is isotopic to the identity.

Disjoint union provides $\mathrm{Disk}_1^{\mathrm{fr}}$ with the structure of a symmetric strict monoidal category, with unit $\emptyset$.

Definition 6.1.2. An E_1-algebra (or $\mathrm{Disk}_1^{\mathrm{fr}}$-algebra) valued in vector spaces is a symmetric monoidal functor

$$A : \mathrm{Disk}_1^{\mathrm{fr}} \to \mathrm{Vect}(k).$$

[1] We will be somewhat imprecise about what respecting framing means. One can think of f respecting the framing, $\{v_i\}$ and $\{w_i\}$, to mean for all x, i the identity $(Df)_x v_i(x) = a w_i(f(x))$ holds for some $a > 0$. We allow ourselves to be imprecise because in ∞-categories one only cares about the homotopy type of the mapping spaces.

Remark 6.1.3. Actually, there is a distinction between $\mathrm{Disk}_1^{\mathrm{fr}}$-algebras and E_1-algebras, the latter are algebras for a certain operad (the little cubes operad see for example [12, ch. 5]). However, as the notion of algebras are equivalent, we allow ourselves to be imprecise about the distinction.

Note that this is a functor of $(\infty, 1)$-categories, meaning that the components of the functor on the hom-spaces are continuous maps. In what sense do the hom sets of $\mathrm{Vect}(k)$ have a topology? None really: we equip the hom sets with the discrete topology. In particular this means that $\pi_n(\mathrm{Map}(V, W)) = 0$ for all $V, W \in \mathrm{Vect}(k)$ and all $n \geq 1$.

6.1.1 What is an E_1-algebra?

Let us explore what an E_1-algebra concretely looks like. We fix some E_1-algebra $A \in E_1(\mathrm{Vect}(k))$, and denote $V := A(\mathbb{R})$ the image of $\mathbb{R}$. As A is a monoidal functor we immediately obtain $A(\mathbb{R}^{\sqcup i}) \cong V^{\otimes i}$ and also $A(\emptyset) = k$. Thus on objects A is completely specified by the vector space $A(\mathbb{R})$.
As $\mathrm{Vect}(k)$ has discrete topology $f \overset{isot.}{\simeq} g \Rightarrow A(f) = A(g)$. Up to isotopy there are two framing preserving embeddings $\mathbb{R} \sqcup \mathbb{R} \hookrightarrow \mathbb{R}$ related by the swap. These are mapped to opposite multiplications maps on V, as drawn in Figure 6.1.

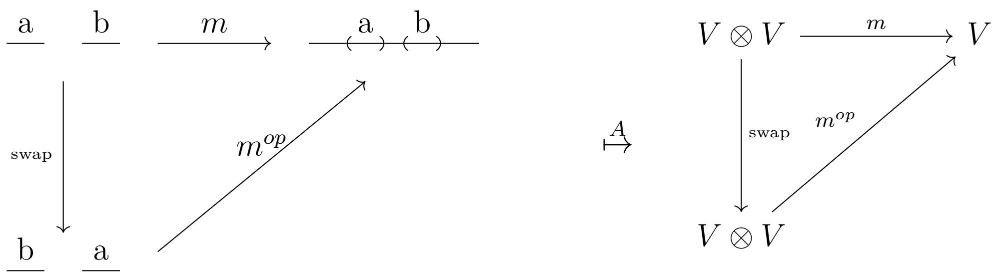


Figure 6.1: Multiplication.

Diagrams that commute up to isotopy are translated to commuting diagrams in $\mathrm{Vect}(k)$. The following two diagrams equip V with the structure of a unital associative k-algebra.

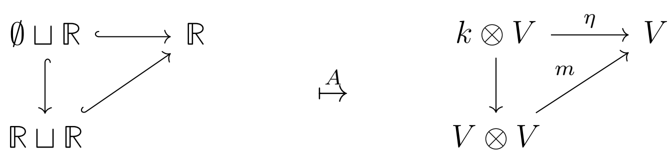


Figure 6.2: Unity.

We can summarize the discussion as follows.

Proposition 6.1.4. *Let $A \in E_1(\mathrm{Vect}(k))$. The vector space $A(\mathbb{R})$ comes naturally equipped with the structure of an associative unital k-algebra.*

□

As we haven't used most of the embbedings $\mathbb{R}^{\sqcup i} \hookrightarrow \mathbb{R}^{\sqcup j}$, one could wonder whether there is more structure on $V = A(\mathbb{R})$. This turns out not to be the case.

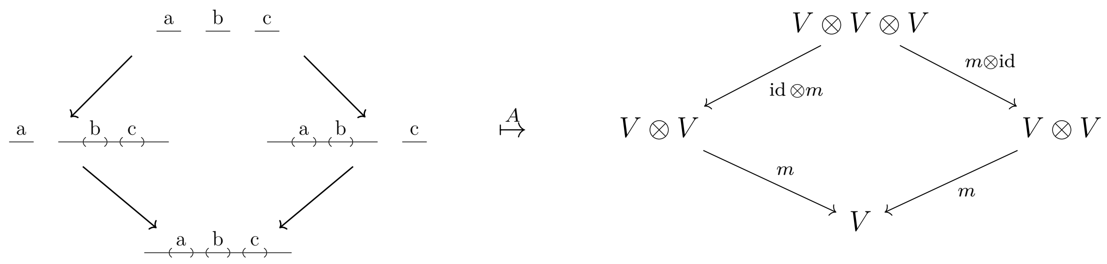


Figure 6.3: Associativity.

Theorem 6.1.5. *The data of an E_1-algebra valued in* Vect(k) *is equivalent to the data of an unital associative algebras.*

□

6.1.2 Generalizing E_1-algebras

We saw that by playing a topological game on the real line, we recaptured the notion of algebras as E_1-algebras valued in vector spaces. We would like to generalize these E_1-algebras, for example by replacing $\mathbb{R}$ by a higher dimensional disk, or by replacing Vect(k) by some more interesting category.
Let us for example fix $\mathcal{C} = \mathcal{C}at$ to be topological category associated to the two category of categories Cat. Recall that Cat is a strict 2-category[2] with categories as objects, functors as morphisms and natural transformations as 2-morphisms. In contrast with Vect(k) one does not obtain $\mathcal{C}at$ by declaring the hom-sets to have discrete topology; the existence of higher morphisms means that there should be interesting topology.
Cat is enriched in categories, but we want to get the associated topological i.e. $(\infty, 1)$-category. We begin by removing all non-invertible 2-morphisms i.e. keep only natural isomorphisms, we obtain a category enriched in groupoids. Then we take the nerve of the hom-sets to obtain a category enriched over simplicial sets. We now take the geometric realization of the hom-sets to obtain $\mathcal{C}at$, enriched over topological spaces. The 2-categorical structure of Cat exhibits itself as follows in $\mathcal{C}at$.

Proposition 6.1.6. *Let $\mathcal{C}at$ be the topological category of categories. Then*

i) Objects of $\mathcal{C}at$ are categories.

ii) $\mathrm{Map}_{\mathcal{C}at}(\mathcal{C}, \mathcal{D})$ *is a topological space glued from simplices whose vertices correspond to functors and whose edges corresponding to natural isomorphisms.*

iii) For all $\mathcal{C}, \mathcal{D} \in \mathcal{C}at$ we have that $\pi_n \mathrm{Map}(\mathcal{C}, \mathcal{D}) = 0$ for $n \geq 2$.

iv) The product $\times$ of categories endows $\mathcal{C}at$ with a symmetric monoidal structure.

□

We could now study an E_1-algebra valued in $\mathcal{C}at$ i.e. symmetric monoidal functor A from $\mathrm{Disk}_1^{\mathrm{fr}}$ to $\mathcal{C}at$. Then $f \overset{isot.}{\simeq} g \Rightarrow A(f) \cong A(g)$, where the functors are naturally isomorphic. Diagrams that commute up to isotopy in $\mathrm{Disk}_1^{\mathrm{fr}}$ are translated to diagrams that commute up to natural isomorphism in $\mathcal{C}at$. It should then come as no big surprise that we have:

[2]A category enriched in categories.

Proposition 6.1.7. *Let $A \in E_1(\mathcal{C}at)$ be an E_1-algebra valued in categories. The category $A(\mathbb{R})$ comes naturally equipped with a monoidal structure.*

□

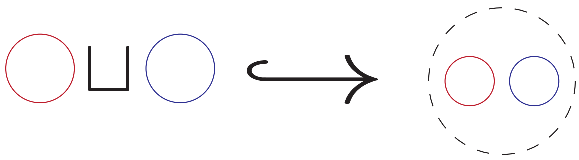
In fact, again the data is equivalent. Besides replacing $\mathrm{Vect}(k)$ with $\mathcal{C}at$, another way to generalize E_1-algebras is by considering higher dimensional disks. We can copy the definition of $\mathrm{Disk}_1^{\mathrm{fr}}$ verbatim, replacing $\mathbb{R}$ by $\mathbb{R}^2$ everywhere, to obtain the topological category $\mathrm{Disk}_2^{\mathrm{fr}}$ of 2 dimensional framed disks.

Definition 6.1.8. An E_2-algebra valued in categories, write $A \in E_2(\mathcal{C}at)$, is a symmetric monoidal functor

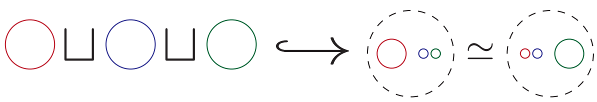
$$A : \mathrm{Disk}_2^{\mathrm{fr}} \to \mathcal{C}at.$$

We will spend the remainder of this subsection answering the question: 'what is an E_2-algebra valued in categories'. As in the one dimensional case, specifying a category $\mathcal{C} =: A(\mathbb{R}^2)$ as the image of $\mathbb{R}^2$ fixes the functor A on objects. Diagrams that commute up to homotopy are transformed to diagrams that commute up to natural isomorphism. Let us draw some of these diagrams, and obtain some structure on the target category $\mathcal{C}$.

The embedding



endows $\mathcal{C}$ with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. The trivial embedding $\emptyset \hookrightarrow \mathbb{R}^2$ endows $\mathcal{C}$ with a unit $\mathbb{1} : * \to \mathcal{C}$. The natural isomorphisms witnessing associativity and (left) unity of the tensor product are obtained from the following isotopies



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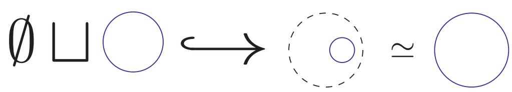

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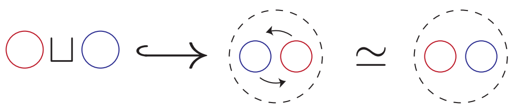
where the isotopies should be thought of as 'shrinking and enlarging disks'. We deduce that the category $\mathcal{C}$ comes naturally equipped with the structure of a monoidal category i.e. an E_1-algebra structure on $\mathcal{C}$. However, there is more structure coming from the extra room we have in two dimensions. The isotopy rotating a pair of disks 180° around eachother



yields a natural isomorphism between $\otimes$ and the opposite tensor product $\otimes^{op}$ i.e. natural isomorphisms $c_{X,Y} : X \otimes Y \cong Y \otimes X$ for all $X, Y \in \mathcal{C}$. We claim this defines the structure of a braiding on the category $\mathcal{C}$.

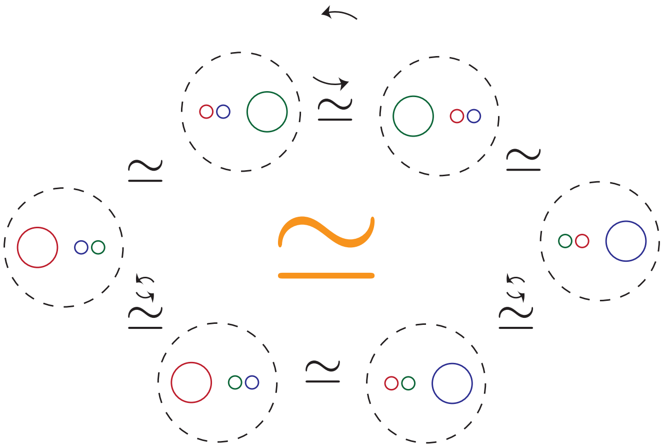
Definition 6.1.9. [8] Let $\mathcal{C}$ be a monoidal category. Denote $\otimes$ the tensor product and $\alpha_{A,B,C} : (A \otimes B) \otimes C \cong A \otimes (B \otimes C)$ the associativity isomorphism. A *braiding on* $\mathcal{C}$ is a family of natural isomorphisms $c_{X,Y} : X \otimes Y \cong Y \otimes X$ satisfying the following coherence equations:

i) $\alpha_{B,C,A} \circ c_{A,B\otimes C} \circ \alpha_{A,B,C} = \mathrm{id}_B \otimes c_{A,C} \circ \alpha_{B,A,C} \circ c_{A,B} \otimes \mathrm{id}_C$,

ii) $\alpha^{-1}_{C,A,B} \circ c_{A\otimes B,C} \circ \alpha^{-1}_{A,B,C} = c_{A,C} \otimes \mathrm{id}_B \circ \alpha^{-1}_{A,C,B} \circ \mathrm{id}_A \otimes c_{B,C}$.

We call a monoidal category $\mathcal{C}$ with a braiding a *braided monoidal category.*

To see that we have indeed defined a braiding on $\mathcal{C} = A(\mathbb{R}^2)$ one only needs to consider two isotopy of isotopies. For example, the (orange coloured) isotopy of isotopies



yields the second coherence equation.

Remark 6.1.10. Note that an isotopy of isotopies corresponds to a diagram of 2-morphisms commuting up to a 3-morphism in $\mathrm{Disk}_2^{\mathrm{fr}}$. Since there are no non-trivial 3-morphisms in $\mathcal{C}at$ the diagram is mapped to a commuting diagram in $\mathcal{C}at$.

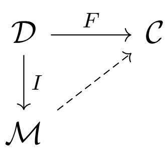
Again, one might wonder whether there is any extra data on the category $\mathcal{C}$.

Theorem 6.1.11. *[12, ex. 5.1.2.4] Endowing a category with the structure of an E_2-algebra is equivalent to endowing it with the structure of a braided monoidal category.*

□

6.2 Left Kan Extensions or Globalization

In mathematics one often comes across situations where one has a functor defined on some subcategory $\mathcal{D} \subset \mathcal{M}$ and you wish to extend your functor is a natural way to the full category $\mathcal{M}$. More generally, one can have a functor $F : \mathcal{D} \to \mathcal{C}$ that you wish to extend along a functor $I : \mathcal{D} \to \mathcal{M}$ as indicated in the figure below.



Expecting a functor to make the square strictly commute would be too restrictive. This is already impossible when $F(a) \neq F(b)$, but $I(a) = I(b)$. Rather we should ask for 'best approximations' to such solutions. Kan extensions, when they exist, provide two canonical answers to this question.

Definition 6.2.1. Let $I : \mathcal{D} \to \mathcal{M}$, $F : \mathcal{D} \to \mathcal{C}$ be functors.

i) A left extension of I along F is a pair $(L, \eta : F \Rightarrow \mathrm{Lan}_F I \circ I)$.

ii) A morphism of extensions $\mu : (L, \eta) \to (L', \eta')$ is a natural transformation $\mu : L \Rightarrow L'$ such that $\eta' = \mu I \circ \eta$.

iii) A left Kan extension $(Lan_I F, \eta)$ is an initial object in the category of extensions.

The universal property of $(Lan_I F, \eta)$ assigns to every natural transformation $\sigma : F \Rightarrow TI$ a natural transformation $\bar{\sigma} : \mathrm{Lan}_I F F \Rightarrow T$ such that $\bar{\sigma} I \circ \eta$. The assignment $\sigma \mapsto \bar{\sigma}$ defines a natural bijection $\mathrm{Nat}(F, TI) \cong \mathrm{Nat}(\mathrm{Lan}_I F, T)$ i.e. the left Kan extension represents the functor $T \mapsto \mathrm{Nat}(F, TI)$. Of course Left Kan extensions need not exist, but it is clear that they are unique (up to unique natural isomorphism) if they exist.

Remark 6.2.2. Those familiar with the yoga of coends should find a pleasent exercise in verifying that if the coend $\int^D \mathcal{M}(I(D), m) \cdot F(D)$ exists, it computes the left Kan extension. Such Left Kan extensions are called pointed. All known interesting Kan extensions are pointed.

Example 6.2.3. i) Let $H \subset G$ be a subgroup of a finite group G, $I : BH \to BG$ the induced inclusion functor and $\rho : BH \to \mathrm{Vect}(k)$ a representation of H. Then $Lan_I \rho = Ind_H(\rho)$ is the induced representation of ρ to G.

ii) Let F be left adjoint to G, then $(G, \epsilon) = (\mathrm{Lan}_F \, \mathrm{id}, \eta)$.

iii) Let $! : \mathcal{C} \to *$ denote the unique functor to the terminal category. A functor $D : J \to \mathcal{C}$ has a colimit iff $Lan_! D$ exists, moreover $\lim_D = Lan_! D(*)$.

The above list of examples should hopefully convince you that left Kan extensions are very natural and fundamental objects. In fact in [13] MacLane has a section called 'All Concepts Are Kan Extensions' where he writes 'The notion of Kan extensions subsumes all the other fundamental concepts of category theory'.

Of course there exists a dual notion of right Kan extensions, so now we have two choices of how to extend an E_n-algebra to the category of manifolds, the two choices come down to whether one wants to consider manifolds as colimits or limits of disks (i.e. affine space). Traditionally, both ways of looking at manifolds have been used.

Example 6.2.4. i) Let $f : \mathbb{R}^2 \to \mathbb{R}, (x, y) \mapsto x^2 + y^2 - 1$. The circle can be represented as the limit $S^1 = eq(\mathbb{R}^2 \underset{0}{\overset{f}{\rightrightarrows}} \mathbb{R})$.

ii) Let I_1, I_2 be two intervals which we image as covering the circle, let I_3, I_4 be two intervals, which we imagine being the connected components of $I_1 \cap I_2$. Then the circle can be represented as the colimit $S_1 = coeq(I_3 \cup I_4 \underset{i_{14} \cup i_{24}}{\overset{i_{13} \cup i_{23}}{\rightrightarrows}} I_1 \cup I_2)$.

Nowadays, the colimit view on manifolds (i.e. using an atlas that represents a manifold as glued from disks) has the upper hand. In a sense that we will make precise, this is a more natural view: any manifold is canonically a colimit of disks.

Definition 6.2.5. i) Let $I : \mathrm{Disk}_1^{\mathrm{fr}} \to \mathrm{Mfld}_n^{\mathrm{fr}}$ denote the inclusion functor of disks into manifolds.

ii) Fix a manifold $M \in \mathrm{Mfld}_n^{\mathrm{fr}}$. The category $I \downarrow M$ has objects $(D, f : I(D) \to M)$ where $D \in \mathrm{Disk}_n^{\mathrm{fr}}$. A morphism from $(D, I(D) \to M) \to (D', I(D') \to M)$ is a morphism $f \in \mathrm{Disk}_n^{\mathrm{fr}}(D, D')$ such that

$$\begin{array}{ccc} I(D) & \longrightarrow & M \\ {\scriptstyle I(f)}\big\downarrow & \nearrow & \\ I(D') & & \end{array}$$

commutes.

iii) Let $P : I \downarrow M \to \mathrm{Disk}_n^{\mathrm{fr}}$ denote the projection functor $(D, I(D) \to M) \mapsto D$, $f \to f$.

Proposition 6.2.6. *(Density of disks in manifolds)*[3] *Let $M \in Mfld$, then*

$$M = \lim(I \downarrow M \xrightarrow{P} Disk \xrightarrow{I} Mfld).$$

Proof. Let us denote the colimit above by C, the colimit expresses C has the following universal property. Let $g_i : I(D_i) \to M'$ be a collection of maps for all $(D_i, I(D_i) \to M) \in I \downarrow M$ such that

$$\begin{array}{ccc} I(D_i) & \xrightarrow{g_i} & M' \\ {\scriptstyle I(f)}\big\downarrow & \nearrow_{g_j} & \\ I(D_j) & & \end{array}$$

for all $f \in Hom_{I\downarrow M}((D_i, I(D_i) \to M), (D_j, I(D_j) \to M))$ then there is a unique map $g : C \to M'$, such that $g_i = I(D_i) \to C \xrightarrow{g} M'$. We claim M itself has this universal property, with maps $I(D_i) \to M$ from $I \downarrow M$ as edges of the cone. The universal property we need to check is: given a collection of maps $g_i|_{D_i \subset M} : D_i \to M'$ such that all maps agree on overlap, these maps glue uniquely to a map $M \to M'$. This is obviously true. □

This colimit expresses any framed manifold is canonically glued from all the framed disks mapping into it. We say that the category $\mathrm{Disk}_n^{\mathrm{fr}}$ is **dense** in the category of manifolds.[4] We can use try to use this colimit to try and extend a functor $F : \mathrm{Disk}_n^{\mathrm{fr}} \to \mathcal{C}$.

Definition 6.2.7. Let $I : \mathcal{D} \to \mathcal{M}$ be a functor. Suppose that for every $M \in \mathcal{M}$ the colimit

$$\lim(I \downarrow M \xrightarrow{P} \mathrm{Disk}_n^{\mathrm{fr}} \xrightarrow{F} \mathcal{C})$$

exists. Then this assignment on objects uniquely extends to a functor $\mathrm{Lan}_I F : \mathcal{M} \to \mathcal{C}$, called the *(pointed) left Kan extension* of I along F.

[3] We are not stating this result in terms of framed manifolds, as we have been sloppy in defining the morphism spaces in the framed case.

[4] In general we say a functor $I : \mathcal{D} \to \mathcal{M}$ is dense if $M = \lim(I \downarrow M \xrightarrow{P} \mathcal{D} \xrightarrow{I} \mathcal{M}$ for all $M \in \mathcal{M}$.

One should verify that this is indeed a left Kan extension, for example by comparing the above colimt to our earlier coend formula. Thus we see that left Kan extensions try to extend functors by approximating objects as colimits of objects in $\mathcal{D}$. Note that in case a functor $I : \mathcal{D} \to \mathcal{M}$ is dense, this is a very natural approximation. In fact, by comparing the two colimit formulas we see that a functor I is dense iff the pointed left Kan extension exists and $\mathrm{Lan}_I \, \mathrm{id}_{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$.

Definition 6.2.8. Let $A : \mathrm{Disk}_n^{\mathrm{fr}} \to \mathcal{C}$ be an E_n-algebra. The left Kan extension of A along $I : \mathrm{Disk}_n^{\mathrm{fr}} \to \mathrm{Mfld}_n^{\mathrm{fr}}$ is called the *factorization homology* of A. We denote the factorization homology as

$$\begin{array}{ccc} \mathrm{Disk}_n^{\mathrm{fr}} & \xrightarrow{A} & \mathcal{C} \\ \big\downarrow{\scriptstyle I} & \nearrow_{\int_{\cdot} A} & \\ \mathrm{Mfld}_n^{\mathrm{fr}} & & \end{array}$$

i.e. $\int_M A := L_I A(M)$.

6.3 Examples and Applications of Factorization Homology

We have seen that factorization homology is defined as a certain colimit. Thus if $\mathcal{C}$ is cocomplete, the factorization homology $\int_{\cdot} A$ of any $E_n(\mathcal{C})$-algebra A exists. What it actually computes can be difficult to determine however, as colimits are defined through maps out of it.

Example 6.3.1. i) $\int_{\mathbb{R}^n} A = A$ for any E_n-algebra A. This holds because if I is a fully faithful functor. Then for any functor F we get $Lan_I F \circ I = F$.

ii) The assignment $U \mapsto \int_U A$ defines a locally constant factorization algebra on $\mathbb{R}^n$.

For more refined examples we need a computational tool: a 'Mayer-Vietoris to our singular homology', if you will.

Let $M = M_R \cup_{M_0 \times \mathbb{R}} M_L$ a gluing of a manifold along a collar. We have embeddings $M_R \sqcup M_0 \times \mathbb{R} \hookrightarrow M_R$. Factorization homology then yields a map $\int_{M_R} \otimes \int_{M_0 \times \mathbb{R}} A \to \int_{M_R} A$ i.e. makes $\int_{M_R} A$ a right $\int_{M_0 \times \mathbb{R}}$ module. Similarly, $\int_{M_L} A$ is a left $\int_{M_0 \times \mathbb{R}} A$ module. If you like, the above maps are induced from the universal properties of the colimits.(*)

Theorem 6.3.2. *(Lurie, Francis) Excision. Let $M = M_R \cup_{M_0 \times \mathbb{R}} M_L$, $\mathcal{C}$ a $\otimes$-presentable symmetric monoidal ∞-category, $A \in \mathrm{Disk}_n^{\mathrm{fr}}(\mathcal{C})$. There exists a canonical isomorphism*

$$\int_M A \cong \int_{M_L} A \otimes_{\int_{M_0 \times \mathbb{R}} A} \int_{M_R} A.$$

□

Remark 6.3.3. Being $\otimes$-presentable is technical requirement on the category $\mathcal{C}$, one can ignore this during the introduction as just a niceness property. Being $\otimes$-presentable roughly means $\mathcal{C}$ is cocomplete and generated by colimits from a collection of 'small' objects. Furthermore, tensor products should distribute over colimits.

The excision property is a powerful computational tool, moreover, as we will see later it is the definining property of factorization homology.

Example 6.3.4. Let $\mathcal{C}$ be the infinity category of chain complexes. Note that the tensor product is the derived tensor product $\overset{L}{\otimes}$. Fix $A \in A_\infty\text{-alg} = E_1(\text{dg-Vect}^{\overset{L}{\otimes}})$. Then by excision we have

$$\begin{aligned}\int_{S^1} A &\cong \int_{\mathbb{R}_R} A \otimes_{\int_{S^0\times\mathbb{R}} A} \int_{\mathbb{R}_L} \\ &\cong A \overset{L}{\otimes}_{A\otimes A^{op}} A \\ &=: CH_*(A).\end{aligned}$$

Thus factorization homology of A over S^1 are the Hochschild chains.

Remark 6.3.5. This also provides with a natural explanation why hochschild homology carries a circle action. Indeed, the S^1 action of the circle on itself induces an action on the factorization homology $\int_S^1 A \cong CH_*(A)$.

We conclude with a classification result of manifold homology theories due to David Ayala and John Francis, very similar in spirit to the Eilenberg-Steenrod classification of homology theories of topological spaces.

Definition 6.3.6. Let $\mathcal{C}$ be some symmetric monoidal ∞-category. An n-dimensional manifold homology theory valued in $\mathcal{C}$ is a symmetric monoidal functor $H : \text{Mfld}_n^{\text{fr}} \to \mathcal{C}$ satisfying the excision property.

Theorem 6.3.7. *([1, theorem 3.24]) Let $\mathcal{C}$ be a $\otimes$-presentable symmetric monoidal infinity category. There is an equivalence*

$$\int : E_n - alg(\mathcal{C}^{\otimes}) \rightleftarrows H(\text{Mfld}_n^{\text{fr}}, \mathcal{C}^{\otimes}) : ev_{\mathbb{R}^n},$$

between E_n-algebras valued in $\mathcal{C}$ and manifold homology theories valued in $\mathcal{C}$. The equivalence is given by factorization homology and evaluation on the n-disk.

□

Chapter 7

Dualizable Tensor Categories

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Date: 2-4-2016

7.1 Introduction

We have seen that the Cobordism hypothesis gives a one-to-one correspondence between fully extended n-dimensional topological field theories (TFTs) valued in some symmetric monoidal (∞, n)-category $\mathcal{C}$ and fully dualizable objects in $\mathcal{C}$. Hence in order to study fully extended three dimensional TFTs one first needs a target $(\infty, 3)$-category. The simplest non-trivial candidate is the "3-category of tensor categories". Roughly speaking objects, morphisms, 1-morphisms and 2-morphisms should correspond to respectively linear monoidal categories, bimodule categories, bimodule functors and bimodule (natural) transformations. To each fully dualizable tensor category there is a corresponding fully extended 3-TFT. In these notes we will identify when a tensor category is fully dualizable and give examples.

Outline

In section 2 we recall the notions of m-dualizability in an (∞, n)-category, and state the cobordism hypothesis. In section 3 we recall the definition of a tensor category and sketch the structure of a 3-category of tensor categories. We conclude by stating a necessary and sufficient condition for a tensor category to be fully dualizable, and give examples.

7.2 Dualizability and the Cobordism Hypothesis

Recall that an ordinary n-TFT, in the sense of Atiyah-Segal, assigns to a closed $n-1$-dimensional manifold some vector space V. When $n = 1$ Zorro's lemma implied that the vector space assigned to a point needs to be finite dimensional. Similarly when $n = 2$ by using the cap and cup we could define a trace that ensured us that the vector space assigned to the circle should be finite dimensional. The targets of higher dimensional TFTs, and fully extended TFTs, are all subject to certain finiteness requirements. These finiteness requirements are expressed as the full dualizability of the target object.

DISCLAIMER. In the following we will disregard discussing what particular model of higher categories we are using. Most of the discussion will be concerned with 2-categories, where one can just think of weak or strict 2-categories.

Definition 7.2.1. A 1-morphism $G : \mathcal{A} \to \mathcal{B}$ in a 2-category admits a *left adjoint* $F : \mathcal{B} \to \mathcal{A}$ if there exists a unit 2-morphisms $\eta : \mathrm{id}_{\mathcal{B}} \to G \circ F$ and a counit 2-morphism $\epsilon : F \circ G \to \mathrm{id}_{\mathcal{A}}$ satisfying the triangle identities:

$$(1_G \diamond \epsilon) \circ (\eta \diamond 1_G) = 1_G,$$
$$(\epsilon \diamond 1_F) \circ (1_F \diamond \eta) = 1_F.$$

Here $f \circ g$ denotes vertical composition, first apply f then apply g, whereas $\diamond$ denotes horizontal composition.

Remark 7.2.2. Note that if in the above we take the strict 2-category of categories we obtain the familiar notion of adjoint functors.

For an (∞, n)-category $\mathcal{C}$, let $-1 \leq k \leq n-2$ we can define the 2-category of k-morphisms, denoted $h_2^{(k)}\mathcal{C}$ with

- Objects: k-morphisms in $\mathcal{C}$;
- 1-morphisms: $(k+1)$-morphisms in $\mathcal{C}$;
- 2-morphisms: equivalence classes of $(k+2)$-morphisms in $\mathcal{C}$.

For $k = -1$ we define $h_2^{(-1)}\mathcal{C}$ to have one object, 1-morphisms being the objects of $\mathcal{C}$, with composition given by the $\otimes$-product in $\mathcal{C}$ and 2-morphisms being equivalence classes of 1-morphisms in $\mathcal{C}$.

Definition 7.2.3. Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category.

i) We say $\mathcal{C}$ has *adjoints for k-morphisms* if every 1-morphisms in $h_2^{(k)}\mathcal{C}$ has a left and right adjoint.

ii) We say $\mathcal{C}$ is *m-dualizable* if it has adjoints for k-morphisms for $0 \leq k \leq m$.

iii) We call $\mathcal{C}$ *fully dualizable* if $\mathcal{C}$ is n-dualizable.

Remark 7.2.4. Note that full dualizability includes 0-dualizability i.e. having adjoints for objects

$$\mathrm{coev} : \mathbb{1} \to \bar{X} \otimes X, \qquad \mathrm{ev} : X \otimes \bar{X} \to \mathbb{1},$$

where $\mathbb{1}$ denotes the tensor unit.

We denote $d\mathcal{C}$ the maximal fully dualizable (∞, n)-subcategory of $\mathcal{C}$. The objects of $d\mathcal{C}$ are called the *fully dualizable objects of* $\mathcal{C}$. Let $\widetilde{d\mathcal{C}}$ denote the space of objects of $d\mathcal{C}$. By viewing the space $\widetilde{d\mathcal{C}}$ is an ∞-groupoid, or rather as an (∞, n)-category that happens to be an $(\infty, 0)$-category, we can state the framed version of the cobordism hypothesis as follows.

Theorem 7.2.5. *(The Cobordism Hypothesis) There is an equivalence of (∞, n)-categories*

$$Fun((Bord_n^{fr})^{\sqcup}, \mathcal{C}^{\otimes}) \cong \widetilde{d\mathcal{C}}$$

established by evaluation of the TFT on the n-framed point.

7.3 Fully Dualizable Tensor Categories

We will now sketch the description of the 3-category of tensor categories as introduced in [5]. We will begin by reminding the reader of the definition of tensor categories.

Definition 7.3.1. i) Let $\overline{\mathrm{Vect}_k}$ denote the category of (possibly infinite dimensional) vector spaces.

ii) Let $\mathrm{Vect}(k)$ denote the category of finite dimensional vector spaces.

iii) A tensor category is an abelian rigid monoidal category enriched in $\overline{\mathrm{Vect}_k}$ in a compatible way.[1]

Example 7.3.2. $\overline{\mathrm{Vect}_k}$ itself is a tensor category.

As discussed in the introduction we wish to introduce a symmetric monoidal 3-category of tensor categories, denoted TC. The objects of TC should be tensor categories and morphisms are module categories, etc. For this purpose we need a tensor product of tensor categories -the monoidal structure on TC and a tensor product of module categories -the composition of morphisms. We need some finiteness restrictions to ensure that these structures indeed exist.

Definition 7.3.3. A tensor category $\mathcal{C}$ is finite if

i) $\mathcal{C}$ is enriched in $\mathrm{Vect}(k)$ i.e. enriched over finite dimensional vector spaces.

ii) Each object has finite length i.e. every decreasing chain of subobjects $X = X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq \dots$ has finite length.

iii) $\mathcal{C}$ has enough projectives i.e. for each object X there is a projective object P that surjects onto X.

iv) Has finitely many isomorphism classes of simple objects.

Example 7.3.4. A linear category (abelian enriched over $\overline{\mathrm{Vect}_k}$) is finite iff it is equivalent to the category of finite dimensional modules of a finite dimensional algebra. In particular $\mathrm{Vect}(k)$ is an example of a finite tensor category.

We will not completely spill out the definition of $\mathcal{C}$-module categories and module functors between them for a tensor category $\mathcal{C}$, they are the obvious categorifications of modules of some algebra, and the natural morphisms between them, see [5]. We are, however, still left the task of defining tensor products, and composition of module categories.

Definition 7.3.5. Let $\mathcal{C}$ be a tensor category, M a right $\mathcal{C}$-module category, N a left $\mathcal{C}$-module category and L a linear category.

i) A *$\mathcal{C}$-balanced functor* $F : M \times N \to L$ is a bilinear functor and a natural isomorphism

$$F(m \otimes c, n) \cong F(m, c \otimes n).$$

[1] For example, the additive structure on hom-spaces needs to agree with the enrichment in vector spaces.

ii) The *relative Deligne tensor product* of M and N is a linear category $M \boxtimes_{\mathcal{C}} N$ together with a $\mathcal{C}$-balanced functor $\boxtimes_{\mathcal{C}} : M \times N \to M \boxtimes_{\mathcal{C}} N$ such that for every linear category L there is an equivalence of functor categories

$$\{F : M \boxtimes_{\mathcal{C}} N \to L : F \text{is a linear functor}\} \cong \{F : M \times N \to L : F \text{ is a } \mathcal{C}\text{-balanced functor}\}.$$

Theorem 7.3.6. *[5, thm 3.2.17] For $\mathcal{C}$ a finite tensor category, M a finite right $\mathcal{C}$-module category and N a left $\mathcal{C}$-module category the relative Deligne tensor product $M \boxtimes_{\mathcal{C}} N$ exists.*

□

Any finite tensor category carries an essentially unique structure of a Vect(k) bimodule category (unique up to unique isomorphism). Hence as a corollary we obtain a tensor product of finite tensor categories $\mathcal{C}$ and $\mathcal{D}$ given by $\mathcal{C} \boxtimes_{\mathrm{Vect}(k)} \mathcal{D}$.

Quasi-Definition 7.3.7. Let TC denote the symmetric monoidal 3-category of tensor categories with

- Objects: finite tensor categories;
- 1-morphisms: finite bimodule categories;
- 2-morphisms: bimodule functors;
- 3-morphisms: bimodule natural transformations.

Remark 7.3.8. The proof of theorem 7.3.6, given as theorem 3.2.18 in [5], requires the base field k of Vect(k) to be perfect. Examples of perfect fields are algebraically closed fields, fields of characteristic zero and finite fields.

Having 'defined' TC this is a possible target category for fully extended 3-dimensional TFTs. Recall that the cobordism hypothesis tells us that such TFTs are in one-to-one correspondence with fully dualizable objects of TC i.e. fully dualizable tensor categories. In [5] the authors provide a complete classification of fully dualizable tensor categories over perfect fields: these are so-called separable tensor categories.

Remark 7.3.9. Discussing separability of tensor categories would require too much space, but one can think of the following analogue. In the 2-category of algberas, an algebra A is fully dualizable if it is finite dimensional and A is projective as an $A - A$ bimodule; such algebras are called separable.
A tensor category is called separable if the $\mathcal{C} - \mathcal{C}$-bimodule category $\mathcal{C}$ is equivalent to the module category of a separable algebra object in the tensor category $\mathcal{C} \boxtimes \mathcal{C}^{mp}$.[2]

Theorem 7.3.10. *Let k be a field of characteristic 0. A tensor category is fully dualizable in TC iff it is a finite semisimple tensor category.*

□

Finite semisimple tensor categories have been extensively studied by mathematicians - most notably by E.N.O. meaning P. Etingof, D. Nikshych and V. Ostrik. These categories are called **fusion categories** and are close to being categories of finite groups.

Example 7.3.11. i) G-mod for finite group G.

[2]Here the superscript *mp* indicates that it is the tensor opposite category i.e. the monoidal category with tensor product $\otimes^{op}$.

ii) Let G be a finite group. Vec_G denotes the category of G-graded vector spaces with graded morphisms. The tensor product is given by $(V \otimes W)_g = \oplus_{xy=g} V_x \otimes V_y$. A complete set of simple objects is given by $\{\delta_g\}_{g \in G}$ with $(\delta_g)_h = k$ if $h = g$ and $(\delta_g)_h = 0$ otherwise. Note that $\mathbb{1} = \delta_e$. Together with the normal associators, unity morphisms and duals of vector spaces this defines a fusion category.

iii) More generally, one can choose a 3-cocycle $\omega \in Z^3(G, k^\times)$ with which we twist Vec_G. The category Vec_G^ω is identical to Vec_G except that we redefine the associators, by extending the twisted associators $\tilde{\alpha}_{\delta_g,\delta_h,\delta_m} = \omega(g,h,m)\alpha_{\delta_g,\delta_h,\delta_m}$ to arbitrary direct sums of the δ_g.

iv) Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{C}$ and q a n^{th} root of unity for $n > 1$. The modular tensor category $\mathcal{M}_q$ associated to the quantum group $\mathcal{U}_q(\mathfrak{g})$ is in particular a fusion category.

Remark 7.3.12. i) Note that the fully extended TFT associated to the assignment of $\mathcal{M}_q$ to a point is the so-called "Turaev-Viro" TQFT, and **not** the Witten-Reshetikhin-Turaev TFT connected to the quantum group. Rather, the WRT TFT is a 3-2-1-TFT which assigns the category $\mathcal{M}_q$ to the circle.

ii) In fact, it is known that if a fusion category is assigned to a point its so-called 'Drinfeld center' will be assigned to the circle. Drinfeld centers have very specific properties, in particular one can show that $\mathcal{M}_q$ cannot be the Drinfeld center of a fusion category. Hence the WRT TFT is not fully extended as a TFT valued in the 3-category of tensor categories.

iii) Fusion categories are also of interest to physicists. These categories should describe the behaviour of anyons, quasi-particles in two-dimensional systems of electrons studied in solid-state physics.

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