KNOT HOMOLOGIES AND MATRIX FACTORIZATIONS

Abstract. These are notes from a lecture series by Tina Kanstrup at the ICMS Summer school "Geometric representation theory and low-dimensional topology" in June 2019. The notes are typed by Corina Keller and Wai-kit Yeung.

1. Lecture 1 - Link homologies and statement of the GNR conjecture

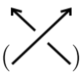
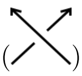
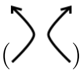
The goal of this lecture is to define Khovanov-Rozansky triply graded link homology and state the Gorsky-Negut-Rasmussen conjecture which provides a way of computing it in terms of sheaves on Hilbert schemes.

Definition 1.1. A *link* is a collection of knots which may be interlinked.

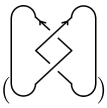
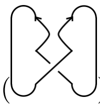
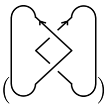
A main question in knot theory is to tell whether two links are isotopic. A partial answer to this question is to use link invariants to distinguish different isotopy classes. One of the most famous ones are the HOMFLY-PT polynomial.

Definition 1.2. The *HOMFLY-PT polynomial* $P_L(a,q)$ is defined by

$$P_L(\text{unknot}) = 1$$

$a\,P_L($  $) - a^{-1}\,P_L($  $) = (q-q^{-1})P_L($  $)$

Example 1.1.

$P($  $) = a^{-1}((q-q^{-1})P($  $) - a^{-1}P($  $))$

$$= a^{-1}(q-q^{-1}) - a^{-2}$$

1.1. Khovanov-Rozansky triply graded homology.

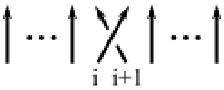
Khovanov-Rozansky triply graded homology assigns to a link a triply vector space $HHH(L)$ (with Q-grading, T-grading and A-grading that will be introduced in definition 1.4). It is a categorification of the HOMFLY-PT polynomial, i.e. it's Euler characteristic χ with respect to the T-grading is the HOMFLY-PT polynomial.

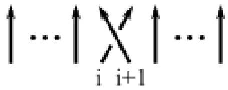
$$\chi(HHH(L)) = P_L(a,q)$$

The group of braids on n strands, with multiplication given by stacking braids on top of each other can be written in terms of generators and relations as follows

$$Br_n = \langle \sigma_1, \ldots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| > 1\rangle.$$

Here the generators represent the following elementary braids.

$\sigma_i =$ 

$\sigma_i^{-1} =$ 

Theorem 1.1 (Markov)**.** *There is a bijection between links up to isotopy and braids up to Markov moves. The bijection is given by closing up the braid (i.e. connecting the kth strand on the top to the kth strand on the bottom).*

- *Markov 1:* $\alpha\beta = \beta\alpha$ *for all* $\alpha, \beta \in Br_n$
- *Markov 2: For* $\beta \in Br_n$ *consider* β *as an element in* Br_{n+1} *by adding an extra strand. Then* $\beta = \beta\sigma_n^{\pm 1}$.

Soergel bimodules. For $n \in \mathbb{Z}_{\geq 1}$, let $R = \mathbb{Q}[x_1, \ldots, x_n]$ with $\deg(x_i) = 2$. The group S_n acts on R by permuting the generators x_i. For each simple reflection $s = (i, i+1) \in S_n$, let $R^s \subset R$ be the subring consisting of elements fixed by the action of s, i.e.

$$R^s = \{f \in R \mid s(f) = f\}.$$

Let B_s be the R-bimodule $B_s = R \otimes_{R^s} R(1)$, where (1) denotes the grading shift defined by $M(1)_i = M_{i+1}$, so that each $f \in M$ of degree i becomes of degree $i-1$ in $M(1)$. The tensor product that defines B_s can be described more explicitly as

$$R \otimes_{R^s} R = \frac{\mathbb{Q}[x_1, \ldots, x_n, x'_1, \ldots, x'_n]}{(x_i + x_{i+1} = x'_i + x'_{i+1}, \quad x_i x_{i+1} = x'_i x'_{i+1}, \quad x_j = x'_j \text{ for } j \neq i, i+1)}.$$

Definition 1.3. The category of *Soergel bimodules*, $\mathbb{S}Bim$, is the smallest full subcategory of the category of R-bimodules containing R and B_s for each simple reflection $s \in S$, which is closed under direct sums, tensor product $\otimes_R$, degree shifts (± 1), and direct summands.

Theorem 1.2 (Soergel)**.** *$\mathbb{S}Bim$ categorifies the Hecke algebra.*

To each simple reflection $s \in S$, one can associate the so called *Rouquier complexes* defined as follows.

$$F_s = [B_s \to R(1)] \qquad\qquad F_s^{-1} = [R(-1) \to B_s]$$
$$1 \otimes 1 \mapsto 1 \qquad\qquad 1 \mapsto x_i \otimes 1 - 1 \otimes x_{i+1}.$$

Theorem 1.3 (Rouquier)**.** *The complexes F_s and F_s^{-1} satisfy the braid relations and $F_s \otimes_R F_s^{-1} \simeq R$.*

Given a braid β, choose an expression $\beta = s_{i_1}^{\epsilon_1} \ldots s_{i_r}^{\epsilon_r}$. Set

$$F(\beta) := F_{s_{i_1}}^{\epsilon_1} \otimes_R \cdots \otimes_R F_{s_{i_r}}^{\epsilon_r},$$

This is well-defined up to canonical isomorphism in the homotopy category $K^b(\mathbb{S}Bim)$ of $\mathbb{S}Bim$.

Exercise 1.1. Show that there is an isomorphism of R-bimodules $B_s \otimes_R B_s \cong B_s(-1) \oplus B_s(1)$.

Example 1.2. Set $\beta = \sigma_s^2$. The corresponding Rouquier complex is

$$F_s \otimes_R F_s \cong \left[\begin{array}{ccccc} & & B_s(1) & & \\ & \nearrow & & \searrow & \\ B_s(1) \oplus B_s(-1) & & & & R(2) \\ & \searrow & & \nearrow & \\ & & B_s(1) & & \end{array} \right] \simeq \left[B_s(-1) \to B_s(1) \to R(2) \right].$$

Hochschield homology is a functor

$$HH : \text{graded } R\text{-bimodules} \to \text{bi-graded } \mathbb{Q}\text{-vector spaces}, \qquad B \mapsto \bigotimes_{i,j} \mathrm{Ext}^j_{R,R}(R, B(i))$$

Definition 1.4. (1) Given $X^\bullet = \cdots \to X^\ell \to X^{\ell+1} \to \ldots$ a complex of graded R-bimodules.

$$HHH^{i,j,k}(X^\bullet) := H_k(\cdots \to HH^{i,j}(X^\ell) \to HH^{i,j}(X^{\ell+1}) \to \ldots).$$

(2) *Triply graded link homology* is defined by $HHH(\beta) := HHH(F(\beta))$. It has 3 gradings:
   (a) Q-grading: grading on all B_s;
   (b) T-grading: homological grading on $F(\beta)$
   (c) A-grading: grading on HHH.

Theorem 1.4 (Khovanov)**.** *$HHH(\beta)$ is an invariant (up to grading shift) under Markov moves, and is therefore a topological invariant of links.*

A problem with this invariant is that it is very difficult to compute. For example, if a braid has m crossings, then the complex $F(\beta)$ has 2^m terms.

1.2. **Relation to Hilbert schemes.** The Gorsky-Negut-Rasmussen conjecture relates HHH to sheaves on Hilbert schemes. Recall the Hilbert scheme of points in the plane,

$$\mathrm{Hilb}_n(\mathbb{C}^2) := \{\text{ideals } I \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I = n\}.$$

The flag Hilbert scheme has the advantage over the usual Hilbert scheme that there exists a natural map $\mathrm{FHilb}_n(\mathbb{C}^2, \mathbb{C}) \to \mathrm{FHilb}_{n-1}(\mathbb{C}^2, \mathbb{C})$.

$$\mathrm{FHilb}_n(\mathbb{C}^2) := \{\text{ideals } I_n \subset \ldots I_1 \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I_k = k\}.$$

For our purpose we will use the the flag Hilbert scheme with support on $(y = 0)$

$$\mathrm{FHilb}_n(\mathbb{C}^2, \mathbb{C}) := \{\text{ideals } I_n \subset \ldots I_1 \subset \mathbb{C}[x,y] \mid \dim \mathbb{C}[x,y]/I_k = k \text{ and } \mathrm{supp}(\mathbb{C}[x,y]/I_k) \subset (y=0)\}.$$

The flag Hilbert scheme (both with and without support condition) has the disadvantage that it is very singular. Thus, we will replace it with its derived version $\mathrm{FHilb}_n^{dg}(\mathbb{C}^2, \mathbb{C})$, to be defined in lecture 3.

Conjecture 1.1 (Goresky-Negut-Rasmussen [GNR])**.** *There exists a pair of adjoint functors*

$$K^b(\mathbb{S}Bim) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^bCoh^{\mathbb{C}^\times \times \mathbb{C}^\times}(FHilb_n^{dg}(\mathbb{C}^2, \mathbb{C}))$$

such that

(1) *i^* is monoidal and fully faithful;*
(2) *there exists a canonical isomorphism $HHH(\beta) \cong H^*(i_*(F(\beta)))$ preserving all three gradings;*
(3) *The full twist $FT_k \in B_k$ corresponds to $(\det \mathcal{T}_k) \otimes \mathcal{O}_{FHilb_n^{dg}(\mathbb{C}^2,\mathbb{C})}$, where $\mathcal{T}_k$ is the tautological rank k vector bundle with $\mathcal{T}_k|_{I_n \subset \cdots \subset I_1} = \mathbb{C}[x,y]/I_k$.*

Oblomkov and Rozansky constructed a link invariant with similar properties. Let D_{per} denote the derived category of 2-periodic complexes of coherent sheaves.

Theorem 1.5 (Oblomkov-Rozansky, [OR])**.** *There is a constructive procedure that assigns to a braid $\beta \in Br_n$ an element $S_\beta \in D_{per}^{\mathbb{C}^\times \times \mathbb{C}^\times}(Hilb_n(\mathbb{C}^2))$, such that*

(1) *the triply graded space $HHH(\beta)$ is an invariant of the link $L(\beta)$;*
(2) *for $\mathbb{C}_a^\times \subset \mathbb{C}^\times \times \mathbb{C}^\times$ the anti-diagonal torus, we have*
$$\sum_i a^i \chi_q(\mathbb{C}_a^\times, H^*(S_\beta \otimes \wedge^i \mathcal{B})) = HOMFLY-PT(L(\beta));$$
(3) $S_{\beta \cdot FT_n} = S_\beta \otimes \det(B)$,

where $\mathcal{B}$ is the bundle dual to the universal quotient bundle $\mathcal{B}^\vee$ whose fiber above $I \in Hilb_n(\mathbb{C}^2)$ is defined to be the n-dimensional vector space $\mathbb{C}[x,y]/I$.

The goal is to relate the [OR] theorem to the [GNR] conjecture.

2. Lecture 2 - Oblomkov-Rozansky link invariant in terms of matrix factorizations

This lecture is an exposition of the paper "Knot homology and sheaves on the Hilbert scheme of points" by Alexei Oblomkov and Lev Rozansky. Alexei also gave a lecture series at CIME in June 2018 on this topic. Notes are available as arXiv:1901.04052.

2.1. Matrix factorizations.

Definition 2.1. Let Z be an affine scheme and F a function on it. A *matrix factorization* on Z with *potential* F is a quadruple (M^0, M^1, D^0, D^1), where

- M^0, M^1 are vector bundles on Z;
- D_0, D_1 are maps $M^0 \xrightarrow{D^0} M^1 \xrightarrow{D^1} M^0$ satisfying $D^0 \circ D^1 = F \cdot id = D^1 \circ D^0$.

Remark 2.1. As a shorthand notation we often write $M = (M^0, M^1)$ and $D = (D^0, D^1)$.

Given two matrix factorization $\mathcal{F}_1 = (M_1, D_1)$, $\mathcal{F}_2 = (M_2, D_2)$, define

$$\underline{Hom}(\mathcal{F}_1, \mathcal{F}_2) := \{f \in \underline{Hom}(M_1, M_2) \mid D^2 \circ f = (-1)^{deg(f)} f \circ D^1\},$$

which is $\mathbb{Z}/2$-graded by $\underline{Hom}^i(M_1, M_2) = Hom(M_1^0, M_2^i) \oplus Hom(M_1^1, M_2^{i+1})$ for $i = 0, 1$. We want a homotopy category version of this. Two elements $\psi, \phi \in \underline{Hom}^0(\mathcal{F}_0, \mathcal{F}_1)$ are homotopic, written $\psi \sim \phi$, if there exists an element $H \in \underline{Hom}^1(\mathcal{F}_0, \mathcal{F}_1)$ such that

$$\psi - \phi = H \circ D_1 + D_2 \circ H.$$

We define the morphisms in the (homotopy category of) matrix factorizations to be

$$Hom_{MF}(\mathcal{F}_1, \mathcal{F}_2) := \underline{Hom}(\mathcal{F}_1, \mathcal{F}_2)/\sim .$$

Denote the resulting category by $MF(Z, F)$.

Theorem 2.1 (Orlov)**.** *$MF(Z, F)$ is a triangulated category.*

Equivariant matrix factorizations. So far we introduced matrix factorizations for affine schemes. Suppose now that the affine scheme Z has an action of the algebraic group G and $F \in \mathbb{C}[Z]^G$. The idea is to explore matrix factorizations on schemes that are group quotients of affine ones.

Remark 2.2. Given a matrix factorization $\mathcal{F} = (M, D) \in MF(Z, F)$, one can naturally require that M is endowed with a G-representation structure and that the differential D is G-equivariant. Such an $\mathcal{F}$ is called *strongy G-equivariant* and the corresponding homotopy category is denoted $MF_G^{strong}(Z, F)$.

For some calculations on strongly equivariant matrix factorizations it can happen that the differentials in the result are not equivariant. Therefore, we introduce a weaker notion of G-equivariance in which we allow "correcting differentials". This notion relies on the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ of G. Recall that the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ is the complex

$$CE(\mathfrak{g}) = (V_\bullet(\mathfrak{g}), d_{CE}), \quad V_p(\mathfrak{g}) = U(\mathfrak{g}) \otimes \Lambda^p \mathfrak{g}, \quad d_{CE} = d_0 + d_1,$$

where

$$d_1(u \otimes v_1 \wedge \cdots \wedge v_n) = \sum_{i=1}^{p} (-1)^i u v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p$$

and

$$d_2(u \otimes v_1 \wedge \cdots \wedge v_n) = \sum_{i<j} u \otimes [v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_p.$$

Here $\hat{}$ indicates that the factor is left out. Consider the map $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ defined by $v \mapsto v \otimes 1 + 1 \otimes v$. If V, W are modules over the Lie algebra $\mathfrak{g}$, we denote by $V \bar{\otimes} W$ the $\mathfrak{g}$-representation that is isomorphic to $V \otimes W$ as a vector space with the $\mathfrak{g}$-action coming from Δ. Recall that the Chevalley-Eilenberg complex $CE(\mathfrak{g})$ is a resolution of the trivial $\mathfrak{g}$-module. A slight modification of this standard result implies that $CE(\mathfrak{g}) \bar{\otimes} M$ is a resolution of the $\mathfrak{g}$-module M.
We are now ready to define a new category $MF_{\mathfrak{g}}(Z, F)$, whose objects are *weakly equivariant matrix factorizations.*

Definition 2.2. Let $MF_{\mathfrak{g}}(Z, F)$ be the category whose objects are triples

$$\mathcal{F} = (M, D, \partial), \quad (M, D) \in MF(Z, F),$$

where $M = (M^0, M^1)$, with $M^i = \mathbb{C}[Z] \otimes V^i$ for V^i a $\mathfrak{g}$-representation, and

$$\partial \in \bigoplus_{i>j} Hom_{\mathbb{C}[Z]}(\Lambda^i \mathfrak{g} \otimes M, \Lambda^j \mathfrak{g} \otimes M).$$

satisfying that the total differential, $D_{tot} := D + d_{CE} + \partial$, is an endomorphism of $CE(\mathfrak{g}) \bar{\otimes} M$ which commutes with the $U(\mathfrak{g})$-action and $D_{tot}^2 = F$.
Given $\mathcal{F} = (M, D, \partial)$ and $\mathcal{F}' = (M', D', \partial')$ define

$\underline{Hom}(\mathcal{F}, \mathcal{F}') :=$

$$\{\psi \in Hom_{\mathbb{C}[Z]}(CE(\mathfrak{g}) \bar{\otimes} M, CE(\mathfrak{g}) \bar{\otimes} M') \mid \psi \circ D_{tot} = D'_{tot} \circ \psi \text{ and } \psi \text{ commutes with the U}(\mathfrak{g})\text{-action}\}.$$

Morphisms ψ and ψ' are homotopic if there exist $h \in Hom_{\mathbb{C}[Z]}(CE(\mathfrak{g})\bar{\otimes}M, CE(\mathfrak{g})\bar{\otimes}M')$ such that $\psi - \psi' = D'_{tot} \circ h - h \circ D_{tot}$ and h commutes with the U($\mathfrak{g}$)-action. We define the morphisms in $MF_{\mathfrak{g}}(Z, F)$ to be

$$Hom_{MF_{\mathfrak{g}}}(\mathcal{F}, \mathcal{F}') := \underline{Hom}(\mathcal{F}, \mathcal{F}')/\sim .$$

Remark 2.3. There is a natural inclusion

$$MF_G^{strong}(Z, F) \hookrightarrow MF_{\mathfrak{g}}(Z, F)$$
$$(M, D) \mapsto (M, D, 0).$$

Functors. Given two $\mathfrak{g}$-equivariant matrix factorizations $\mathcal{F}_1 = (M_1, D_1, \partial_1)$ and $\mathcal{F}_2 = (M_2, D_2, \partial_2)$. We define their *tensor product* as

$$\mathcal{F}_1 \otimes \mathcal{F}_2 = (M_1 \otimes M_2, D_1 \otimes 1 + 1 \otimes D_2, \partial_1 \otimes 1 + 1 \otimes \partial_2).$$

Suppose we have an equivariant morphism $f : Z_1 \to Z_2$ of affine schemes and $f^*F_2 = F_1$. Since the pullback of a free module is free, we get a *pullback* functor

$$f^* : MF_{\mathfrak{g}}(Z_2, F_2) \to MF_{\mathfrak{g}}(Z_1, f^*Z_2).$$

Moreover, for $\pi : X \times Y \to Y$ a $\mathfrak{g}$-equivariant projection map, we can define a *pushforward* functor along the projection

$$\pi_* : MF_{\mathfrak{g}}(X \times Y, \pi^*F) \to MF_{\mathfrak{g}}(Y, F).$$

Remark 2.4. A pushforward functor can also be defined for embeddings $Z_1 \hookrightarrow Z_2$, where Z_1 is the common zero of an ideal $I = (t_0, \dots, t_n)$, which is such that the t_i's form a regular sequence. The definition is a bit cumbersome so we refer to the Oblomkov-Rozansky paper "Knot homology and sheaves on the Hilbert scheme of points".

Finally, we want to define the quotient map. To that end, let $\mathcal{F} \in (M, D, \partial) \in MF_{\mathfrak{g}}(Z, F)$. The 'derived' version of taking the $\mathfrak{g}$-invariant part of $\mathcal{F}$ is

$$CE_{\mathfrak{g}}(\mathcal{F}) = (CE_{\mathfrak{g}}(M), D + d_{CE} + \partial) \in MF(Z/G, F),$$

where by Z/G we mean $Spec(\mathbb{C}[X]^{\mathfrak{g}})$ and $CE_{\mathfrak{g}}(M) = Hom_{\mathfrak{g}}(CE(\mathfrak{g}), CE(\mathfrak{g})\widetilde{\otimes}M)$.

2.2. **Braid group action.** We want to construct an action of the affine braid group on matrix factorizations. We begin by explaining a construction of the convolution algebra on matrix factorizations.
Convolution product. Let G be a reductive algebraic group over $\mathbb{C}$, $B \subset G$ a Borel subgroup and $T \subset B$ a maximal torus. Denote $\mathfrak{g}$, $\mathfrak{b}$ and $\mathfrak{t}$ the corresponding Lie algebras. Moreover, let us denote by $\mathfrak{n} \subset \mathfrak{b}$ the nilpotent radical.

Remark 2.5. In the case of interest, we can take $G = GL_n$, then we can think of B as the upper triangular matrices and of T as the diagonal matrices. The nilpotent radical $\mathfrak{n}$ is the Lie algebra of strictly upper triangular matrices.

We introduce a convolution operation in the category of matrix factorizations on the space $\mathscr{X}_2 = \mathfrak{g} \times (G \times \mathfrak{n})^2$. Let $p_1, p_2 : \mathscr{X}_2 \to \mathfrak{g} \times G \times \mathfrak{n}$ be the two projections. Notice that the space $\mathscr{X}_1 = \mathfrak{g} \times G \times \mathfrak{n}$ has an action of G and B given by

$$(h, b) \cdot (X, g, Y) = (Ad_h(X), hgb, Ad_b^{-1}(Y)), \quad (h, b) \in G \times B.$$

Furthermore, $\mathbb{C}^\times \times \mathbb{C}^\times$ acts on $\mathscr{X}_1$ by

$$(\lambda, \nu) \cdot (X, g, Y) = (\lambda^2 X, g, \lambda^{-2}\nu^2 Y).$$

And lastly, there is a moment map $\mu : \mathscr{X}_1 \to \mathbb{C}$ defined by

$$\mu(X, g, Y) = Tr(X Ad_g Y).$$

We define our *convolution category* to be

$$MF_{G \times B^2}(\mathscr{X}_2, W), \quad W = p_1^*(\mu) - p_2^*(\mu),$$

Since $B = TU$ is not reductive we require strong $G \times T^2$-equivariance, but only weak U^2-equivalence. There are three $G \times B^3$-equivariant maps $\pi_{ij} : \mathscr{X}_3 = \mathfrak{g} \times (G \times \mathfrak{n})^3 \to \mathscr{X}_2$ for $1 \leq i < j \leq 3$, which we use to define the associative *convolution product*

$$* : MF_{G\times B^2}(\mathscr{X}_2, W) \times MF_{G\times B^2}(\mathscr{X}_2, W) \to MF_{G\times B^2}(\mathscr{X}_2, W)$$
$$(\mathcal{F}, \mathcal{G}) \mapsto \mathcal{F} * \mathcal{G} = \pi_{13*}(CE_{\mathfrak{n}^{(2)}}(\pi_{12}^*\mathcal{F} \otimes \pi_{23}^*\mathcal{G})^{T^{(2)}}).$$

Knorrer reduction. There is a procedure called Knorrer reduction which for certain kinds of matrix factorization categories gives an equivalence with a matrix factorization category on a smaller space. In our case of interest it gives an equivalence of categories

$$MF_{G\times B^2}(\mathscr{X}_2, W) \simeq MF_{B^2}(\overline{\mathscr{X}}, \overline{W}),$$

where $\overline{\mathscr{X}} = \mathfrak{b} \times G \times \mathfrak{n}$ and $\overline{W}(X, g, Y) = Tr(XAd_gY)$. Here the B^2-equivariance on the right hand side is also strong T^2-equivariance and weak U^2-equivariance. The equivalence in particular gives a convolution product on $MF_{B^2}(\overline{\mathscr{X}}, \overline{W})$. The advantage is that convolution products are easier to compute in $MF_{B^2}(\overline{\mathscr{X}}, \overline{W})$, since it involves fewer matrices. We'll establish an equivalence between these two categories by other means in lecture 3.
Braid group action on matrix factorizations. For a subgroup $H \subset G$, set $\overline{\mathscr{X}}(H) = \mathfrak{b} \times H \times \mathfrak{n}$. The embedding $i : \overline{\mathscr{X}}(B) \hookrightarrow \overline{\mathscr{X}}$ gives a pushforward $i_* : MF_{B^2}(\overline{\mathscr{X}}(B), 0) \to MF_{B^2}(\overline{\mathscr{X}}, \overline{W})$.
Let $\mathbb{C}[\overline{\mathscr{X}}(B)]$ be the matrix factorization sitting only in homological degree 2. Then $\mathbb{1} := i_*(\mathbb{C}[\overline{\mathscr{X}}(B)])$ is the unit of convolution.

Example 2.1. *($n = 2$)* Consider the matrices

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}, \qquad X = \begin{bmatrix} x_{11} & x_{12} \\ 0 & x_{22} \end{bmatrix}, \qquad Y = \begin{bmatrix} 0 & y_{12} \\ 0 & 0 \end{bmatrix}$$

and take

$$\overline{W}(X, g, Y) := \frac{1}{\det(g)}(y_{12}(2g_{11}x_{11} + g_{21}x_{12})g_{21})$$

The matrix factorization for the positive crossing is

$$\overline{\mathcal{C}}_+ := (\mathbb{C}[\overline{\mathscr{X}}] \otimes \wedge\langle\theta\rangle, D, 0, 0) \in MF_{B^2}(\overline{\mathscr{X}}, \overline{W})$$

$$D = \frac{g_{12}y_{12}}{\det(g)}\theta + (g_{11}(x_{11} - x_{22}) + g_{21}x_{12})\frac{\partial}{\partial\theta}$$

Take the characters

$$\chi_1\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = a\,, \qquad \chi_2\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = c$$

then $\overline{\mathcal{C}}_- := \overline{\mathcal{C}}_+\langle -\chi_1, \chi_2\rangle$

Induction functors. One can use the induction functors to get all simple reflections.
Let $P_k \subset G$ be the k-th standard parabolic subgroup (i.e., its Lie algebra is generated by $\mathfrak{b}$ and $E_{i,i+1}$ for $i \neq k$). Consider the diagram

$$\begin{array}{ccc} P_k & \hookrightarrow & G \\ \downarrow & & \\ G_k \times G_{n-k} & & \end{array}$$

This induces maps

$$\bar{i}_k : \overline{\mathscr{X}}(P_k) \to \overline{\mathscr{X}} \quad \text{and} \quad \bar{p}_k : \overline{\mathscr{X}}(P_k) \to \overline{\mathscr{X}}_k \times \overline{\mathscr{X}}_{n-k}$$

which then induces

$$\overline{ind}_k := \bar{i}_k \circ \bar{p}_k : MF_{B_k^2}(\overline{\mathscr{X}}_k, \overline{W}) \times MF_{B_{n-k}^2}(\overline{\mathscr{X}}_{n-k}, \overline{W}) \to MF_{B_n^2}(\overline{\mathscr{X}}_n, \overline{W})$$

For the positive crossing at the k-th strand, we associate

$$\overline{\mathcal{C}}_+^{(k)} := \overline{ind}_{k+1}(\overline{ind}_{k-1}(\mathbb{1}_{k-1} \times \overline{\mathcal{C}}_+) \times \mathbb{1}_{n-k-1})$$

Proposition 2.1. *These satisfy braid relations.*

As a result, for any braid $\beta = \sigma_{i_1}^{\epsilon_1} \dots \sigma_{i_r}^{\epsilon_r}$, the object $\overline{\mathcal{C}}_\beta := \overline{\mathcal{C}}_{\epsilon_1}^{(i_1)} \dots \overline{\mathcal{C}}_{\epsilon_r}^{(i_r)}$ is well-defined. This constitutes a braid group action.

2.3. **Knot invariants.** We specialize to $G = GL_n$. The Hilbert schemes that we considered in the first lecture can be alternatively written in the following form.

$$\begin{aligned}
\mathrm{Hilb}_n(\mathbb{C}^2) &:= \{(x,y,v) \in \mathfrak{g} \times \mathfrak{g} \times \mathbb{C}^n \mid [x,y]=0, \mathbb{C}\langle x,y\rangle v = \mathbb{C}^n\}/G \\
\mathrm{FHilb}_n(\mathbb{C}^2) &:= \{(x,y,v) \in \mathfrak{b} \times \mathfrak{b} \times \mathbb{C}^n \mid [x,y]=0, \mathbb{C}\langle x,y\rangle v = \mathbb{C}^n\}/B \\
\mathrm{FHilb}_n(\mathbb{C}^2,\mathbb{C}) &:= \{(x,y,v) \in \mathfrak{b} \times \mathfrak{n} \times \mathbb{C}^n \mid [x,y]=0, \mathbb{C}\langle x,y\rangle v = \mathbb{C}^n\}/B
\end{aligned}$$

Oblomkov and Rozansky work with a Hilbert scheme $\widetilde{\mathrm{FHilb}}^{free}$ defined as

$$\widetilde{\mathrm{FHilb}}^{free} := \{(x,y,v) \in \mathfrak{b} \times \mathfrak{n} \times \mathbb{C}^n \mid \mathbb{C}\langle x,y\rangle v = \mathbb{C}^n\}.$$

Notice that this Hilbert scheme admits a $\mathbb{C}^\times \times \mathbb{C}^\times$-action by rescaling of matrices. Finally, define

$$\mathrm{FHilb}^{free} := \widetilde{\mathrm{FHilb}}^{free}/B.$$

Note that there is an embedding

$$\begin{aligned}
j_e : \widetilde{\mathrm{FHilb}}^{free} &\to \overline{\mathscr{X}} \\
(X,Y,v) &\mapsto (X,e,Y),
\end{aligned}$$

and hence one gets a *'closure of the braid'* map. Note that matrix factorizations with potential zero is just the usual the usual derived category of 2-periodic complexes D_{per}.

$$L : MF_{B^2\times\mathbb{C}^*\times\mathbb{C}^*}(\overline{\mathscr{X}}, \overline{W}) \xrightarrow{j_e^*} MF_{B\times\mathbb{C}^*\times\mathbb{C}^*}(\widetilde{\mathrm{FHilb}}^{free}, 0) \to D^{per}_{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FHilb}^{free}),$$

We can now state the main theorem due to Oblomkov and Rozansky.

Theorem 2.2 (Oblomkov, Rozansky [OR])**.** *For any $\beta \in Br_n$ the doubly graded vector space*

$$H^k(\beta) = \mathbb{H}(L(\bar{C}_\beta) \otimes \Lambda^{(k+d(\beta)-n-1)/2}\mathcal{B})$$

is an isotopy invariant of the braid closure $L(\beta)$, where

1. $\mathbb{H}$ *denotes hypercohomology;*
2. $L(\bar{C}_\beta) \in D^{per}_{\mathbb{C}^\times\times\mathbb{C}^\times}(FHilb^{free})$*;*
3. $d(\beta)$ *is the difference between positive and negative crossings;*
4. $\mathcal{B}$ *is the dual bundle to the universal quotient bundle on $FHilb^{free}$.*

3. Lecture 3 - Relating OR theorem and GNR conjecture

The goal of this lecture if to relate the Oblomkov-Rozansky (OR) theorem and the Gorsky-Negut-Rasmussen (GNR) conjecture, where the latter is recalled below.

Conjecture 3.1 (Goresky-Negut-Rasmussen)**.** *There exists a pair of adjoint functors*

$$K^b(\mathbb{S}Bim) \underset{i^*}{\overset{i_*}{\rightleftarrows}} D^bCoh^{\mathbb{C}^\times\times\mathbb{C}^\times}(FHilb_n^{dg}(\mathbb{C}^2,\mathbb{C})) \ ,$$

such that

1. i^* *is monoidal and fully faithful;*
2. *there exists a canonical isomorphism $HHH(\beta) \cong H^*(i_*(F(\beta)))$ preserving all three gradings.*

3.1. **Coherent sheaves on dg-schemes.** Throughout let X be a smooth scheme and $\mathcal{A} = \bigoplus_i \mathcal{A}^i$ a $\mathbb{Z}$-graded sheaf of $\mathcal{O}_X$-dg-algebras.

Definition 3.1. An $\mathcal{A}$-*dg-module* $\mathcal{F} = \bigoplus_i \mathcal{F}^i$ is a sheaf of $\mathbb{Z}$-graded left $\mathcal{A}$-modules. We call $\mathcal{F}$

(1) *quasi-coherent* is each $\mathcal{F}^i$ is a quasi-coherent $\mathcal{O}_X$-module;
(2) *coherent* if it is quasi-coherent and its cohomology $H(\mathcal{F})$ is coherent over $H(\mathcal{A})$.

In the following, let G be an algebraic group.

Definition 3.2. Let X and $\mathcal{A}$ be as above and let $\mathcal{F}$ be an $\mathcal{A}$-dg-module. Moreover, let X be endowed with a G-action and assume that each $\mathcal{A}^i$ is G-equivariant and the differential and multiplication is also G-equivariant. Then $\mathcal{F}$ is called G-*equivariant* if each $\mathcal{F}^i$ is G-equivariant and the differential and the action are G-equivariant.

The derived category of G-equivariant coherent left dg-modules over the $\mathcal{O}_X$-dg-algebra $\mathcal{A}$ is denoted by $DCoh^G(\mathcal{A})$.

Example 3.1. *(G-equivariant coherent left dg-modules)*

(1) $DCoh^G(\mathcal{O}_X) \simeq D^bCoh(X)$
(2) Let G act on two smooth complex algebraic varieties X, Y and on a vector space V. Moreover, let $s_X : X \to V \leftarrow Y : s_Y$ be two G-equivariant maps. If $X \times_V Y$ is smooth then there is an exact sequence

$$0 \to \mathcal{O}_{X\times Y} \otimes \Lambda^{dim(V)}V^* \to \cdots \to \mathcal{O}_{X\times Y} \otimes \Lambda^2 V^* \to \mathcal{O}_{X\times Y} \otimes V^* \to \mathcal{O}_{X\times Y} \to \mathcal{O}_{(X\times Y)\times_V\{0\}}.$$

where the differential is given by $d(f) = f(s)$, with $s = s_X - s_Y$, and then extended by Leibniz rule. In the case we're interested in this is not the case, so instead we take the exact part and define the DG-algebra $\mathcal{O}_{X\times^R_V Y} := \mathcal{O}_X \otimes \mathcal{O}_Y \otimes \Lambda^\bullet V^*$. We then define

$$DCoh^G(X \times^R_V Y) := DCoh^G(X \times_V Y, \mathcal{O}_{X\times^R_V Y}).$$

The derived version of the flag Hilbert scheme with support $(y = 0)$ defined in lecture 1 that appears in the GNR conjecture is defined as

$$\mathrm{FHilb}^{dg}_n(\mathbb{C}^2, \mathbb{C}) = \mathrm{FHilb}^{free} \times^R_{\mathfrak{n}} \{0\},$$

where the map $\mathrm{FHilb}^{free} \to \mathfrak{n}$ is given by $(x, y, v) \mapsto [x, y]$.

3.2. **Relation between dg-schemes and matrix factorizations.** In the following, let X be a smooth complex algebraic variety and G a complex reductive algebraic group acting on X. Moreover, let $\pi : E \to X$ be a G-equivariant vector bundle, $s : X \to E$ a G-equivariant section and $\pi^\vee : E^\vee \to X$ the dual bundle. Set

$$W : E^\vee \xrightarrow{(id,(\pi^\vee)^* s)} E^\vee \times_X E \xrightarrow{\langle -,-\rangle} \mathbb{C},$$

and we assume that W is not a zero divisor. One can define a derived category $DMF^G(E^\vee, W)$ of matrix factorizations on $E^\vee$ with potential W as

$$DMF^G(E^\vee, W) = H^0(MF^{G,strong}(E^\vee, W))/\langle \text{locally contractible} \rangle,$$

where locally contracible means that there exists an open cover $\{U_i\}$ in the smooth topology such that $M|_{U_i} = 0$.

Remark 3.1. If X is affine, the quotient is not necessary.

Theorem 3.1 (Arkhipov, Kanstrup). *There is an equivalence of categories*

$$D^bCoh^G(s^{-1}(0)) \simeq DMF^{G\times\mathbb{C}^\times}(E^\vee, W),$$

where $s^{-1}(0)$ is the derived fiber product.

Remark 3.2. This is a G-equivariant version of a result by Isik [Is]

The key ingredient for the proof of the above theorem is the linear Koszul duality of Mirkovic and Riche [MR]. For simplicity, take E to be the trivial vector bundle $V \times X \to X$. Following the construction of Mirkovic and Riche we get an equivalence

$$DCoh^{G\times\mathbb{C}^\times}(\Lambda V \otimes \mathcal{O}_X[t]) \simeq DCoh^{G\times\mathbb{C}^\times}(\underbrace{\epsilon SymV \otimes \mathcal{O}_X \xrightarrow{s} SymV \otimes \mathcal{O}_X}_{\mathcal{D}}),$$

where t has internal degree -2 and homological degree 0, ϵ has internal degree 2 and homological degree -1 and $\epsilon^2 = 0$. Then, one has the following.

$$\begin{aligned} DCoh^G(s^{-1}(0)) &\simeq DCoh^{G\times\mathbb{C}^\times}(s^{-1}(0) \otimes_X \mathcal{O}_X[t, t^{-1}]) \\ &\simeq DCoh^{G\times\mathbb{C}^\times}(\mathcal{D})/\mathrm{Perf}, \end{aligned}$$

where Perf denotes the full subcategory of perfect complexes, i.e. complexes quasi-isomorphic to bounded complexes of vector bundles. The quotient category of coherent sheaves by perfect complexes is called the singularity category and is denoted D_{sing}. At this point notice that one can complete $\mathcal{D}$ to an exact sequence

$$0 \to \epsilon SymV \otimes \mathcal{O}_X \xrightarrow{s} SymV \otimes \mathcal{O}_X \to \pi_*^\vee \mathcal{O}_{W^{-1}(0)} \to 0.$$

Hence, we get

$$\begin{aligned} DCoh^{G\times\mathbb{C}^\times}(\mathcal{D})/\mathrm{Perf} &\simeq D_{sing}^{G\times\mathbb{C}^\times}(\pi_*^\vee \mathcal{O}_{W^{-1}(0)}) \\ &\simeq D_{sing}^{G\times\mathbb{C}^\times}(W^{-1}(0)) \\ &\simeq DMF^{G\times\mathbb{C}^\times}(E^\vee, W), \end{aligned}$$

where the last line follows from a result of Orlov, Polishchuk and Vaintrob [PV].

Example 3.2. *(Applications of theorem 3.2).*

(1) Let $X = \mathrm{FHilb}^{free}$, $E = \mathrm{FHilb}^{free} \times \mathfrak{n}$ and $s(x, y, v) = [x, y]$. Then the theorem tells us that
$$D^bCoh^{\mathbb{C}^*}(\mathrm{FHilb}^{dg}(\mathbb{C}^2, \mathbb{C})) \simeq DMF^{\mathbb{C}^*\times\mathbb{C}^*}(\mathrm{FHilb}^{free} \times \mathfrak{n}^*, W).$$

(2) Let $X = G \times \mathfrak{n}$, $E = \mathfrak{b}^* \times G \times \mathfrak{n}$ and $s(g, Y) = Tr(-Ad_g Y)$. Notice that $Tr(-Ad_g Y) = 0$ precisely if $Ad_g Y \in \mathfrak{n}$. Hence, we find
$$\begin{aligned} DMF^{B^2\times\mathbb{C}^*\times\mathbb{C}^*}(\mathfrak{b} \times G \times \mathfrak{n}, W) &\simeq D^bCoh^{B^2\times\mathbb{C}^*}((G \times \mathfrak{n}) \times^R_{\mathfrak{g}} \mathfrak{n}) \\ &\simeq D^bCoh^{G\times\mathbb{C}^*}((G \times \mathfrak{n})/B \times^R_{\mathfrak{g}} (G \times \mathfrak{n})/B). \end{aligned}$$

The variety $St := (G \times \mathfrak{n})/B \times^R_g (G \times \mathfrak{n})/B$ is called the Steinberg variety and it is a central object of study in geometric representation theory.

(3) Let $X = G\times\mathfrak{n}\times G\times\mathfrak{n}$, $E = \mathfrak{g}^*\times G\times\mathfrak{n}\times G\times\mathfrak{n}$ and $s(g_1, y_1, g_2, y_2) = Tr(-(Ad_{g_1} y_1) - Ad_{g_2} y_2)$. In this example we find again the Steinberg variety
$$\begin{aligned} DMF^{G\times B^2\times\mathbb{C}^*}(\mathfrak{g} \times G \times \mathfrak{n} \times G \times \mathfrak{n}, W) &\simeq D^bCoh^{G\times B^2\times\mathbb{C}^*}((G \times \mathfrak{n}) \times^R_{\mathfrak{g}} (G \times \mathfrak{n})) \\ &\simeq D^bCoh^{G\times\mathbb{C}^*}(St). \end{aligned}$$

The category $D^bCoh^{G\times\mathbb{C}^*}(St)$ categorifies the affine Hecke algebra (see [CG]) and it is often called the affine Hecke category. Another known categorification is the category of affine Soergel bimodules.

Theorem 3.2 (Bezrukavnikov-Yun [Be], [BY])**.** *There is an equivalence of categories*

$$D^bCoh^{G\times\mathbb{C}^*\times\mathbb{C}^*}(St) \simeq H^b(\mathit{Aff}\, \mathbb{S}Bim).$$

Remark 3.3. The category of affine Soergel bimodules contains Soergel bimodules but it is not a full subcategory. This is a major issue since HHH is calculated in terms of Hom in Soergel bimodules.

Action of the extended affine braid group. We have seen in the previous section that G-equivariant coherent sheaves on the Steinberg variety are related to affine Hecke algebras. It turns out to have an action of the affine braid group Br^{aff}. The latter has a (Bernstein) presentation with generators (T_s, θ_x) indexed by simple reflections s and weights $x \in X(T) = Hom(T, \mathbb{C}^\times)$, where $T \subset B$ is the maximal torus. Consider the map

$$\varphi : St \xrightarrow{proj} G/B \times G/B$$

For $s \in W$ a simple reflection consider the G-orbit $Y_s = G(B/B, sB/B) \subset G/B \times G/B$. Then define

$$Z_s = \overline{\varphi^{-1}(Y_s)}.$$

Moreover, let $\widetilde{g} = (G \times \mathfrak{n})/B$ and consider the projection ψ from the diagonal $\Delta(\widetilde{g}$ in St to G/B. For $x \in X(T)$ there is a canonical line bundle $\mathcal{O}_{G/B}(x)$ on G/B. We define

$$\mathcal{O}_\Delta(x) := \psi^* \mathcal{O}_{G/B}(x).$$

Recall that we have a convolution product, which in the case at hand is given by

$$* : D^b Coh^G(\widetilde{g} \times_{\mathfrak{g}}^R \widetilde{g}) \times D^b Coh^G(\widetilde{g} \times_{\mathfrak{g}}^R \widetilde{g}) \to D^b Coh^G(\widetilde{g} \times_{\mathfrak{g}}^R \widetilde{g})$$
$$(\mathcal{F}, \mathcal{G}) \mapsto p_{13*}(p_{12}^* \mathcal{F} \otimes p_{23}^* \mathcal{G}),$$

where $p_{ij} : \widetilde{g} \times_{\mathfrak{g}}^R \widetilde{g} \times_{\mathfrak{g}}^R \widetilde{g} \to \widetilde{g} \times_{\mathfrak{g}}^R \widetilde{g}$ is the projection to the i, j-factor.

Theorem 3.3 (Bezrukavnikov-Riche [BR]). *There is an action of the affine braid group on $D^b Coh(St)$ where T_s acts by convolution with $\mathcal{O}_{Z_s}$ and θ_x acts by convolution with $\mathcal{O}_\Delta(x)$.*

Proposition 3.1 (Kanstrup). *The equivalence $DMF^{B^2 \times \mathbb{C}^\times}(\mathfrak{b} \times G \times \mathfrak{n}, W) \simeq D^b Coh^G(St)$ from example 2 is monoidal. Under the equivalence the generator $\overline{C}_s$ in* [OR] *goes to the generator $\mathcal{O}_{Z_s}$ in* [BR].

The rest of the functor in [OR] (which corresponds to closing the braid) can also be translated into the setting of coherent sheaves.

Proposition 3.2 (Kanstrup). *The functor $j_e^* : MF^{B^2 \times \mathbb{C}^2 \times \mathbb{C}^2}(\mathfrak{b} \times G \times \mathfrak{n}, W) \to MF^{\mathbb{C}^2 \times \mathbb{C}^2}(FHilb^{free}, 0)$ becomes*

$$\widetilde{j_e^*} : D^b Coh^{B^2 \times \mathbb{C}^\times}((G \times \mathfrak{n}) \times_{\mathfrak{g}} \mathfrak{n}) \to D^b Coh^{\mathbb{C}^\times}(FHilb^{free})$$

with $\widetilde{j_e}(x, y, v) = (e, y, y)$.

3.3. **Understanding results in terms of derived algebraic geometry.** Let X be a derived stack. Recall that the loop space $\mathcal{L}(X)$ of X is the derived stack

$$\mathcal{L}(X) = Map(S_1, X) \simeq X \times_{X \times X}^R X,$$

where we use that $S^1 \simeq pt \amalg_{pt \amalg pt} pt$.

Example 3.3. *(Loop spaces)*

1. If X is a derived scheme, then $\mathcal{L}(X) \simeq \mathbb{L}_X[-1]$ (shifted cotangent complex).
2. Let BG denote the classifying space of G then $\mathcal{L}(BG) = G/G$.

For our purpose we'll need the notion of unipotent loop space. Note that the affinization of S^1 is $B\mathbb{G}_a$.

Definition 3.3. The *unipotent loop space* $\mathcal{L}^u(X) := Map(B\mathbb{G}_a, X)$ is the space of loops that factor through $B\mathbb{G}_a$.

Example 3.4. $\mathcal{L}^u(BG) \simeq G^u/G$, where G^u is completion along unipotent elements.

Steinberg varieties as loop spaces. Let $G^u \subset G$ be the completion along unipotent elements, $\widetilde{G}^u = \{(g, F) \in G^u \times G/B \mid g \in F\}$. Then Ben-Zvi and Nadler observed that

$$\begin{aligned} St_G/G &= \widetilde{G}^u/G \times_{\widetilde{G}/G} \widetilde{G}^u/G \\ &\simeq \mathcal{L}^u(B\backslash G/B). \end{aligned}$$

Theorem 3.4 (Ben-Zvi - Francis - Nadler [BFN]). *Consider the correspondence*

$$St_G/G \leftarrow \mathcal{L}(G^u/G) \times_{G^u/G} \widetilde{G}^u/G \rightarrow \mathcal{L}(G^u/G) = (G^u \times G) \times^R_G \{0\} =: Com^u/G,$$

where the map $G^u \times G \to G$ is the commutator so Com^u/G is the commuting variety. Pulling back and pushing forward gives an equivalence

$$Z(D^bCoh(St_G/G), pt) \simeq D^bCoh(Com^u_G/G),$$

where Z denotes the Drinfeld center.

Notice that passing to the Drinfeld center Z assures that Markov 1 is satisfied. However, Markov 2 does not hold in the affine setting. In order for Markov 2 to hold, we would need to pass to the finite Hecke algebra setting. Soergel bimodules are equivalent to D-modules on $B\backslash G/B$. The result below by Ben-Zvi and Nadler gives an equivalence with a category defined in terms of coherent sheaves on a unipotent loop space. Coherent sheaves on loop spaces comes with an S^1-action and hence it is a $\mathcal{O}(BS^1) = k[u]$-linear category, where u is a variable sitting in degree 2. Inverting u by tensoring the category by $\otimes_{k[u]} k[u, u^{-1}]$ gives a 2-periodic category which we denote by subscript loc.

Theorem 3.5 (Ben-Zvi, Nadler [BN]). *There is an equivalence of ∞-categories*

$$Coh_{[B\backslash G/B]}(\mathcal{L}^u(B\backslash G/B))^{B\mathbb{G}_a \times \mathbb{C}^*}_{loc} \simeq D\text{-}mod(B\backslash G/B),$$

where subscript $[B\backslash G/B]$ indicates the subcategory of coherent sheaves, which remain coherent when restricted to $B\backslash G/B$.

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