Introduction to MDPs and RL

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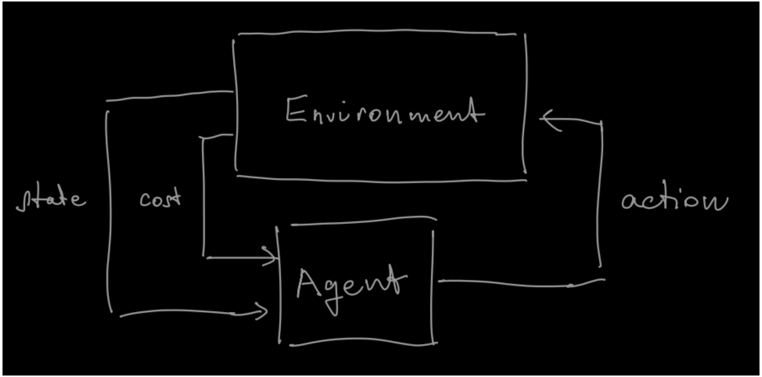
Outline

- MDPs and entropy regularized MDPs
  - Bellman principle
  - Policy and value iteration methods
- Multi-armed bandits, regret and other basic notions
- Q-learning
- Classical Policy Gradient (PG)
  - Performance difference lemma and policy gradient theorem
  - Difficulty of convergence analysis due to lack of convexity
  - Polyak–Łojasiewicz (PL) gradient dominance condition
- Mirror descent
  - Role of Performance difference as convexity and L-smoothness
  - Convergence rate MDP case with inexact advantage



RL Aim: learn to interact with an environment in an optimal (cost minimizing) way.

Data: $(s_t, a_t, c_t, s_{t+1}, a_{t+1}, \ldots)$.

Mathematical abstraction: MDP.

Key MDP / RL results

Overview of RL [Sutton and Barto, 2018] and results in discrete state-action spaces

- Classical policy gradient [Sutton et al., 1999].
- Natural policy gradient [Kakade, 2001].
- Actor-critic method [Haarnoja et al., 2018].
- Mirror descent method [Tomar et al., 2020].
- Convergence of classical PG in tabular setting [Mei et al., 2021].

Continuous state-action spaces: [Doya, 2000], [Van Hasselt, 2012], [Manna et al., 2022].

Entropy regularised: [Haarnoja et al., 2017, Geist et al., 2019].

MDP

Infinite-horizon Markov decision problem (S, A, P, c, γ):

- S is the state space, A is the action space
- $P \in \mathcal{P}(S|S \times A)$ is the transition probability kernel
- $c \in B_b(S \times A)$ is a cost function, and γ discount factor
- $H_n := (S \times A)^n \times S$ is the space of admissible histories

Aim: minimise the objective over policies $\alpha = (\alpha_n)_{n \in \mathbb{N}}$ s.t. $\alpha_n : H_n \to A$ measurable:

$$V^{\alpha}(s) = \mathbb{E}_s^{\alpha} \sum_{n=0}^{\infty} \gamma^n c(s_n, a_n) \in \mathbb{R} \cup \{+\infty\}, \tag{1}$$

with $a_n := \alpha_n(h_n)$, $h_n = (s_0, a_0, \ldots, s_{n-1}, a_{n-1}, s_n)$ and $s_{n+1} \sim P(\cdot|s_n, a_n)$, $s_0 = s$.

Infinite-horizon Markov decision model (S, A, P, c, γ):

- S is the state space, A is the action space,
- $P \in \mathcal{P}(S|S \times A)$ is the transition probability kernel,
- $c \in B_b(S \times A)$ is a cost function, and γ a discount factor,
- $H_n := (S \times A)^n \times S$ is the space of admissible histories,

Aim: minimise over relaxed policies $\pi = (\pi_n)_{n\in\mathbb{N}}$ s.t. $\pi_n : H_n \to P(A)$ measurable the objective:

$$V^\pi(s) = \mathbb{E}_s^\pi \sum_{n=0}^{\infty} \gamma^n \int_A c(s_n, a)\, \pi_n(da) \in \mathbb{R} \cup \{+\infty\}\,, \tag{2}$$

with $\pi_n := \pi_n(h_n)$, $h_n = (s_0, a_0, \ldots, s_{n-1}, a_{n-1}, s_n)$ and $s_{n+1} \sim \int_A P(\cdot|s_n, a)\, \pi_n(da)$, $s_0 = s$.

Optimal value is:

$$V^*(s) := \sup_\pi V^\pi(s)\,.$$

Bellman principle aka Dynamic programming for relaxed MDPs

Dynamic Programming Principle (DPP)

Assumption 1

1. *The kernel* $P \in \mathcal{P}(S|S \times A)$ *is strongly continuous, that is: for every* $v \in B_b(S)$ *(bounded and measurable) the function* $w(s,a) = \int_S v(s')P(\mathrm{d}s'|s,a)$ *is bounded and measurable as a function from* $S \times A$ *to* $\mathbb{R}$.
2. *The cost function* $c \in B_b(S \times A)$ *is lower semi-continuous and inf-compact on* $S \times A$ *i.e. for any* $s \in S$ *and any* $l \in \mathbb{R}$ *the set* $\{a \in A : c(s,a) \leq l\}$ *is compact.*

Theorem 1 (Dynamic programming principle)

Let Assumption 1 hold. Then the optimal value function $V^* \in B_b(S)$ *is the unique solution of the Bellman equation*

$$V^*(s) = \min_{a \in A} \left[c(s,a) + \gamma \int_S V^*(s')P(\mathrm{d}s'|s,a) \right]. \tag{3}$$

Moreover, writing $Q^*(s,a) = c(s,a) + \gamma \int_S V^*(s')P(\mathrm{d}s'|s,a)$*, there exists a measurable function* $f^* : S \to A$ *called a selector such that* $f^*(s) \in \operatorname{argmin}_{a \in A} Q^*(s,a)$ *and the induced policy* $\pi^* \in \mathcal{P}(A|S)$ *defined by* $\pi^*(\mathrm{d}a|s) = \delta_{f^*(s)}(\mathrm{d}a)$ *for all* $s \in S$ *satisfies* $V^* = V^{\pi^*}$.

Proof. [Hernández-Lerma and Lasserre, 2012, Theorem 4.2.3].

Proposition 2

Let $\pi(da|s) \in \mathcal{P}(A|S)$. *Then*

$$\|V_0^\pi\|_{B_b(S)} \leq \frac{\|c\|_{B_b(S\times A)}}{1-\gamma}\,.$$

Proof. *Exercise, start with* (1) *which is definition of* V_0^π.

Lemma 2

Let $\pi \in \mathcal{P}(A|S)$. *The value function* V_0^π *is the unique bounded solution of the on-policy Bellman equation:*

$$V_0^\pi(s) = \int_A \left(c(s,a) + \gamma \int_S V_0^\pi(s')P(ds'|s,a) \right) \pi(da|s), \quad \forall s \in S\,.$$

Recall the Bellman operator $T : B_b(S) \mapsto B_b(S)$ given by

$$(Tu)(s) = \inf_{m \in \mathcal{P}(A)} \int_A \Big(c(s,a) + \gamma \int_S u(s')P(ds'|s,a) \Big) m(da)\,.$$

Value iteration

1: Choose stopping tolerance $\delta > 0$.
2: Take initial guess of value e.g. $V^{(0)}(s) := 0$ for all $s \in S$.
3: Set $n = 0$.
4: **repeat**
5: $\quad n \leftarrow n + 1$
6: $\quad$ For each $s \in S$ evaluate $V^{(n)}(s) := (TV^{(n-1)})(s)$.
7: **until** $\|V^{(n)} - V^{(n-1)}\|_{B_b(s)} < \delta$
8: For each $s \in S$ set $\hat{V} := V^{(n)}$
9: For each $s \in S$ set

$$\hat{\pi}(da|s) \in \operatorname*{argmin}_{m \in \mathcal{P}(A)} \int_A \Big(c(s,a) + \gamma \int_S \hat{V}(s')P(ds'|s,a) \Big) m(da)$$

10: **return** $(\hat{V}, \hat{\pi})$

We know that $\|V^{(n)} - V^*\|_{B_b(s)} \leq \gamma^n \|V^{(0)} - V^*\|_{B_b(s)}$.

Policy iteration

Policy iteration

Choose stopping tolerance $\delta > 0$.

Take initial guess of policy $\pi^{(0)}(da|s)$ for all $s \in S$.

Set $n = 0$.

repeat

  $n \leftarrow n + 1$

  Solve the on-policy Bellman equation

$$V^{\pi^{(n-1)}}(s) = \int_A \left(c(s,a) + \gamma \int_S V_0^{\pi^{(n-1)}}(s')P(ds'|s,a) \right) \pi^{(n-1)}(da|s), \quad \forall s \in S\,.$$

  For each $s \in S$ set

$$\pi^{(n)}(da|s) \in \operatorname*{argmin}_{m \in \mathcal{P}(A)} \int_A \left(c(s,a) + \gamma \int_S V^{\pi^{(n-1)}}(s')P(ds'|s,a) \right) m(da)$$

until $\|V^{\pi^{(n)}} - V^{\pi^{(n-1)}}\|_{B_b(s)} < \delta$

For each $s \in S$ set $\hat{V}(s) := V^{\pi^{(n)}}(s)$ and $\hat{\pi}(\cdot|s) := \pi^{(n)}(\cdot|s)$

return $(\hat{V}, \hat{\pi})$

Can prove linear convergence (typically with a better rate than value iteration).

Function approximation

If S and A are finite $\rightsquigarrow$ tabular case and no need. Otherwise we may need to approximate functions $V : S \to \mathbb{R}$ or $a : A \to \mathbb{R}$ or $p : S \times A \to \mathbb{R}$ or ...

Approximating $f : X \to \mathbb{R}$ with X some general (high dimensional) space.

Linear:

- Fix a set of "features" or "basis functions" $(\phi_k : X \to \mathbb{R})_{k=1,\dots,M}$ linearly independent.
- Write $\hat{f}(x;\theta) := \sum_{k=1}^{M} \theta_k \phi_k(x)$ with some $\theta \in \mathbb{R}^M$.

One hidden layer feed-forward NN:

- Fix an activation $\varphi^N : \mathbb{R}^N \to \mathbb{R}^N$ so that for some $\varphi : \mathbb{R} \to \mathbb{R}$ we have $(\varphi^N(x))_i = \varphi(x_i)$, $i = 1, \dots, N$.
- Write $\hat{f}(x;\theta) := \hat{f}(x; W^{(1)}, B^{(1)}, W^{(2)}, B^{(2)})$ with $\theta = (W^{(1)}, B^{(1)}, W^{(2)}, B^{(2)})$, where

$$\hat{f}(x; W^{(1)}, B^{(1)}, W^{(2)}, B^{(2)}) = W^{(2)}\varphi(W^{(1)}x + B^{(1)}) + B^{(2)}\,.$$

- If $x \in \mathbb{R}^d$ and hidden width is $d_1 \in \mathbb{N}$ then $W^{(1)} \in \mathbb{R}^{d_1 \times d}$, $B^{(1)} \in \mathbb{R}^{d_1}$, $W^{(2)} \in \mathbb{R}^{1 \times d_1}$ and $B^{(2)} \in \mathbb{R}$.

More complicated NN architectures:

- Deep feed-forward NNs, RNNs, LSTMs, Transformers, CNNs, Vision transformers, ...

Learning and bandits

Multiple one-armed bandit problem

Choose machine $k = \{1, 2, \ldots, K\}$ to "spin" with reward per spin $r_k \sim \mathcal{D}_k$; $\mathcal{D}_k$ unknown r_k indep.

Regret

Imgine you play repeatedly forever.

Policy at each step n is $\pi_n \in \mathcal{P}(A)$ with $A = \{1, \dots, K\}$.

There is $k^* := \text{argmax}_{k \in K} \mathbb{E} r_k$ unknown to you and so $\pi^* := \delta_{k^*}$.

Regret:

$$R^\pi(N) := \mathbb{E} \sum_{n=0}^{N} \left(r_{k^*} - \sum_k r_k \pi_n(k) \right) .$$

Clearly

- $R^\pi(N) \geq 0$
- If $aN \leq R^\pi(N) \leq AN$ for some $0 < a < A$ then (linear regret) we are not learning.
- It can be shown that best regret is logarithmic.

Classical bandit methods

- Explore than exploit: Try each arm M-times, then forever play the one with best sample average.
- ε-greedy: Given some estimate of each arm average reward choose the one with best expected reward with probability $1 - \varepsilon$ and play an arm chosen uniformly at random with probability $\varepsilon > 0$. Update estimate of average reward from observation.
- Upper confidence bound (UCB): Play the arm with highest upper confidence arm. Update estimate of average reward from observation and confidence bound (from assumed variance and number of times played).

Kullback–Leibler divergence aka relative entropy

Relative entropy - definition and basics

If $\nu, \mu \in \mathcal{P}(A)$ and if $\mu(B) = 0 \implies \nu(B) = 0$ for every $B \in \mathcal{B}(A)$ then we say ν is absolutely continuous w.r.t. μ (notation $\nu << \mu$).

For $\mu \in \mathcal{P}(A)$ define

$$\mathcal{P}(A) \ni \nu \mapsto \mathrm{KL}(\nu|\mu) = \begin{cases} \int_A \ln \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\, \nu(da) & \text{if } \nu << \mu\,, \\ +\infty & \text{otherwise}\,. \end{cases}$$

Note that

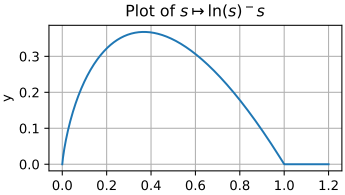
$$\int_A \left(\ln \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^- \nu(da) = \int_A \left(\ln \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)^- \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\, \mu(da)$$

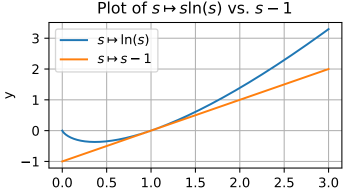
and $s \mapsto (\ln s)^- s \geq 0$ is bounded for $s \geq 0$, so KL is well defined.

Moreover $s \ln s \geq s - 1$ for $s \geq 0$ (with equality only if $s = 1$) and so

$$\mathrm{KL}(\nu|\mu) = \int_A \left(\ln \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right) \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\, \mu(da) \geq \int_A \left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu} - 1\right) \mu(da) = 0\,,$$

with equality only if $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = 1$ i.e. if $\nu = \mu$.





Useful identity

$$\mathrm{KL}(\nu|\mu) - \mathrm{KL}(\nu'|\mu) = \mathrm{KL}(\nu|\nu') + \int_A \ln \frac{\mathrm{d}\nu'}{\mathrm{d}\mu}(a)(\nu - \nu')(da)\,. \tag{4}$$

which holds for any $\nu, \nu' \in \mathcal{P}(A)$ for which the quantities in the identity are finite.

Variational formula: for $f \in B_b(A)$:

$$\inf_{\nu \in \mathcal{P}(A)} \left(\int_A f\, d\nu + \mathrm{KL}(\nu|\mu) \right) = -\ln \int_A e^{-f} \mu(da)\,,$$

and if

$$\frac{\mathrm{d}\nu^*}{\mathrm{d}\mu}(a) = \frac{e^{-f(a)}}{\int_A e^{-f(a')}\mu(da')}$$

then $\nu^* = \mathrm{argmin}_{\nu \in \mathcal{P}(A)} \left(\int_A f\, d\nu + \mathrm{KL}(\nu|\mu) \right)$.

Relative entropy - dual formulation and convexity

Donsker–Varadhan variational formula

$$\mathrm{KL}(\nu|\mu) = \sup_{g \in C_b(A)} \left(\int_A g(a)\,\nu(da) - \ln \int_A e^{g(a)}\,\mu(da) \right)$$

and

$$\mathrm{KL}(\nu|\mu) = \sup_{\psi \in B_b(A)} \left(\int_A \psi(a)\,\nu(da) - \ln \int_A e^{\psi(a)}\,\mu(da) \right) .$$

N.B. for fixed g

$$(\nu, \mu) \mapsto \int_A g(a)\,\nu(da) - \ln \int_A e^{g(a)}\,\mu(da)$$

is convex. As a supremum over such g

- $\mathcal{P}(A) \times \mathcal{P}(A) \ni (\nu, \mu) \mapsto \mathrm{KL}(\nu|\mu)$ is convex, lower-semicontinuous.

Moreover

- For fixed $\mu \in \mathcal{P}(A)$ we have

$$\{\nu \in \mathcal{P}(A) : \mathrm{KL}(\nu|\mu) < \infty\} \ni \nu \mapsto \mathrm{KL}(\nu|\mu)$$

  strictly convex, from strict convexity of $[0, \infty) \ni s \mapsto s \ln s \in \mathbb{R}$.

All from [Dupuis and Ellis, 1997, Ch. 1, Sec. 4].

Relaxed and regularized formulation of the MDP

Infinite-horizon Markov decision model (S, A, P, c, γ):

- S is the state space, A is the action space,
- $P \in \mathcal{P}(S|S \times A)$ is the transition probability kernel,
- $c \in B_b(S \times A)$ is a cost function, and γ a discount factor,
- $H_n := (S \times A)^n \times S$ is the space of admissible histories,
- $\tau \geq 0$ strenght of entropy regularizer,
- for $\mu', \mu \in \mathcal{P}(A)$ define $\mathrm{KL}(\mu'|\mu) = \int_A \ln \frac{\mathrm{d}\mu'}{\mathrm{d}\mu}(a)\mu'(da)$ if $\mu' \ll \mu$, and $+\infty$ otherwise.

Aim: minimise over relaxed policies $\pi = (\pi_n)_{n \in \mathbb{N}}$ s.t. $\pi_n : H_n \to P(A)$ measurable the objective:

$$V_\tau^\pi(s) = \mathbb{E}_s^\pi \left[\sum_{n=0}^\infty \gamma^n \left(\int_A c(s_n, a)\, \pi_n(da) + \tau\, \mathrm{KL}(\pi_n|\mu) \right) \right] \in \mathbb{R} \cup \{+\infty\}\,, \tag{5}$$

with $\pi_n := \pi_n(h_n)$, $h_n = (s_0, a_0, \ldots, s_{n-1}, a_{n-1}, s_n)$ and $s_{n+1} \sim \int_A P(\cdot|s_n, a)\, \pi_n(da)$, $s_0 = s$.

Optimal value is $\tau \geq 0$ dependent:

$$V_\tau^*(s) := \inf_\pi V_\tau^\pi(s)\,.$$

N.B. $V_0^* = V^*$.

Recall $H_n := (S \times A)^n \times S$ is the space of admissible histories.

Let $V_\tau^* : S \to \mathbb{R}$ be

$$V_\tau^*(s) = \inf_\pi V_\tau^\pi(s), \quad \forall s \in S\,, \tag{6}$$

where infimum is over policies $\pi = (\pi_n)_{n\in\mathbb{N}}$ s.t. $\pi_n : H_n \to P(A)$ is measurable.

Theorem 3 (Dynamic programming principle)

Let $\tau > 0$. The optimal value function V_τ^* is the unique bounded solution of

$$V_\tau^*(s) = \inf_{m\in\mathcal{P}(A)} \int_A \left(c(s,a) + \tau \ln \frac{\mathrm{d}m}{\mathrm{d}\mu}(a) + \gamma \int_S V_\tau^*(s')P(ds'|s,a) \right) m(da), \quad \forall s \in S\,.$$

DPP consequences for $\tau > 0$

For all $s \in S$,

$$V^*_\tau(s) = -\tau \ln \int_A \exp\left(-\frac{1}{\tau} Q^*_\tau(s,a)\right) \mu(da),$$

where $Q^* \in B_b(S \times A)$ is defined by

$$Q^*_\tau(s,a) = c(s,a) + \gamma \int_S V^*_\tau(s') P(ds'|s,a)\,, \quad \forall (s,a) \in S \times A.$$

Moreover, there is an optimal policy $\pi^*_\tau \in \mathcal{P}_\mu(A|S)$ given by

$$\pi^*_\tau(da|s) = \exp\left(-(Q^*_\tau(s,a) - V^*_\tau(s))/\tau\right) \mu(da)\,, \quad \forall s \in S. \tag{7}$$

Let

$$\Pi_\mu = \{\pi \in \mathcal{P}(A|S) : \ln \tfrac{d\pi}{d\mu} \in B_b(S \times A)\}\,.$$

Then

$$\inf_\pi V^\pi_\tau = V^*_\tau(s) = \inf_{\pi \in \Pi_\mu} V^\pi_\tau\,.$$

Finally, for each $\pi \in \Pi_\mu$, we define the Q-function $Q^\pi_\tau \in B_b(S \times A)$ by

$$Q^\pi_\tau(s,a) = c(s,a) + \gamma \int_S V^\pi_\tau(s') P(ds'|s,a)\,. \tag{8}$$

DPP consequences for $\tau > 0$

Proposition 3

Let $f \in B_b(S \times A)$ and $\pi \in \Pi_\mu$ be such that $\pi(da|s) = \frac{\exp(f(s,a))\mu(da)}{\int_A \exp(f(s,a'))\mu(da')}$ for all $s \in S$. Then

$$\left\| \ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu} \right\|_{B_b(S\times A)} \leq 2\|f\|_{B_b(S\times A)}, \quad \|V_\tau^\pi\|_{B_b(S)} \leq \frac{1}{1-\gamma}\left(\|c\|_{B_b(S\times A)} + 2\tau\|f\|_{B_b(S\times A)}\right).$$

Proof. As $\mu(A) = 1$, for all $g \in B_b(S \times A)$ and $s \in S$,

$$\ln \int_A \exp(g(s,a'))\mu(da') \leq \ln\left(e^{\|g\|_{B_b(S\times A)}}\mu(A)\right) = \|g\|_{B_b(S\times A)},$$
$$\ln \int_A \exp(g(s,a'))\mu(da') \geq \ln\left(e^{-\|g\|_{B_b(S\times A)}}\mu(A)\right) = -\|g\|_{B_b(S\times A)}.$$

Then, for all $(s,a) \in S \times A$, using $\ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s) = f(s,a) - \ln \int_A \exp(f(s,a'))\mu(da')$,

$$\left|\ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s)\right| \leq |f(s,a)| + \left|\ln \int_A \exp(f(s,a'))\mu(da')\right| \leq 2\|f\|_{B_b(S\times A)},$$

which implies that

$$\left|\mathbb{E}_s^\pi\left[\sum_{t=0}^\infty \gamma^t\left(\tau \ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(a_t|s_t)\right)\right]\right| \leq 2\tau\|f\|_{B_b(S\times A)}\sum_{t=0}^\infty \gamma^t = \frac{2\tau\|f\|_{B_b(S\times A)}}{1-\gamma}.$$

The rest follows as usual. □

DPP consequences for $\tau > 0$

Lemma 4

Let $\tau > 0$ and $\pi \in \Pi_\mu$. The value function V_τ^π is the unique bounded solution of the on-policy Bellman equation:

$$V_\tau^\pi(s) = \int_A \Big(c(s,a) + \tau \ln \tfrac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s) + \gamma \int_S V_\tau^\pi(s') P(ds'|s,a) \Big) \pi(da|s), \quad \forall s \in S\,.$$

Note that from this and defn. of the Q-function (8) we have for all $\pi \in \Pi_\mu$ and $s \in S$ that

$$V_\tau^\pi(s') = \int_A \Big(Q_\tau^\pi(s',a') + \tau \ln \tfrac{\mathrm{d}\pi}{\mathrm{d}\mu}(a'|s') \Big)\, \pi(da'|s'), \quad \forall s \in S\,. \tag{9}$$

Using this in the defn. of the Q-function (8) we have the on policy Q-Bellman equation

$$Q_\tau^\pi(s,a) = c(s,a) + \gamma \int_S \int_A \Big(Q_\tau^\pi(s',a') + \tau \ln \tfrac{\mathrm{d}\pi}{\mathrm{d}\mu}(a'|s') \Big)\, \pi(da'|s') P(ds'|s,a)\,,\ \forall (s,a)\,. \tag{10}$$

Q-learning

What does “solving our RL problem” mean?

We will say we've “solved our RL problem” if we can find a near optimal policy for the MDP under the assumptions that:

- We do **not** have access to costs c and transitions $P \in \mathcal{P}(S|S \times A)$.
- We choose $\gamma > 0$, $\tau \geq 0$
- We have access to a simulator of the environment and we can repeatedly use it cost-free.
- The simulator will initialise at $s \sim \rho \in \mathcal{P}(S)$ of its choice and will run until termination or until we reset it.

Q-learning in tabular setting: ε-greedy

Tabular ε-greedy Q-learning

Initialize environment, schedule $(\delta_k)_k$ state-action value $Q_0 = Q_0(s, a)$,
for $k = 0, 1, \ldots$ **do**
    Observe state s_k
    Take $a_k \in \operatorname{argmin}_{a \in A} Q_k(s_k|a)$ with prob $1 - \varepsilon$ and choose $a_k \sim \mu$ with probability $\varepsilon > 0$.
    Execute a_k in environment, accept cost c_k, new state s_{k+1}
    Update Q: Values for state not observed in this step are unchanged: $Q_{k+1} = Q_k$ while for the observed state and action

$$Q_{k+1}(s_k, a_k) = Q_k(s_k, a_k) + \delta_k \left[c_k + \gamma Q_k(s_{k+1}, a_{k+1}) - Q_k(s_k, a_k) \right],$$

    $s_k \leftarrow s_{k+1}$
end for

Q-learning in tabular setting: ε-greedy

Tabular softmax Q-learning

Initialize environment, schedule $(\delta_k)_k$ state-action value $Q_0 = Q_0(s, a)$,
for $k = 0, 1, \ldots$ **do**
    Observe state s_k
    Set $\pi_k(\cdot|s_k) \propto \exp(-\frac{1}{\tau} Q_k(s_k|\cdot))\mu$ and take $a_k \sim \pi_k(\cdot|s_k)$.
    Execute a_k in environment, accept cost c_k, new state s_{k+1}
    Update Q: Values for state not observed in this step are unchanged: $Q_{k+1} = Q_k$ while for the observed state and action

$$Q_{k+1}(s_k, a_k) = Q_k(s_k, a_k) + \delta_k \left[c_k + \gamma \Big(Q_k(s_{k+1}, a_{k+1}) + \tau \ln \tfrac{\mathrm{d}\pi_k}{\mathrm{d}\mu}(a_{k+1}|s_{k+1}) \Big) - Q_k(s_k, a_k) \right],$$

    $s_k \leftarrow s_{k+1}$
end for

DPP equation for Q^*_τ:

$$Q^*_\tau(s,a) = c(s,a) + \gamma \int_S \inf_{m\in\mathcal{P}(A)} \left(Q^*_\tau(s',a') + \tau \ln \tfrac{\mathrm{d}m}{\mathrm{d}\mu}(a') \right) m(da') \Big) P(ds'|s,a)\,.$$

Re-write:

$$0 = \mathbb{E}^{s'\sim P(\cdot|s,a)}_{a'\sim\pi^*_\tau(\cdot|s')} \left[c(s,a) + \gamma\Big(Q^*_\tau(s',a') + \tau \ln \tfrac{\mathrm{d}\pi^*_\tau}{\mathrm{d}\mu}(a'|s') \Big) \Big) - Q^*_\tau(s,a) \right].$$

Q-learning:

$$Q_{k+1}(s_k,a_k) = Q_k(s_k,a_k) + \delta_k \left[c(s_k,a_k) + \gamma\Big(Q_k(s_{k+1},a_{k+1}) + \tau \ln \tfrac{\mathrm{d}\pi_k}{\mathrm{d}\mu}(a_{k+1}|s_{k+1}) \Big) - Q_k(s_k,a_k) \right],$$

where $s_{k+1} \sim P(\cdot|s_k,a_k)$, $\pi_k(da'|s_k) \propto \exp(-\tau^{-1} Q_k(s_k,a'))\mu(da')$, $a_{k+1} \sim \pi_k(\cdot|s_k)$.

Convergence: like value iteration + stochastic approximation (Robbins–Monro).

Q-learning with function approximation: softmax

Q-learning: softmax with function approximation

Initialize environment, parametrized state-action value Q_θ,

for $n = 0, 1, \ldots, N_{\text{episodes}}$ **do**

  Make space in memory buffer

  for $t = 0, 1, \ldots, N_{\text{steps in episode}}$ **do**

    Observe state s_t

    Take $a_t \sim \exp(-\frac{1}{\tau} Q_\theta(s_t|a))\mu(da)$

    Execute a_t in environment, accept cost c_t, new state s_{t+1}

    Store (s_t, a_t, c_t, s_{t+1}) in memory

    $t \leftarrow t+1$, $s_t \leftarrow s_{t+1}$

  end for

  Sample $(s_j, a_j, c_j, s_{j+1})_{j=1}^N$

  For $j = 1, \ldots, N$ set

$$v_j = c_j - \gamma\tau \ln \int_A \exp(-\tfrac{1}{\tau} Q_{\theta_n}(s_{j+1}, a))\mu(da)\,.$$

  Let $L(\theta) := \sum_{j=1}^N |v_j - Q_\theta(s_j, a_j)|^2$ and update policy parameters

$$\theta_{n+1} = \theta_n - \eta \nabla_\theta L(\theta)\,.$$

end for

Classical policy gradient

Policy gradient meta-algorithm

Policy gradient (PG)

Initialize environment, parametrized policy π_θ,
for $n = 0, 1, \ldots, N_{\text{episodes}}$ **do**
  Clear memory buffer
  for $t = 0, 1, \ldots, N_{\text{steps in episode}}$ **do**
    Observe state s_t
    Sample action $a_t \sim \pi_{\theta_n}(a_t | s_t)$
    Execute a_t in environment, accept cost c_t, new state s_{t+1}
    Store $(s_t, a_t, c_t, \log \pi_{\theta_n}(a_t | s_t), V(s_t))$
    $t \leftarrow t + 1$, $s_t \leftarrow s_{t+1}$ in memory.
  end for
  Estimate $\nabla_\theta V^{\pi_\theta}$ from memory data
  Update policy parameters

$$\theta_{n+1} = \theta_n - \eta \nabla_\theta V^{\pi_\theta} \,.$$

end for

Policy gradient (PG) methods

Recall we are minimizing

$$\Pi_\mu \ni \pi \mapsto V_\tau^\pi(\rho) \in \mathbb{R}\,.$$

A "gradient" update would be

$$\pi_{n+1} = \pi_n - \eta \nabla_\pi V_\tau^\pi(\rho).$$

But even if S and A are of finite cardinality and

$$\nabla_\pi V_\tau^\pi(\rho) := (\nabla_{\pi(s,a)} V_\tau^\pi(\rho))_{(s,a)\in S\times A}$$

with $(\nabla_{\pi(s,a) V_\tau^\pi(\rho)})_{(s,a)\in S\times A} \in \mathbb{R}^{N_S \times N_A}$ is a gradient in $\mathbb{R}^{N_S\times N_A}$ **not in** $\mathcal{P}(A|S) \equiv \Delta(A)^{N_S}$.

Policy gradient (PG) methods

$$\pi_{n+1} = \pi_n - \eta \nabla_\pi V_\tau^\pi(\rho).$$

Parametrize:

- Direct: $\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) \propto e^{\theta(s,a)}$,
- Log-linear: $\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) \propto e^{\langle \theta, g(s,a)\rangle}$ with $g : S \times A \to \mathbb{R}^p$ basis.
- Neural-net: $\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) \propto e^{g_\theta(s,a)}$.

Then

$$\theta_{n+1} = \theta_n - \eta \nabla_\theta V_\tau^{\pi_\theta}(\rho)$$

classical gradient descent: [Cauchy, 1847][1] seems fine.

1. How to get $\nabla_\theta V_\tau^{\pi_\theta}(\rho)$ **from data?**
2. Convergence: e.g. is $\theta \mapsto V_\tau^{\pi_\theta}(\rho)$ convex?

[1] From [Lemaréchal, 2012] Cauchy and the gradient method, *Doc. Math. Extra*, 251–254.

Let

$$d^\pi(ds'|s) = (1-\gamma)\sum_{n=0}^{\infty}\gamma^n P_\pi^n(ds'|s) \quad \text{and} \quad d_\rho^\pi(ds) = \int_S d^\pi(ds|s')\rho(ds')\,. \tag{11}$$

We will refer to d^π as the occupancy kernel.

Lemma 5

Let $\pi \in \mathcal{P}(A|S)$ and $f, g \in B_b(S)$ such that for all $s \in S$,

$$f(s) = \gamma \int_A \int_S f(s')P(ds'|s,a)\pi(da|s) + g(s)\,. \tag{12}$$

Then $f(s) = \frac{1}{1-\gamma}\int_S g(s')d^\pi(ds'|s)$ for all $s \in S$.

Proof of stochastic representation for solutions of certain linear equations

Proof. A kernel $k \in b\mathcal{M}(S|S)$ induces a linear operator $L_k \in \mathcal{L}(B_b(S))$ by

$$B_b(S) \ni h \mapsto L_k h = \int_S h(s')k(ds'|\cdot)\,.$$

Since $\|L_k h\|_{B_b(S)} \leq \|h\|_{B_b(S)}\|k\|_{b\mathcal{M}(S|S)}$ for all $h \in B_b(S)$, $\|L_k\|_{\mathcal{L}(B_b(S))} \leq \|k\|_{b\mathcal{M}(S|S)}$.

Consider the kernel $\gamma P_\pi \in b\mathcal{M}(S|S)$ defined by $(\gamma P_\pi)(B) = \gamma \int_B \int_A P(ds'|s,a)\pi(da|s)$ for all $B \in \mathcal{B}(S)$. Then as $P_\pi \in \mathcal{P}(S|S)$ and $\|P_\pi\|_{b\mathcal{M}(S|S)} = 1$,

$$\|L_{\gamma P_\pi}\|_{\mathcal{L}(B_b(S))} \leq \|\gamma P_\pi\|_{b\mathcal{M}(S|S)} = \gamma\|P_\pi\|_{b\mathcal{M}(S|S)} = \gamma < 1\,.$$

The linear equation (12) that f satisfies g is equivalent to

$$(\mathrm{id} - L_{\gamma P_\pi})f = g\,.$$

The operator $\mathrm{id} - L_{\gamma P_\pi} \in \mathcal{L}(B_b(S))$ is invertible, and the inverse operator is given by the Neumann series

$$(\mathrm{id} - L_{\gamma P_\pi})^{-1} = \sum_{n=0}^\infty L_{\gamma P_\pi}^n\,.$$

Thus, $f = \sum_{n=0}^\infty L_{\gamma P_\pi}^n g$. Observe that $L_{\gamma P_\pi}^n = L_{\gamma^n P_\pi^n}$ for all $n \in \mathbb{N}_0$, where P_π^n is the n-times product of the kernel P_π with $P_\pi^0(ds'|s) := \delta_s(ds')$. Then by the definition (11) of $d^\pi \in \mathcal{P}(S|S)$,

$$f = \sum_{n=0}^\infty L_{\gamma P_\pi}^n g = \frac{1}{1-\gamma}\int_S g(s')d^\pi(ds'|\cdot)$$

which is the desired identity. □

Lemma 6 (Performance difference[1])

For all $\rho \in \mathcal{P}(S)$ *and* $\pi, \pi' \in \Pi_\mu$,

$$V_\tau^\pi(\rho) - V_\tau^{\pi'}(\rho) = \tfrac{1}{1-\gamma} \int_S \int_A \Big(Q_\tau^{\pi'}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi'}{\mathrm{d}\mu}(a|s) - V_\tau^{\pi'}(s) \Big)(\pi - \pi')(da|s) d_\rho^\pi(ds)$$
$$+ \tfrac{\tau}{1-\gamma} \int_S \mathrm{KL}(\pi(\cdot|s)|\pi'(\cdot|s)) d_\rho^\pi(ds)\,.$$

[1] Tabular case [Howard, 1960, Ch. 7, p. 87], re-discovered in RL context [Kakade and Langford, 2002], Polish spaces + entropy [Kerimkulov et al., 2025a]

Proof. By (9), for all $s \in S$,

$$
\begin{aligned}
& V_\tau^\pi(s) - V_\tau^{\pi'}(s) \\
&= \int_A \left(Q_\tau^\pi(a|s) + \tau \ln \tfrac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s) \right) \pi(da|s) - \int_A \left(Q_\tau^{\pi'}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi'}{\mathrm{d}\mu}(a|s) \right) \pi'(da|s) \\
&= \int_A \left(Q_\tau^{\pi'}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi'}{\mathrm{d}\mu}(a|s) \right) (\pi - \pi')(da|s) \\
&\quad + \int_A \left(Q_\tau^\pi(s,a) + \tau \ln \tfrac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s) - Q_\tau^{\pi'}(s,a) - \tau \ln \tfrac{\mathrm{d}\pi'}{\mathrm{d}\mu}(a|s) \right) \pi(da|s) .
\end{aligned}
$$

Hence for all $s \in S$ we have

$$
\begin{aligned}
V_\tau^\pi(s) - V_\tau^{\pi'}(s) &= \int_A \left(Q_\tau^{\pi'}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi'}{\mathrm{d}\mu}(a|s) \right) (\pi - \pi')(da|s) \\
&\qquad + \gamma \int_A \int_S \left(V_\tau^\pi(s') - V_\tau^{\pi'}(s') \right) P(ds'|s,a)\pi(da|s) + \tau \, \mathrm{KL}(\pi(\cdot|s)|\pi'(\cdot|s)) ,
\end{aligned}
$$

where the last equality used def. of Q. fn (8) and KL identity (4). Hence, by Fubini's theorem and Lemma 5, for all $s \in S$,

$$
\begin{aligned}
& V_\tau^\pi(s) - V_\tau^{\pi'}(s) \\
&= \tfrac{1}{1-\gamma} \int_S \left[\int_A \left(Q_\tau^{\pi'}(s',a) + \tau \ln \tfrac{\mathrm{d}\pi'}{\mathrm{d}\mu}(a|s') \right) (\pi - \pi')(da|s') + \tau \, \mathrm{KL}(\pi(\cdot|s')|\pi'(\cdot|s')) \right] d^\pi(ds'|s).
\end{aligned}
$$

Integrating both sides with respect to ρ yields the desired identity. □

Proposition 4

Let $\pi, \pi' \in \Pi_\mu$ *be such that* $\pi(da|s) = \frac{\exp(f(s,a))\mu(da)}{\int_A \exp(f(s,a'))\mu(da')}$ *for all* $s \in S$. *Then*

$$\|Q_\tau^{\pi'} - Q_\tau^{\pi}\|_{B_b(S\times A)} \leq \frac{\gamma}{(1-\gamma)^2}\left(\|c\|_{B_b(S\times A)} + 2\tau\|f\|_{B_b(S\times A)}\right)\|\pi - \pi'\|_{b\mathcal{M}(A|S)} + \frac{\tau\gamma}{1-\gamma}\left\|\ln\frac{\mathrm{d}\pi'}{\mathrm{d}\pi}\right\|_{B_b(S\times A)}.$$

Proof. Start by getting the estimate for $\|V_\tau^{\pi'} - V_\tau^{\pi}\|_{B_b(S)}$ using Lemma 6 (performance difference).

Proposition 5

Let $\tau \geq 0$ and $\rho \in \mathcal{P}(S)$. For all $\pi, \pi' \in \Pi_\mu \subset \mathcal{P}(A|S)$

$$\lim_{\varepsilon \searrow 0} \frac{V_\tau^{(1-\varepsilon)\pi+\varepsilon\pi'}(\rho) - V_\tau^\pi(\rho)}{\varepsilon} = \frac{1}{1-\gamma}\int_S \int_A \left(Q_\tau^\pi(s,a) + \tau \ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s) - V_\tau^\pi(s) \right) (\pi' - \pi)(da|s) d_\rho^\pi(ds) \,. \tag{13}$$

Proof. Let $\pi^\varepsilon = (1-\varepsilon)\pi + \varepsilon\pi' = \pi + \varepsilon(\pi' - \pi)$ and note that $\pi - \pi^\varepsilon = -\varepsilon(\pi' - \pi) = \varepsilon(\pi - \pi')$. Then

$$\begin{aligned}
\frac{1}{\varepsilon}(V_\tau^\pi(\rho) - V_\tau^{\pi^\varepsilon}(\rho)) &= \frac{1}{\varepsilon}\frac{1}{1-\gamma}\int_S \int_A \left(Q_\tau^{\pi^\varepsilon}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^\varepsilon}{\mathrm{d}\mu}(a|s) \right) (\pi - \pi^\varepsilon)(da|s) d_\rho^\pi(ds) \\
&\quad + \frac{1}{\varepsilon}\frac{\tau}{1-\gamma}\int_S \mathrm{KL}(\pi(\cdot|s)|\pi^\varepsilon(\cdot|s)) d_\rho^\pi(ds) \\
&= \frac{1}{1-\gamma}\int_S \int_A \left(Q_\tau^{\pi^\varepsilon}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^\varepsilon}{\mathrm{d}\mu}(a|s) \right) (\pi - \pi')(da|s) d_\rho^\pi(ds) \\
&\quad + \frac{1}{\varepsilon}\frac{\tau}{1-\gamma}\int_S \mathrm{KL}(\pi(\cdot|s)|\pi^\varepsilon(\cdot|s)) d_\rho^\pi(ds) \,.
\end{aligned}$$

Proof of PG theorem for general state and action spaces

From the KL identity (4) we get

$$\frac{1}{\varepsilon}(V_\tau^\pi(\rho) - V_\tau^{\pi^\varepsilon}(\rho)) = \frac{1}{1-\gamma}\int_S \int_A Q_\tau^{\pi^\varepsilon}(s,a)(\pi - \pi')(da|s) d_\rho^\pi(ds) + \frac{1}{\varepsilon}\frac{\tau}{1-\gamma}\int_S \Big(\mathrm{KL}(\pi(\cdot|s)|\mu(\cdot|s)) - \mathrm{KL}(\pi^\varepsilon(\cdot|s)|\mu(\cdot|s)) \Big) d_\rho^\pi(ds)\,.$$

Thus

$$\frac{1}{\varepsilon}(V_\tau^{\pi^\varepsilon}(\rho) - V_\tau^\pi(\rho)) = \frac{1}{1-\gamma}\int_S \int_A Q_\tau^{\pi^\varepsilon}(s,a)(\pi' - \pi)(da|s) d_\rho^\pi(ds) + \frac{\tau}{1-\gamma}\int_S \frac{1}{\varepsilon}\Big(\mathrm{KL}(\pi^\varepsilon(\cdot|s)|\mu(\cdot|s)) - \mathrm{KL}(\pi(\cdot|s)|\mu(\cdot|s)) \Big) d_\rho^\pi(ds)\,.$$

The first integral on the right hand side converges to $\frac{1}{1-\gamma}\int_S \int_A Q_\tau^\pi(s,a)(\pi'-\pi)(da|s)d_\rho^\pi(ds)$ as $\varepsilon \to 0$ due to Proposition 4. Moreover, as $\pi, \pi' \in \Pi_\mu$, for all $s \in S$, by [Kerimkulov et al., 2025b, Lemma 3.8],

$$\lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon}\Big(\mathrm{KL}(\pi^\varepsilon(\cdot|s)|\mu(\cdot|s)) - \mathrm{KL}(\pi(\cdot|s)|\mu(\cdot|s)) \Big) = \int_A \ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(a|s)(\pi' - \pi)(da|s)\,,$$

which along with Proposition 3 and the dominated yields the desired limit. □

First variation and chain rule

For a fixed $\nu \in \mathcal{P}(S)$ define $\langle \cdot, \cdot \rangle_\nu : B_b(S \times A) \times b\mathcal{M}(A|S) \to \mathbb{R}$ by

$$\langle Z, m \rangle_\nu = \frac{1}{1-\gamma} \int_S \int_A Z(s,a) m(da|s) \nu(ds) , \quad (Z, m) \in B_b(S \times A) \times b\mathcal{M}(A|S).$$

As a consequence of Proposition 5, given $\nu \in \mathcal{P}(S)$ satisfying $d_\rho^\pi \ll \nu$,

$$\lim_{\varepsilon \searrow 0} \frac{V_\tau^{(1-\varepsilon)\pi + \varepsilon \pi'}(\rho) - V_\tau^\pi(\rho)}{\varepsilon} = \left\langle \frac{\delta V_\tau^\pi(\rho)}{\delta \pi}\bigg|_\nu , \pi' - \pi \right\rangle_\nu ,$$

with

$$\frac{\delta V_\tau^\pi(\rho)}{\delta \pi}\bigg|_\nu (s,a) = \left(Q_\tau^\pi(s,a) + \tau \ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(s,a) - V_\tau^\pi(s) \right) \frac{\mathrm{d}d_\rho^\pi}{\mathrm{d}\nu}(s) . \tag{14}$$

Let $(\mathbb{H}, (\cdot,\cdot)_{\mathbb{H}})$ be a Hilbert space.

Lemma 7 (Chain rule)

Let $\pi : \mathbb{H} \to \Pi_\mu$ be given. Then $\partial_{\theta_i} V_\tau^{\pi_\theta}(\rho) = \left\langle \frac{\delta V_\tau^\pi(\rho)}{\delta \pi}, \partial_{\theta_i} \pi_\theta \right\rangle_{d_\rho^\pi}$.

Proof. Similar to [Kerimkulov et al., 2025a, Proposition 3.8].

Policy gradient theorem

Theorem 8 (PG for parametrization)

Let $\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) := \frac{e^{g_\theta(s,a)}}{Z_\theta(s)}$, *where* $Z_\theta(s) := \int_A e^{g_\theta(s,a')}\mu(da')$. *Then*

$$\nabla_\theta V_\tau^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\mathbb{E}^{s\sim d_\rho^{\pi_\theta}}_{a\sim\pi_\theta(\cdot|s)}\left[\frac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\right].$$

Proof. From Lemma 7 (chain rule) we have:

$$\nabla_\theta V_\tau^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\int_S\int_A \frac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\nabla_\theta \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\mu(da)\, d_\rho^{\pi_\theta}(ds)\,.$$

Taking the gradient of the logarithm and re-arranging we see that

$$\nabla_\theta \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) = \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\,. \tag{15}$$

Hence

$$\nabla_\theta V_\tau^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\int_S\int_A \frac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\pi_\theta(da|s)\, d_\rho^{\pi_\theta}(ds)\,.$$

We just need to rewrite this in terms of expectation to get the conclusion. □

Remark on baseline. We can take any $b \in B_b(S)$. Then

$$\int_A b(s)\nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\pi_\theta(da|s) = b(s)\int_A \nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\pi_\theta(da|s) = b(s)\int_A \nabla_\theta \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\mu(da)$$
$$= b(s)\nabla_\theta \int_A \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\mu(da) = b(s)\nabla_\theta 1 = 0\,.$$

Hence

$$\nabla_\theta V_\tau^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}^{s\sim d_\rho^{\pi_\theta}}\left[\left(\frac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a) + b(s)\right)\nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\right].$$

Some remarks on PG

Remark on estimating the advantage function. First variation:

$$\frac{\delta V^{\pi\theta}_\tau}{\delta \pi}(s,a) = \underbrace{Q^\pi_\tau(s,a) - V^\pi_\tau(s)}_{=:A^\pi_\tau(s,a)\ \text{“advantage”}} + \tau \ln \frac{\mathrm{d}\pi}{\mathrm{d}\mu}(s,a)\,.$$

Advantage $A^\pi_\tau(s,a)$ can be estimated from data: $(s_t, a_t, c_t, s_{t+1}, a_{t+1}, \ldots)$.

$$\hat{A}^\pi_\tau := c_t + \gamma \hat{V}^\pi_\tau(s_{t+1}) - \hat{V}^\pi_\tau(s_t)\,,$$

where $\hat{V}_\tau \approx V^\pi_\tau$. N.B.

$$\mathbb{E}_{s_{t+1}\ \sim P(\cdot|s_t,a_t)} \hat{A}^\pi_\tau = c(s_t,a_t) + \gamma \int_S \hat{V}_\tau(s') P(ds'|s_t,a_t) - \hat{V}_\tau(s_t)$$

would be equal to $A^\pi_\tau(s_t,a_t)$ if $\hat{V}_\tau = V^\pi_\tau$ in which case it would be unbiased. Alternative

$$\hat{A}^\pi_\tau := \sum_{l=0}^{\infty} \gamma^{t+l} c_{t+l} - \hat{V}(s_t)\,.$$

Generalised advantage estimation (GAE) formula [Schulman et al., 2015] allows efficient variance vs bias tradeoffs.

Corollary 9 (to Policy Gradient Theorem)

Let $\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) := \frac{e^{g_\theta(s,a)}}{Z_\theta(s)}$, $Z_\theta(s) := \int_A e^{g_\theta(s,a')}\mu(da')$. *Then*

$$\nabla_\theta V_\tau^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\mathbb{E}^{s\sim d_\rho^{\pi_\theta}}_{a\sim\pi_\theta(\cdot|s)}\left[\frac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\left(\nabla_\theta g_\theta(s,a) - \int_A(\nabla_\theta g_\theta)(s,a')\pi_\theta(da'|s)\right)\right].$$

Proof. Note that

$$\ln\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) = g_\theta(s,a) - \ln Z_\theta(s)$$

and so

$$\nabla_\theta \ln\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(s,a) = \nabla_\theta g_\theta(s,a) - \nabla_\theta Z_\theta(s)\frac{1}{Z_\theta(s)} = \nabla_\theta g_\theta(s,a) - \int_A(\nabla_\theta g_\theta)(s,a')\frac{e^{g_\theta(s,a')}}{Z_\theta(s)}\mu(da').$$

Hence we have an expression for gradient of the log-density:

$$\nabla_\theta \ln\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(s,a) = \nabla_\theta g_\theta(s,a) - \int_A(\nabla_\theta g_\theta)(s,a')\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a'|s)\mu(da') \qquad (16)$$

which concludes the calculation. □

Some remarks on PG

If the state and action spaces are finite and we take the direct (tabular) parametrizations so that $g_\theta(s,a) := \theta(s,a)$ then

$$\partial_{\theta_{\hat s,\hat a}} g_\theta(s,a) - \sum_{a'} \partial_{\theta_{\hat s,\hat a}} g_\theta(s,a')\pi_\theta(a'|s) = \delta_{\hat s,\hat a}(s,a) - \sum_{a'} \delta_{\hat s,\hat a}(s,a')\pi(a'|s)$$

$$= \delta_{\hat s,\hat a}(s,a) - \delta_{\hat s}(s)\pi(\hat a|s) = \delta_{\hat s}(s)\big(\delta_{\hat a}(a) - \delta_{\hat s}(s)\pi(\hat a|s)\big)\,.$$

Hence

$$\partial_{\theta_{\hat s,\hat a}} V_\tau^{\pi_\theta}(\rho) = \tfrac{1}{1-\gamma}\sum_{s,a} \tfrac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\delta_{\hat s}(s)\delta_{\hat a}(a)\pi_\theta(a|s)d_\rho^{\pi_\theta}(s)$$

$$- \tfrac{1}{1-\gamma}\sum_{s,a} \tfrac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\delta_{\hat s}(s)\pi(\hat a|s)\pi_\theta(a|s)d_\rho^{\pi_\theta}(s)\,.$$

But

$$\sum_{s,a} \tfrac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\delta_{\hat s}(s)\pi(\hat a|s)\pi_\theta(a|s)d_\rho^{\pi_\theta}(s) = \sum_s \delta_{\hat s}(s)\pi(\hat a|s)\sum_a \tfrac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\pi_\theta(a|s)d_\rho^{\pi_\theta}(s) = 0$$

and so

$$\partial_{\theta_{\hat s,\hat a}} V_\tau^{\pi_\theta}(\rho) = \tfrac{1}{1-\gamma}\tfrac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(\hat s,\hat a)\pi_\theta(\hat a|\hat s)d_\rho^{\pi_\theta}(\hat s)\,.$$

This is (for the $\tau = 0$ case) exactly Lemma C.1 in [Agarwal et al., 2019].

If $g_\theta(s,a) = (\theta, \phi(s,a))_{\mathbb{H}}$ then $\partial_{\theta_i} g_\theta(s,a) = \theta_i(s,a)$ and so

$$\nabla_\theta V_\tau^{\pi_\theta}(\rho) = \frac{1}{1-\gamma}\mathbb{E}^{s\sim d_\rho^{\pi_\theta}}_{a\sim\pi_\theta(\cdot|s)}\left[\frac{\delta V_\tau^{\pi_\theta}}{\delta\pi}(s,a)\left(\phi(s,a) - \int_A \phi(s,a')\pi_\theta(da'|s)\right)\right]. \quad (17)$$

Summary of PG so far:

- We have expression for the gradient.
- It can be estimated from data.
- For some simple parametrizations it's nice and simple.

Next: what about convergence?

Lack of convexity in softmax parametrization

Consider *minimizing*, over $\pi \in \mathcal{P}(A)$ the objective

$$v^\pi := \int_A c(a)\pi(da)\,.$$

We trivially have, for $\pi, \pi' \in \mathcal{P}(A)$ that

$$v^{(1-\varepsilon)\pi+\varepsilon\pi'} \le (1-\varepsilon)v^\pi + \varepsilon v^{\pi'}$$

so $\mathcal{P}(A) \ni \pi \mapsto v^\pi \in \mathbb{R}$ is convex.

Consider *minimizing*, over $\theta \in \mathbb{R}^{|A|}$ the objective

$$v^{\pi_\theta} := \int_A c(a)\pi_\theta(da),$$

with $\pi_\theta(a) = \frac{e^{\theta(a)}}{\sum_{a'} e^{\theta(a')}}$.

The map $\mathbb{R}^p \ni \theta \mapsto v^{\pi_\theta} \in \mathbb{R}$ is *not* convex [Mei et al., 2020, Propn. 1].

Performance difference vs convexity

Definition (Convexity)

If for some $\tau \geq 0$ we have all $m, m' \in \mathcal{P}(A)$ that

$$F(m) - F(m') \geq \left\langle \frac{\delta F(m')}{\delta m}, m - m' \right\rangle + \tau \, \mathrm{KL}(m|m') \,,$$

then F is convex ($\tau = 0$) or strongly convex ($\tau > 0$).

Equiv.: $\frac{\delta F}{\delta m}(m, \cdot)$ exists and $F((1-\varepsilon)m + \varepsilon m') \leq (1-\varepsilon)F(m) + \varepsilon F(m')$ for all $m, m' \in \mathcal{P}(A)$, $\varepsilon \in [0,1]$.

Performance difference

$$(V_\tau^{\pi} - V_\tau^{\pi'})(\rho) = \left\langle \frac{\delta V_\tau^{\pi'}}{\delta \pi}, \pi - \pi' \right\rangle_{\rho,\pi} + \frac{\tau}{1-\gamma} \int_S \mathrm{KL}(\pi|\pi')(s) d_\rho^{\pi}(ds) \,,$$

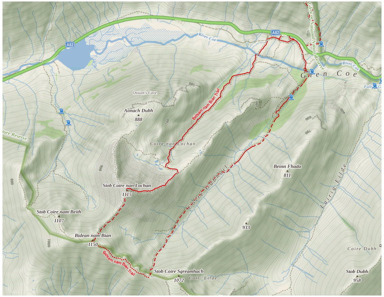
where

$$\langle h, \hat{\pi} \rangle_{\rho,\pi} := \frac{1}{1-\gamma} \int_S \int_A h(s,a) \hat{\pi}(da|s) \, d_\rho^{\pi}(ds)$$

The map $\Pi_\mu \ni \pi \mapsto V_\tau^{\pi}(\rho) \in \mathbb{R} \cup \{+\infty\}$ is **not** convex, e.g. [Giegrich et al., 2024, Proposition 2.4] **even if underlying dynamics is linear and costs convex.**

Continuous time gradient flow

$$\frac{d}{ds}\theta_s = -\nabla f(\theta_s) \implies \frac{d}{ds}\left[f(\theta_s) - f(\theta^*)\right] = -|\nabla f(\theta_s)|^2 .$$



Non-uniform Polyak–Łojasiewicz: there is $\mu : \mathbb{R}^p \to (0, \infty)$ s.t. for all $\theta \in \mathbb{R}^p$

$$0 \le f(\theta) - f(\theta^*) \le \mu(\theta)|\nabla f(\theta)|^2 .$$

Hence

$$\frac{d}{ds}\left[f(\theta_s) - f(\theta^*)\right] = -|\nabla f(\theta_s)|^2 \le -\mu^{-1}(\theta_s)\left[f(\theta_s) - f(\theta^*)\right]$$

Grönwall:

$$0 \le f(\theta_s) - f(\theta^*) \le \left[f(\theta_0) - f(\theta^*)\right] \exp\left(- \int_0^s \mu^{-1}(\theta_r)\, dr \right) .$$

Q: Is $\inf_r \mu^{-1}(\theta_r) \ge \alpha > 0$?

Convergence of classical PG in MDPs

- Discrete time LQR: Polyak–Łojasiewicz (PL) / gradient dominance established and so PG has linear convergence [Fazel et al., 2018, Bu et al., 2019, Hu et al., 2023].
- In general discrete state-action setting best PL result is non-uniform [Mei et al., 2020] but shown lower bounded along PG and hence convergence.

Mirror descent

Mirror descent

Static optimization mirror descent:

- Goes back to at least [Nemirovski, 1979].
- Modern proximal point form [Beck and Teboulle, 2003].
- For general probability measures [Aubin-Frankowski et al., 2022].

Mirror descent for MDPs

Discrete space MDPs and constants **dependent** on $|S|$ and $|A|$:

- [Cen et al., 2022], entropy regularised, show linear convergence for disc. time. mirror descent
- [Cayci et al., 2021] same setting i.e natural policy gradient, log-linear policies i.e. mirror desc with func. approx.
- [Xiao, 2022] and [Khodadadian et al., 2022] achieved *linear convergence for unregularised MDPs* with inexact policy evaluation by employing geometrically increasing step sizes in NPG.

Discrete space MDPs and constants **independent of** of $|S|$ and $|A|$:

- [Lan, 2023] linear convergence of policy mirror descent with arbitrary convex regularisers and [Zhan et al., 2023] convergence rates independent of action space dimension.

MDPs with **general** S and A:

- Discrete step mirror descent and Fisher–Rao flow: Exponential convergence for entropy regularized MDPs in Polish state & action spaces [Kerimkulov et al., 2025a].

Deriving mirror descent

Aim: find

$$\pi^*(\cdot|s) = \arg\min_\pi V_\tau^\pi(s)\,.$$

Let's say we have π_{old}. Fix $\rho \in \mathcal{P}(S)$ and write $V_\tau^\pi = V_\tau^\pi(\rho)$. By perf. diff., Lemma 6,

$$V_\tau^\pi = V_\tau^{\pi_{\text{old}}} + \left\langle \tfrac{\delta V_\tau^{\pi_{\text{old}}}}{\delta\pi}, \pi - \pi_{\text{old}} \right\rangle_{\rho,\pi} + \tfrac{\tau}{1-\gamma}\int_S \mathrm{KL}(\pi|\pi_{\text{old}})(s)\, d_\rho^\pi(ds)\,.$$

Linearize and penalize with $\lambda \geq \tau$ to not move too far

$$L^\pi := V_\tau^{\pi_{\text{old}}} + \left\langle \frac{\delta V_\tau^{\pi_{\text{old}}}}{\delta\pi}, \pi - \pi_{\text{old}} \right\rangle_{\rho,\pi_{\text{old}}} + \tfrac{\lambda}{1-\gamma}\int_S \mathrm{KL}(\pi|\pi_{\text{old}})(s)\, d_\rho^{\pi_{\text{old}}}(ds)\,.$$

Mirror descent optimizes $\pi \mapsto L^\pi(x)$ giving

$$\pi_{\text{new}}(da|s) = \arg\min_\pi \left(V_\tau^{\pi_{\text{old}}} + \int_A \frac{\delta V_\tau^{\pi_{\text{old}}}}{\delta\pi}(s,a)(\pi - \pi_{\text{old}})(da|s) + \lambda\, \mathrm{KL}(\pi|\pi_{\text{old}})(s) \right).$$

Motivation for studying mirror descent

Policy gradient: introduce parametrized densities $\pi_\theta(da, s) \propto e^{g_\theta(a,s)}\mu(da)$. Step $\eta > 0$:

$$\theta_{\text{new}} = \theta_{\text{old}} + \eta \nabla_\theta V_\tau^{\pi_{\theta_{\text{old}}}},$$

$$\nabla_\theta V_\tau^{\pi_{\theta_{\text{old}}}}(\rho) = \frac{1}{1-\gamma}\mathbb{E}^{s\sim d_\rho^{\pi_{\theta_{\text{old}}}}}_{a\sim\pi_{\theta_{\text{old}}}(\cdot|s)}\left[\frac{\delta V_\tau^{\pi_{\theta_{\text{old}}}}}{\delta\pi}(s,a)\nabla_\theta \ln\frac{\mathrm{d}\pi_{\theta_{\text{old}}}}{\mathrm{d}\mu}(a|s)\right].$$

Problem: Even if θ_{new} and θ_{old} are close $\pi_{\theta_{\text{old}}}$ and $\pi_{\theta_{\text{new}}}$ may be *very different!*

Instead, re-write the mirror descent objective:

$$\begin{aligned} L_{\text{MD}}(\theta) &= \left\langle \frac{\delta V_\tau^{\pi_{\theta_{\text{old}}}}}{\delta\pi}, \pi_\theta\right\rangle_{\rho,\pi_{\theta_{\text{old}}}} + \lambda\int_S \text{KL}(\pi_\theta|\pi_{\theta_{\text{old}}})(s)\, d_\rho^{\pi_{\theta_{\text{old}}}}(ds) \\ &= \mathbb{E}^{s\sim d_\rho^{\pi_{\theta_{\text{old}}}}}_{a\sim\pi_\theta(\cdot|s)}\left[\frac{\delta V_\tau^{\pi_{\theta_{\text{old}}}}}{\delta\pi}(s,a) + \lambda\,\text{KL}(\pi_\theta|\pi_{\theta_{\text{old}}})(s)\right] \\ &= \mathbb{E}^{s\sim d_\rho^{\pi_{\theta_{\text{old}}}}}_{a\sim\pi_{\theta_{\text{old}}}(\cdot|s)}\left[\frac{\delta V_\tau^{\pi_{\theta_{\text{old}}}}}{\delta\pi}(s,a)\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\pi_{\theta_{\text{old}}}}(a|s) + \lambda\,\text{KL}(\pi_\theta|\pi_{\theta_{\text{old}}})(s)\right]. \end{aligned}$$

Step $\eta > 0$:

$$\theta_{\text{new}} = \theta_{\text{old}} + \eta\nabla_\theta L_{\text{MD}}(\theta_{\text{old}}).$$

Mirror descent policy improvement (with exact update)

Mirror descent update

$$\pi^{n+1}(\cdot|s) = \operatorname*{argmin}_{m\in\mathcal{P}(A)} \int_A \frac{\delta V_\tau^{\pi^n}}{\delta\pi}(s,a)(m(da) - \pi^n(da|s)) + \lambda\, \mathrm{KL}(m|\pi^n(\cdot|s))\,. \tag{18}$$

From the performance difference lemma, see Lemma (6), we see that

$$\begin{aligned}(V_\tau^{n+1} - V_\tau^n)(\rho) &= \tfrac{1}{1-\gamma}\int_S \Big(\int_A \tfrac{\delta V_\tau^n}{\delta\pi}(s,a)(\pi^{n+1}-\pi^n)(da|s) + \tau\,\mathrm{KL}(\pi^{n+1}|\pi^n)(s)\Big)\, d_\rho^{\pi^{n+1}}(ds)\\ &\le \tfrac{1}{1-\gamma}\int_S \Big(\int_A \tfrac{\delta V_\tau^n}{\delta\pi}(s,a)(\pi^{n+1}-\pi^n)(da|s) + \lambda\,\mathrm{KL}(\pi^{n+1}|\pi^n)(s)\Big)\, d_\rho^{\pi^{n+1}}(ds)\,.\end{aligned} \tag{19}$$

From the mirror descent update (18) we have, for all $\pi \in \Pi_\mu$ and $s \in S$ that

$$\int_A \frac{\delta V_\tau^n}{\delta\pi}(s,a)(\pi - \pi^n)(da|s) + \lambda\,\mathrm{KL}(\pi|\pi^n)(s) \ge \int_A \frac{\delta V_\tau^n}{\delta\pi}(s,a)(\pi^{n+1} - \pi^n)(da|s) + \lambda\,\mathrm{KL}(\pi^{n+1}|\pi^n)(s)\,.$$

This with $\pi = \pi^n$ allows us to conclude that for all $s \in S$ we have

$$\int_A \frac{\delta V_\tau^n}{\delta\pi}(s,a)(\pi^{n+1} - \pi^n)(da|s) + \lambda\,\mathrm{KL}(\pi^{n+1}|\pi^n)(s) \le 0\,. \tag{20}$$

From (19) we have

$$(V_\tau^{n+1} - V_\tau^n)(\rho) \le 0\,.$$

Recall that $\frac{\delta V_\tau^{\pi_n}}{\delta \pi} = A_\tau^{\pi_n} + \tau \ln \frac{\mathrm{d}\pi^n}{\mathrm{d}\mu} = Q_\tau^{\pi_n} - V_\tau^{\pi_n} + \tau \ln \frac{\mathrm{d}\pi^n}{\mathrm{d}\mu}$.

Updates can only be made with an approximation $\hat{A}_n(s,a) = A_\tau^{\pi_n}(s,a) + \mathcal{E}_n(s,a)$.

Consider the scheme

$$\pi^{n+1}(da|s) = \operatorname*{argmin}_{m \in \mathcal{P}(A)} \int_A \left(\hat{A}_n(s,a) + \tau \ln \frac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s) \right) (m(da) - \pi^n(da|s)) + \lambda \, \mathrm{KL}(m|\pi^n(\cdot|s)\,. \tag{21}$$

This is from [Lan, 2023].

Lemma 10

Let $F : S \to \mathbb{R}$ be such that $F \le 0$. Then for any π and any $s \in S$

$$\frac{1}{1-\gamma}\int_S F(s')\, d_s^\pi(ds') \le F(s)\,. \tag{22}$$

Proof. From (11) and the fact that $P_\pi^0(ds'|s) = \delta_s(ds')$ we have for all $s \in S$ that

$$\begin{aligned}\frac{1}{1-\gamma}\int_S F(s')\, d_s^\pi(ds') &= \int_S F(s')P_\pi^0(ds'|s) + \sum_{k=1}^\infty \int_S \gamma^k F(s')P_\pi^k(ds'|s)\\ &\le \int_S F(s')\delta_s(ds') = F(s)\,.\end{aligned} \tag{23}$$

This concludes the proof. □

Lemma 11 (L-smoothness for exact update)

Let $\pi, \pi' \in \Pi_\mu$ *satisfy* $\int_A \frac{\delta V_\tau^{\pi'}}{\delta \pi}(s,a)(\pi - \pi')(da|s) + \tau \, \mathrm{KL}(\pi|\pi')(s) \leq 0$ *for all* $s \in S$*. Then for all* $s \in S$*,*

$$(V_\tau^\pi - V_\tau^{\pi'})(s) \leq \int_A \frac{\delta V_\tau^{\pi'}}{\delta \pi}(s,a)(\pi - \pi')(da|s) + \tau \, \mathrm{KL}(\pi|\pi')(s) \, .$$

In particular with $\pi' = \pi_{old}$ *and* $\pi = \pi_{new}$ *given by the exact update* (18) *satisfy this.*

Proof. From perf. diff. lemma and Lan's trick:

$$\begin{aligned}(V_\tau^\pi - V_\tau^{\pi'})(s) &\leq \tfrac{1}{1-\gamma} \int_S \Big(\int_A \tfrac{\delta V_\tau^{\pi'}}{\delta \pi}(s',a)(\pi - \pi')(da|s') + \tau \, \mathrm{KL}(\pi|\pi')(s') \Big) d_s^\pi(ds') \\ &\leq \int_A \tfrac{\delta V_\tau^{\pi'}}{\delta \pi}(s,a)(\pi - \pi')(da|s) + \tau \, \mathrm{KL}(\pi|\pi')(s) \, .\end{aligned}$$

Convergence of mirror descent with approximate advantage

Ingredients for convergence of mirror descent

- Convexity (strong for "linear" rate)
- L-smoothness
- Three point lemma

Lemma 12 (Three point lemma / Bregman proximal inequality)

Let $G : M_\mu \to \mathbb{R}$ be convex. For all $m' \in M_\mu$ let

$$m^* = \operatorname*{argmin}_{m \in M_\mu} \left\{ G(m) + \mathrm{KL}(m|m') \right\} . \tag{24}$$

Then, for all $m \in M_\mu$, we have

$$G(m) + \mathrm{KL}(m|m') \geq G(m^*) + \mathrm{KL}(m|m^*) + \mathrm{KL}(m^*|m') . \tag{25}$$

The proof of Lemma 12 can be found e.g., in [Aubin-Frankowski et al., 2022] noting that the flat derivative of KL is well defined on M_μ, see e.g. [Kerimkulov et al., 2025b, Lemma 3.8].

Let π^n be generated by inductive application of the approximate mirror descent step (21). Let $V_\tau^n := V_\tau^{\pi^n}$ for $n \in \mathbb{N}$. We begin with an application of Bregman proximal inequality, see Lemma 12. Fix $s \in S$ and $\pi^n \in \Pi_\mu$ and define $G : M_\mu \to \mathbb{R}$ by

$$G(m) = \frac{1}{\lambda} \int_A \left(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s) \right) (m(da) - \pi^n(da|s)) .$$

It is linear and thus clearly convex and hence due to the mirror descent update (21) is equivalent to (24) and so we have, for all $\pi \in \Pi_\mu$, $s \in S$ and $n \in \mathbb{N}$ that

$$\begin{aligned}
&\frac{1}{\lambda} \int_A \left(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s) \right) (\pi - \pi^n)(da|s) + \mathrm{KL}(\pi|\pi^n)(s) \\
&\geq \frac{1}{\lambda} \int_A \left(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s) \right) (\pi^{n+1} - \pi^n)(da|s) + \mathrm{KL}(\pi|\pi^{n+1})(s) + \mathrm{KL}(\pi^{n+1}|\pi^n)(s) .
\end{aligned}$$

Re-arranging this leads to

$$\begin{aligned}
&\mathrm{KL}(\pi|\pi^{n+1})(s) - \mathrm{KL}(\pi|\pi^n)(s) \\
&\leq \frac{1}{\lambda} \int_A \left(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s) \right) (\pi - \pi^n)(da|s) \\
&\qquad - \frac{1}{\lambda} \int_A \left(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s) \right) (\pi^{n+1} - \pi^n)(da|s) - \mathrm{KL}(\pi^{n+1}|\pi^n)(s) .
\end{aligned} \tag{26}$$

From the performace difference, Lemma 6, we have

$$
\begin{aligned}
&(V_\tau^{n+1} - V_\tau^n)(s) \\
&= \tfrac{1}{1-\gamma} \int_S \big(\int_A \big(\hat{A}_n - \mathcal{E}_n + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}\big)(s,a)(\pi^{n+1} - \pi^n)(da|s) + \tau \, \mathrm{KL}(\pi^{n+1}|\pi^n)(s)\big) \, d_\rho^{\pi^{n+1}}(ds) \,.
\end{aligned}
$$

Note that (21), together with $\lambda \geq \tau$ guarantees that

$$
0 \geq \int_A \big(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s)\big)(\pi^{n+1} - \pi^n)(da|s) + \tau \, \mathrm{KL}(\pi^{n+1}|\pi^n)(s) =: F(s)
$$

for all $s \in S$. Thus we may apply Lemma 10 and get

$$
(V_\tau^{n+1} - V_\tau^n)(s) \leq F(s) - \tfrac{1}{1-\gamma} \int_S \int_A \mathcal{E}_n(s,a)(\pi^{n+1} - \pi^n)(da|s) d_\rho^{\pi^{n+1}}(ds) \,.
$$

Assume that $\|\mathcal{E}\|_{B_b(S \times A)} = \delta_n < \infty$. Then we have the following approximate L-smoothness:

$$
(V_\tau^{n+1} - V_\tau^n)(s) \leq F(s) + \frac{2\delta_n}{1-\gamma} \,, \;\; s \in S.
$$

Applying this in (26) and taking we thus have, for all $s \in S$, that

$$
\begin{aligned}
\mathrm{KL}(\pi_\tau^*|\pi^{n+1})(s) - \mathrm{KL}(\pi_\tau^*|\pi^n)(s) \leq \frac{1}{\lambda} \int_A \big(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s)\big)(\pi_\tau^* - \pi^n)(da|s) \\
- \tfrac{1}{\lambda}(V_\tau^{n+1} - V_\tau^n)(s) + \tfrac{2\delta_n}{(1-\gamma)\lambda} \,.
\end{aligned}
\tag{27}
$$

Summing up over $n = 0, 1, \ldots, N-1$ we see (spotting the telescoping sums) that for all $s \in S$,

$$
\begin{aligned}
\mathsf{KL}(\pi^*_\tau|\pi^N)(s) - \mathsf{KL}(\pi^*_\tau|\pi^0)(s) \leq \sum_{n=0}^{N-1} \frac{1}{\lambda} \int_A \Big(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s)\Big)(\pi^*_\tau - \pi^n)(da|s) \\
- \frac{1}{\lambda}(V^N_\tau - V^0_\tau)(s) + \frac{2}{(1-\gamma)\lambda} \sum_{n=0}^{N-1} \delta_n \,.
\end{aligned}
$$

We wish to apply performance difference in due course and so we observe that the above is equivalent to

$$
\begin{aligned}
\mathsf{KL}(\pi^*_\tau|\pi^N)(s) - \mathsf{KL}(\pi^*_\tau|\pi^0)(s) \leq \sum_{n=0}^{N-1} \frac{1}{\lambda} \int_A \Big(A^{\pi^n}_\tau(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s)\Big)(\pi^*_\tau - \pi^n)(da|s) \\
+ \sum_{n=0}^{N-1} \frac{1}{\lambda} \int_A \mathcal{E}_n(s,a)(\pi^*_\tau - \pi^n)(da|s) - \frac{1}{\lambda}(V^N_\tau - V^0_\tau)(s) + \frac{2}{(1-\gamma)\lambda} \sum_{n=0}^{N-1} \delta_n \,.
\end{aligned}
\tag{28}
$$

Notice that $V_\tau^N(s) \geq V_\tau^*(s)$ and so $(V_\tau^N - V_\tau^0)(s) \geq (V_\tau^* - V_\tau^0)(s)$ for all $N \in \mathbb{N}$. Let

$$y^n := \int_S \mathrm{KL}(\pi_\tau^*|\pi^n)(s) d_\rho^{\pi_\tau^*}(ds) \text{ and } \alpha := -\int_S (V_\tau^* - V^0)(s) d_\rho^{\pi_\tau^*}(ds)$$

so that, after integrating (28) over $d_\rho^{\pi_\tau^*}$ and using $\|\mathcal{E}\|_{B_b(S\times A)} = \delta_n < \infty$ we have

$$y^N - y^0 \leq \sum_{n=0}^{N-1} \frac{1}{\lambda} \int_S \int_A \frac{\delta V_\tau^n}{\delta \pi}(s,a)(\pi_\tau^* - \pi^n)(da|s) d_\rho^{\pi_\tau^*}(ds) + \frac{2}{\lambda} \sum_{n=0}^{N-1} \delta_n + \frac{\alpha}{\lambda} + \frac{2}{(1-\gamma)\lambda} \sum_{n=0}^{N-1} \delta_n \,.$$

Using the performance difference lemma, see Lemma 6, and upper bounding the approximation error terms we get

$$y^N - y^0 \leq \sum_{n=0}^{N-1} \left[\frac{1-\gamma}{\lambda} (V^{\pi_\tau^*} - V^{\pi^n})(\rho) - \frac{\tau}{\lambda} \int_S \mathrm{KL}(\pi_\tau^*|\pi^n)(s) d_\rho^{\pi_\tau^*}(ds) \right] + \frac{\alpha}{\lambda} + \frac{4}{(1-\gamma)\lambda} \sum_{n=0}^{N-1} \delta_n \,.$$

Since since $\mathrm{KL}(\cdot|\cdot) \geq 0$ we get that

$$y^N - y^0 \leq N \frac{1-\gamma}{\lambda} \Big(V_\tau^{\pi_\tau^*}(\rho) - \min_{n=0,1,\dots,N-1} V_\tau^{\pi^N}(\rho) \Big) + \frac{\alpha}{\lambda} + \frac{4}{(1-\gamma)\lambda} \sum_{n=0}^{N-1} \delta_n \,.$$

Hence

$$N \frac{1-\gamma}{\lambda} \Big(\min_{n=0,1,\dots,N-1} V_\tau^{\pi^N}(\rho) - V_\tau^{\pi_\tau^*}(\rho) \Big) \leq \frac{\alpha}{\lambda} + y^0 + \frac{4}{(1-\gamma)\lambda} \sum_{n=0}^{N-1} \delta_n \,.$$

and so

$$0 \leq \min_{n=0,1,\dots,N-1} V_\tau^{\pi^N}(\rho) - V_\tau^{\pi_\tau^*}(\rho) \leq \frac{1}{N} \frac{\alpha + \lambda y^0}{1-\gamma} + \frac{1}{N} \frac{4}{(1-\gamma)^2} \sum_{n=0}^{N-1} \delta_n \,.$$

Convergence of mirror descent with approximate advantage

Theorem 13

Given $\pi_0 \in \Pi_\mu$*, let* $(\pi_n)_{\mathbb{N}}$ *be given by*

$$\pi^{n+1}(da|s) = \operatorname*{argmin}_{m\in\mathcal{P}(A)} \int_A \Big(\hat{A}_n(s,a) + \tau \ln \tfrac{\mathrm{d}\pi^n}{\mathrm{d}\mu}(a|s)\Big)(m(da) - \pi^n(da|s)) + \lambda \,\mathrm{KL}(m|\pi^n(\cdot|s)\,.$$

where $\hat{A}_n(s,a) = A^{\pi_n}_\tau(s,a) + \mathcal{E}_n(s,a)$ *and* $\|\mathcal{E}\|_{B_b(S\times A)} = \delta_n < \infty$ *for all* $n \in \mathbb{N}$*. Then*

$$0 \le \min_{n=0,1,\dots,N-1} V^{\pi^N}_\tau(\rho) - V^{\pi^*_\tau}_\tau(\rho) \le \frac{1}{N}\frac{\alpha + \lambda y^0}{1-\gamma} + \frac{1}{N}\frac{4}{(1-\gamma)^2}\sum_{n=0}^{N-1}\delta_n\,,$$

where $\alpha := -\int_S (V^*_\tau - V^0)(s) d^{\pi^*_\tau}_\rho(ds)$ *and* $y^0 := \int_S \mathrm{KL}(\pi^*_\tau|\pi^0)(s)\; d^{\pi^*_\tau}_\rho(ds)$*.*

This is a small extension of results in [Kerimkulov et al., 2025a], [Lan, 2023].

Natural policy gradient is mirror descent

Natural policy gradient (NPG)

Let $\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) := \frac{e^{g_\theta(s,a)}}{Z_\theta(s)}$, $Z_\theta(s) := \int_A e^{g_\theta(s,a')}\mu(da')$ with $g_\theta(s,a) = (\theta, \phi(s,a))_{\mathbb{H}}$.

Fisher information matrix

$$F(\theta) := \int_S \int_A \nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu} \otimes \nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\pi_\theta(da|s)d_\rho^{\pi_\theta}(da|s)\,,$$

where for $\theta, \theta' \in \mathbb{H}$ we have $(\theta \otimes \theta')_{jk} = \theta_j \theta'_k$. Let

$$\phi_{\pi_\theta} := \phi(s,a) - \int_A \phi(s,a')\pi_\theta(da'|s)\,.$$

Recalling (16) we have that $\nabla_\theta \ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s) = \nabla_\theta g_\theta(s,a) - \int_A (\nabla_\theta g_\theta)(s,a')\frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a'|s)\mu(da') = \phi_{\pi_\theta}(s,a)$. Hence

$$F(\theta) = \int_S \int_A \phi_{\pi_\theta} \otimes \phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(da|s)\,.$$

Natural policy gradient (NPG) updates are

$$\theta_{n+1} = \theta_n - \eta F(\theta)^\dagger \nabla_\theta V_\tau^{\pi_{\theta^n}}(\rho)\,, \quad n = 0, 1, \dots\,, \quad \theta^0 \in \mathbb{H} \text{ given.} \tag{29}$$

Here, for $M \in \mathcal{L}(\mathbb{H},\mathbb{H})$ we use $M^\dagger$ to denote the Moore–Penrose pseudo-inverse (which coincides with M^{-1} for invertible M).

NPG in RL is due to [Kakade, 2001].

Proposition 6

If given $\theta \in \mathbb{H}$ we take $\ln \frac{\mathrm{d}\pi_{\pi_\theta}}{\mathrm{d}\mu}(a|s) = (\theta, \phi_\theta)_{\mathbb{H}}$ *and thus obtain π_{θ_n} corresponding to θ_n then $\pi_{\theta_{n+1}}$ with θ_{n+1} given by the NPG update* (29) *is equal to π^{n+1} given by*

$$\pi_{\theta_{n+1}}(\cdot|s) = \operatorname*{argmin}_{m\in\mathcal{P}(A)} \int_A \left(\hat{w}(\theta_n) + \tau\theta_n, \phi_{\pi_{\theta_n}}(s,a)\right)_{\mathbb{H}}(m(da) - \pi_{\theta_n}(da|s)) + \lambda\,\mathrm{KL}(m|\pi_{\theta_n}(\cdot|s))$$

which is the mirror descent update (18) *where the flat derivative is replaced by its approximation* $\hat{A}_n = (\hat{w}(\theta) + \tau\theta, \phi_{\pi_\theta})_{\mathbb{H}}$.

Remark: Let[2]

$$L^{\pi_\theta}(w) := \tfrac{1}{2}\int_S \int_A |A^{\pi_\theta}_\tau(s,a) - (w, \phi_{\pi_\theta}(s,a))_{\mathbb{H}}|^2 \pi_\theta(da|s) d^{\pi_\theta}_\rho(ds)\,, \tag{30}$$

where $A^{\pi_\theta}_\tau(s,a) = Q^{\pi_\theta}_\tau(s,a) - V^{\pi_\theta}_\tau(s)$. So NPG updates are:

$$\theta_{n+1} = \theta_n, -\tfrac{1}{\lambda}\left(\hat{w}(\theta_n) + \tau\theta_n\right)\,,$$

where $\hat{w}(\theta_n)$ is the minimizer for (30).

[2] We are not including the $\ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}$ term. It's just an additive term we can trivially see that $|\ln \frac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu} - (y, \phi_{\pi_\theta})_{\mathbb{H}}|^2$ is minimized by $y = \theta$.

Proof. Notice that

$$\nabla_w L^{\pi_\theta}(w) = \int_S \int_A (A_\tau^{\pi_\theta}(s,a) - (w, \phi_{\pi_\theta}(s,a))_{\mathbb{H}})\phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(ds)$$

and so the first order condition for any minimizer $\hat{w}$ of (30) is

$$\int_S \int_A (\hat{w}, \phi_{\pi_\theta}(s,a))_{\mathbb{H}}\phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(ds) = \int_S \int_A A_\tau^{\pi_\theta}(s,a)\phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(ds)\,.$$

Moreover, for any $w \in \mathbb{H}$ we have $F(\theta)w = \int_S \int_A (w, \phi_{\pi_\theta}(s,a))_{\mathbb{H}}\phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(ds)$. Noting also that the minimizer above depends on θ we have

$$F(\theta)\hat{w}(\theta) = \int_S \int_A A_\tau^{\pi_\theta}(s,a)\phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(ds)\,.$$

Note that the Moore–Penrose pseudo-inverse provides the smallest norm solution to this i.e.

$$\hat{w}(\theta) = F(\theta)^\dagger \int_S \int_A A_\tau^{\pi_\theta}(s,a)\phi_{\pi_\theta}(s,a)\pi_\theta(da|s)d_\rho^{\pi_\theta}(ds)\,.$$

This, together with (17) leads to

$$\begin{aligned} F(\theta)^\dagger \nabla_\theta V_\tau^{\pi_\theta}(\rho) &= \tfrac{1}{1-\gamma} F(\theta)^\dagger \mathbb{E}_{a\sim\pi_\theta(\cdot|s)}^{s\sim d_\rho^{\pi_\theta}}\Big[\Big(A_\tau^{\pi_\theta}(s,a) + \tau \ln \tfrac{\mathrm{d}\pi_\theta}{\mathrm{d}\mu}(a|s)\Big)\phi_{\pi_\theta}(s,a)\Big] \\ &= \tfrac{1}{1-\gamma}\big(\hat{w}(\theta) + \tau\theta\big)\,. \end{aligned}$$

Have

$$F(\theta)^\dagger \nabla_\theta V_\tau^{\pi_\theta}(\rho) = \tfrac{1}{1-\gamma}\big(\hat{w}(\theta) + \tau\theta\big)\,.$$

So the NPG stepping scheme (29) becomes

$$\theta_{n+1} = \theta_n - \tfrac{\eta}{1-\gamma}\big(\hat{w}(\theta_n) + \tau\theta_n\big)\,,\quad n = 0,1,\dots\,,\quad \theta_0 \in \mathbb{H} \text{ given.}$$

Letting $\lambda = \eta(1-\gamma)^{-1}$ we have

$$(\theta_{n+1}, \phi)_{\mathbb{H}} = (\theta_n, \phi)_{\mathbb{H}} - \tfrac{1}{\lambda}\big(\hat{w}(\theta_n) + \tau\theta_n, \phi(s,a)\big)_{\mathbb{H}}\,.$$

Since $\ln \frac{\mathrm{d}\pi_{\pi_{\theta_n}}}{\mathrm{d}\mu}(a|s) = (\theta_n, \phi)_{\mathbb{H}} - \Big(\theta_n, \int_A \phi(\cdot, a')\pi_{\theta_n}(da'|\cdot)\Big)_{\mathbb{H}}$ and collecting all the terms constant in a in some $b = b(s)$ we then have

$$\ln \frac{\mathrm{d}\pi_{\theta_{n+1}}}{\mathrm{d}\mu}(a|s) = \ln \frac{\mathrm{d}\pi_{\pi_{\theta_n}}}{\mathrm{d}\mu}(a|s) - \tfrac{1}{\lambda}\big(\hat{w}(\theta_n) + \tau\theta_n, \phi_{\pi_{\theta_n}}(s,a)\big)_{\mathbb{H}} + b(s)\,,$$

with b chosen such that $\pi_{\theta_{n+1}} \in \mathcal{P}(A|S)$. Hence

$$\ln \frac{\mathrm{d}\pi_{\theta_{n+1}}}{\mathrm{d}\pi_{\pi_{\theta_n}}}(a|s) = -\tfrac{1}{\lambda}\big(\hat{w}(\theta_n) + \tau\theta_n, \phi_{\pi_{\theta_n}}(s,a)\big)_{\mathbb{H}} + b(s)\,.$$

And so

$$\frac{\mathrm{d}\pi_{\theta_{n+1}}}{\mathrm{d}\pi_{\pi_{\theta_n}}}(a|s) = \exp\Big(- \tfrac{1}{\lambda}\big(\hat{w}(\theta_n) + \tau\theta_n, \phi_{\pi_{\theta_n}}(s,a)\big)_{\mathbb{H}} + b(s)\Big)\,.$$

Then

$$\pi_{\theta_{n+1}}(\cdot|s) = \operatorname*{argmin}_{m\in\mathcal{P}(A)} \int_A \big(\hat{w}(\theta_n) + \tau\theta_n, \phi_{\pi_{\theta_n}}(s,a)\big)_{\mathbb{H}}(m(da) - \pi_{\theta_n}(da|s)) + \lambda\,\mathrm{KL}(m|\pi_{\theta_n}(\cdot|s))\,,$$

due to [Dupuis and Ellis, 1997], Lemma 1.4.3. □

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