Skein Categories

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These notes were taken from a talk given by Juliet Cooke at the 2019 Geometric Representation Theory and Low-Dimensional Topology Summer School hosted at the ICMS in Edinburgh. First, the notion of a Skein Category will be defined. After this, a connection to factorization homology will be sketched.

1 Ribbon Categories and Skein Categories

We begin by stating a few necessary definitions.

Definition. We define a *ribbon category* $(\mathscr{V}, \otimes, \mathbb{1}, B_{x,y})$ to be a braided monoidal category which is rigid (i.e. $\mathscr{V}$ has dual objects x^* and morphisms $\mathbb{1} \to x \otimes x^*$ and $x \otimes x^* \to \mathbb{1}$ satisfying certain compatibility conditions) which is equipped with a morphism $\mathscr{O}_x : x \to x$ for all $x \in \mathrm{Ob}\mathscr{V}$ called a *twist map* such that

1. $\mathscr{O}_{x\otimes y} = B_{y,x}B_{x,y}(\mathscr{O}_x \otimes \mathscr{O}_y)$
2. $\mathscr{O}_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}$
3. $\mathscr{O}_{x^*} = \mathscr{O}_x^* : x^* \to x^*$

Now, fix $\Bbbk$ to be a commutative ring, which we will usually think of as $\mathbb{C}$. From now on, let $\mathscr{V}$ be a $\Bbbk$-linear ribbon category which has a strict monoidal structure. The example to keep in mind here is the category $\mathrm{Rep}_q^{\mathrm{fd}}(G)$ of finite dimensional representations of $\mathcal{U}_q(\mathfrak{g})$ where $\mathfrak{g}$ is a semisimple Lie algebra and $q \in \mathbb{C}$ not a root of unity.

Definition. A *ribbon graph* is a topological space R built from a finite number of the following:

1. Framed strands called *ribbons* with distinguished top and bottom bases and a specified direction which does not depend on the bases. If the direction points towards the top, then the strand is said to have *positive direction.* Otherwise, the strand has *negative direction.*
2. *Coupons* $C \cong [0, 1]^2$ with distinguished top and bottom bases.

where a top (resp. bottom) base of a ribbon may be attached either to its bottom (resp. top) base, or to the bottom (resp. top) base of a coupon.

Definition. A $\mathscr{V}$*-colored ribbon graph* is a ribbon graph such that

1. Strands are each labeled by a choice of object $v \in \mathrm{Ob}\mathscr{V}$

2. Coupons are labeled by morphisms f in the following manner. The domain of f is a tensor product of the labels of the each of the strands attached to the bottom base of the coupon, with the caveat that if the direction of a strand labeled by x is negative, then the tensorand of the domain corresponding to that strand is x^*. Similarly, a codomain of f is specified in the same way.

Definition. Let R be a $\mathscr{V}$-colored ribbon graph and let X be the set consisting of the unattached bases of the strands of R. A *$\mathscr{V}$-ribbon diagram* of a surface Σ is an embedding $d : R \hookrightarrow \Sigma \times [0,1]$ with restrictions $d\big|_X : X \hookrightarrow \Sigma \times \{0,1\}$ and $d\big|_{R-X} : R - X \hookrightarrow \Sigma \times [0,1]$.

Definition. $\mathrm{Ribbon}_{\mathscr{V}}(\Sigma)$ is the $\Bbbk$-linear category with

- Objects: finite sets of colored framed points with orientation.
- Morphisms: $\Bbbk$-linear combinations of $\mathscr{V}$-colored ribbon graphs R up to ambient isotopy of of R in $\Sigma \times [0,1]$ which fix the boundary $\Sigma \times \{0,1\}$ pointwise.
- Composition: given by scaling and stacking ribbon graphs in the $[0,1]$ coordinate, thought of as a vertical direction.

If we set $\Sigma = [0,1]^2$, then $\mathrm{Ribbon}_{\mathscr{V}}([0,1]^2)$ is a ribbon category, where $\otimes$ is given by scaling and stacking ribbon graphs in either coordinate of Σ, thought of as a horizontal stacking.

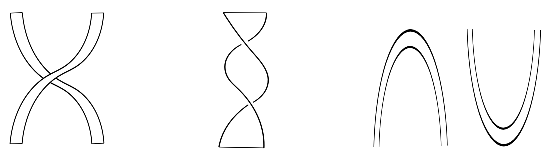
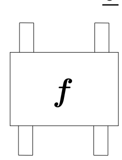


Figure 1: The braiding, twist, and duality morphisms

Theorem. (Turaev)
There is a surjective and full $\Bbbk$-linear ribbon functor

$$\mathrm{eval} : \mathrm{Ribbon}_{\mathscr{V}}([0,1]^2) \to \mathscr{V}$$

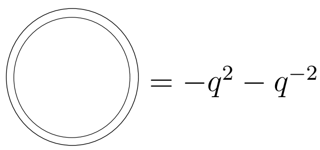
$$\underline{v} \mapsto v$$

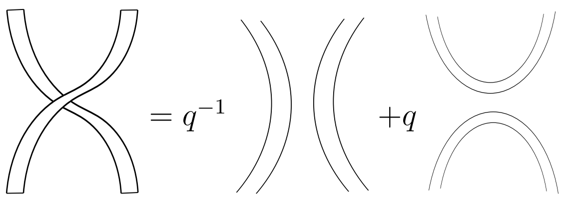


$$\mapsto f$$

Definition. The $\mathscr{V}$-skein category on Σ, $\mathbf{Sk}_{\mathscr{V}}(\Sigma)$, is the category $\mathrm{Ribbon}_{\mathscr{V}}(\Sigma)$ with the following relation imposed on morphisms: $F \sim G$ if F and G are equal outside of a cube and $\mathrm{eval}(F\big|_{cube}) = \mathrm{eval}(G\big|_{cube})$. Furthermore, the boundary of this cube intersects F and G with only a finite number of transverse strands on the top and bottom faces.

Example. Let $\mathscr{V} = \mathrm{Rep}_q^{fd}(\mathrm{SL}_2)$ be the category of finite dimensional representations of the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$. The relation imposed removes coupons, directions and colorings. Furthermore, the following relations are implied:





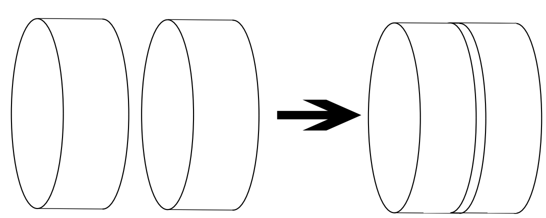
Thus, the category $\mathbf{Sk}_{\mathscr{V}}(\Sigma)$ is a category of tangles up to isotopy fixed in $\Sigma \times \{0,1\}$ subject to the Kauffman bracket skein relations. Furthermore, the endomorphism algebra $\mathrm{Hom}(\phi, \phi)$ is the Kauffman bracket skein algebra of Σ.

2 Factorization Homology

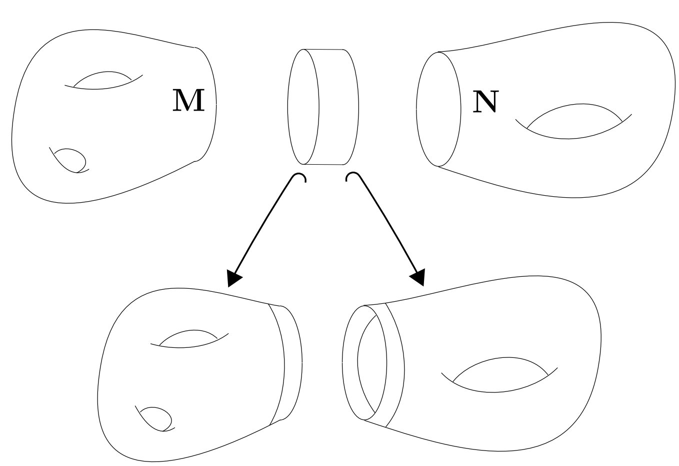
We have a monoidal functor

$$\mathbf{Sk}_{\mathscr{V}}(-) : \mathrm{Mfld}^{\amalg}_{2,\mathrm{fr}} \to \mathrm{Cat}^{\times}_{\Bbbk}$$

$\mathrm{Mfld}^{\amalg}_{2,\mathrm{fr}}$ is the category with objects framed, oriented surfaces and morphisms embeddings. $\mathrm{Cat}^{\times}_{\Bbbk}$ is the 2-category with objects small $\Bbbk$-linear categories with morphisms functors and 2-morphisms natural isomorphisms. We would like to talk about the excision property of factorization homology. Let C be a 1-manifold and let $A = C \times [0,1]$. $\mathbf{Sk}_{\mathscr{V}}(A)$ is a monoidal category with monoidal product induced by the topological gluing in the interval coordinate, with some small overlap:



If we have two objects M and N of $\mathrm{Mfld}^{\amalg}_{2,\mathrm{fr}}$ with embeddings of A as below, then $\mathbf{Sk}_{\mathscr{V}}(M)$ is a right $\mathbf{Sk}_{\mathscr{V}}(A)$-module and $\mathbf{Sk}_{\mathscr{V}}(N)$ is a left $\mathbf{Sk}_{\mathscr{V}}(A)$-module.



So we can define the relative tensor product $\mathbf{Sk}_{\mathscr{V}}(M) \bigotimes_{\mathbf{Sk}_{\mathscr{V}}(A)} \mathbf{Sk}_{\mathscr{V}}(N)$ as the colimit in $\mathrm{Cat}_{\Bbbk}^{\times}$ of the 2-sided bar construction of the diagram

$$\mathcal{M} \times \mathcal{N} \leftleftarrows \mathcal{M} \times \mathcal{A} \times \mathcal{N} \Lleftarrow \mathcal{M} \times \mathcal{A} \times \mathcal{A} \times \mathcal{N} \cdots$$

where $\mathcal{M} = \mathbf{Sk}_{\mathscr{V}}(M), \mathcal{N} = \mathbf{Sk}_{\mathscr{V}}(N), \mathcal{A} = \mathbf{Sk}_{\mathscr{V}}(A)$.

Theorem. (Cooke)

$$\mathbf{Sk}_{\mathscr{V}}(M \underset{A}{\cup} N) \simeq \mathbf{Sk}_{\mathscr{V}}(M) \bigotimes_{\mathbf{Sk}_{\mathscr{V}}(A)} \mathbf{Sk}_{\mathscr{V}}(N)$$

By [Turaev], we have $\mathbf{Sk}_{\mathscr{V}}([0,1]^2) \simeq \mathscr{V}$ and $\mathbf{Sk}_{\mathscr{V}}([0,1]^2)$ has a canonical monoidal structure gien by inclusion on manifolds. This yields an E_1-structure. These facts, together with the above theorem, characterize a factorization homology so the functor

$$\mathbf{Sk}_{\mathscr{V}}(-) : \mathrm{Mfld}_{2,\mathrm{fr}}^{\amalg} \to \mathrm{Cat}_{\Bbbk}^{\times}$$

is the $\Bbbk$-linear functor

$$\int_{-} \mathscr{V} : \mathrm{Mfld}_{2,\mathrm{fr}}^{\amalg} \to \mathrm{Cat}_{\Bbbk}^{\times}$$

of surfaces with coefficients in $\mathscr{V}$.

Note that the free cocompletion of $\mathbf{Sk}_{\mathrm{Rep}_q^{\mathrm{fd}}(G)}$ gives $\displaystyle\int_{-}^{\mathrm{Pr}} \mathrm{Rep}_q^{\mathrm{fd}}(G)$.