2017/18 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 4 - Friday 2rd March 2018[1] - TopHat: ????

Note that these exercises are not in the lecture notes.

Exercise 4.1 (Duality approach, complete market, no consumption with power utility)**.** Consider the problem of maximising expected utility of terminal wealth, with no consumption, for the power utility function

$$U(x) = \frac{x^p}{p},\ p < 1,\ p \neq 0,\ x \in \mathbb{R}^+,$$

in the standard d-dimensional complete market model with stock price vector S following

$$\mathrm{d}S_t = \mathbf{diag}(S_t)[\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t],$$

with unique market price of risk process $\lambda_t = \sigma_t^{-1}(\mu_t - r)$ and deflator Y. The value function is

$$u(x) := \sup_{\theta \in \mathcal{U}} \mathbb{E}[U(X_T)],$$

where $\mathcal{U}$ denotes admissible strategies which represent fraction of wealth invested in each of the assets.
You may use the results derived in lectures for maximisation of expected utility of terminal wealth using duality.
We will need to define the process H by

$$H_t := \mathbb{E}[Y_T^q | \mathcal{F}_t], 0 \leq t \leq T, \ \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

a) Show that the optimal terminal wealth, value function and optimal wealth process are given by

$$\hat{X}_T = \frac{x}{H_0} Y_T^{-(1-q)},\ u(x) = \frac{x^p}{p} H_0^{1-p},\ \hat{X}_t = \frac{x}{H_0}\frac{H_t}{Y_t},\ 0 \leq t \leq T.$$

b) Explain why the process H must have dynamics of the form

$$\mathrm{d}H_t = H_t \gamma_t^* \mathrm{d}W_t,$$

for some adapted d-dimensional vector γ, and deduce that the optimal portfolio proportion process $\hat{\theta}$ can be written in the form

$$\hat{\theta}_t = (\sigma_t^*)^{-1}(\lambda_t + \gamma_t)\ ,\ 0 \leq t \leq T.$$

c) Decompose Y^q according to $Y^q = \Lambda M$, for some adapted processes Λ, M, where $M := \mathcal{E}(-q\lambda \cdot W)$ is an exponential martingale, and deduce a formula for the process Λ.

[1] Last updated 21st March 2018

d) Hence show that H may be written as $H = ML$, where L is the process defined by

$$L_t := \mathbb{E}_M[\Lambda_T | F_t], \ 0 \leq t \leq T,$$

and where $\mathbb{E}_M$ denotes expectation with respect to the measure $\mathbb{P}_M$ defined by

$$\frac{\mathrm{d}\mathbb{P}_M}{\mathrm{d}\mathbb{P}} = M_T.$$

e) Explain why the $\mathbb{P}$-dynamics of L must be of the form

$$\mathrm{d}L_t = L_t \nu_t^* (q\lambda_t \mathrm{d}t + \mathrm{d}W_t) \ ,$$

for some d-dimensional adapted process ν.

f) By considering the dynamics of $H = ML$, deduce that

$$\gamma_t = \nu_t - q\lambda_t, \ 0 \leq t \leq T,$$

and hence that the optimal proportion of wealth in the stock is given by

$$\hat{\theta}_t = (\sigma_t^*)^{-1}[(1-q)\lambda_t + \nu_t], \ 0 \leq t \leq T.$$

g) If r and λ are deterministic, deduce that the formula for the optimal proportion of wealth in stock reduces to

$$\hat{\theta}_t = (\sigma_t^*)^{-1}(1-q)\lambda_t = \frac{1}{1-p}(\sigma_t^*)^{-1}\lambda_t, \quad 0 \leq t \leq T,$$

and derive formulae for the value function and optimal wealth process in this case (i.e. no ν term).

Solution (to Exercise 4.1)**.**

a) From the lectures, the optimal terminal wealth is given by $U'(\hat{X}_T) = yY_T$, with y determined via $\mathbb{E}[Y_T\hat{X}_T] = x$ (since Y is the deflator of X). Since $U'(x) = x^{p-1}$ and $U'(\hat{X}_T) = yY_T$ we obtain

$$\hat{X}_T = (yY_T)^{-(1-q)},$$

where we have used $1 - q = 1/(1-p)$. The parameter y is thus determined by

$$\mathbb{E}[Y_T(yY_T)^{-(1-q)}] = x,$$

which yields (since y constant)

$$y^{(1-q)} = \frac{H_0}{x},$$

where $H_0 = \mathbb{E}[Y_T^q]$. Hence we arrive at

$$\hat{X}_T = \frac{x}{H_0} Y_T^{-(1-q)},$$

which give the first term.

By definition, the value function is given by $u(x) = \mathbb{E}[\hat{X}_T^p/p]$, and using the previous result yields

$$u(x) = \frac{x^p}{p} H_0^{1-p}.$$

Finally, using that XY is a martingale and hence $\mathbb{E}[\hat{X}_T Y_T|\mathcal{F}_t] = \hat{X}_t Y_t$, we may write the dynamics of the optimal wealth process as

$$\hat{X}_t = \frac{1}{Y_t}\mathbb{E}[Y_T\hat{X}_T|\mathcal{F}_t] = \frac{x}{H_0}\frac{H_t}{Y_t}, \quad 0 \leq t \leq T,$$

where

$$H_t := \mathbb{E}[Y_T^q|\mathcal{F}_t], \ 0 \leq t \leq T.$$

b) Noting that Y is non-negative, then process H is a non-negative $\mathbb{P}$-martingale (this does NOT imply Y is a martingale), thus it has dynamics of the form

$$\mathrm{d}H_t = H_t\gamma_t^*\mathrm{d}W_t,$$

for some adapted d-dimensional vector γ.

From the previous part we have $\hat{X}_tY_t = (x/H_0)H_t$, so

$$\mathrm{d}(\hat{X}_tY_t) = \frac{x}{H_0}\mathrm{d}H_t = \frac{x}{H_0}H_t\gamma_t^*\mathrm{d}W_t = \hat{X}_tY_t\gamma_t^*\mathrm{d}W_t.$$

Comparing this with the dynamics (which is the known form of XY)

$$\mathrm{d}(\hat{X}_tY_t) = \hat{X}_tY_t(\hat{\theta}_t^*\sigma_t - \lambda_t^*)\mathrm{d}W_t,$$

yields

$$\hat{\theta}_t = (\sigma_t^*)^{-1}(\lambda_t + \gamma_t)\ ,\ 0 \le t \le T. \tag{1}$$

c) Using the fact that Y is a deflator we may write, $Y = DZ$, with $Z = \mathcal{E}(-\lambda \cdot W)$ and $D_t = \exp(-\int_0^t r_s\mathrm{d}s)$. Hence,

$$\begin{aligned} Y_t^q &= D_t^q \exp(-q\int_0^t \lambda_s^*\mathrm{d}W_s - \frac{1}{2}q\int_0^t \|\lambda_s\|^2\mathrm{d}s) \\ &= D_t^q \exp(-\frac{1}{2}q(1-q)\int_0^t \|\lambda_s\|^2\mathrm{d}s)\mathcal{E}(-q\lambda\cdot W)_t =: \Lambda_tM_t, \end{aligned}$$

with Λ and M defined by

$$\begin{aligned} \Lambda_t &:= D_t^q \exp(-\frac{1}{2}q(1-q)\int_0^t \|\lambda_s\|^2\mathrm{d}s) \\ &= \exp(-q\int_0^t r_s\mathrm{d}s - \frac{1}{2}q(1-q)\int_0^t \|\lambda_s\|^2\mathrm{d}s), \quad 0 \le t \le T., \\ M_t &:= \mathcal{E}(-q\lambda\cdot W)_t, \quad 0 \le t \le T, \end{aligned}$$

and note that M is an exponential $\mathbb{P}$-martingale.

d) A more general version of Baye's formula says that for an integrable $\mathcal{F}_T$-measurable random variable V we have

$$\mathbb{E}_M[V|\mathcal{F}_t] = \frac{1}{M_t}\mathbb{E}[VM_T|\mathcal{F}_t],\ 0 \le t \le T,$$

where $\mathbb{E}_M$ denotes expectation with respect to the probability measure $\mathbb{P}_M$ with density process

$$\frac{\mathrm{d}\mathbb{P}_M}{\mathrm{d}\mathbb{P}}\mathcal{F}_t = M_t,\ 0 \le t \le T.$$

Therefore using the previous part and then Baye's formula we can write

$$H_t = \mathbb{E}[Y_T^q|\mathcal{F}_t] = \mathbb{E}[\Lambda_TM_T|\mathcal{F}_t] = M_t\mathbb{E}_M[\Lambda_T|\mathcal{F}_t] =: M_tL_t, \quad 0 \le t \le T,$$

where L is defined by

$$L_t := \mathbb{E}_M[\Lambda_T|\mathcal{F}_t], \quad 0 \le t \le T.$$

e) Similar to H, one can also note that L is a non-negative $\mathbb{P}_M$-martingale. Hence it has $\mathbb{P}_M$-dynamics of the form

$$\mathrm{d}L_t = L_t \nu_t^* \mathrm{d}W_t^M,$$

for some d-dimensional adapted process v, where W^M is a d-dimensional $\mathbb{P}_M$-Brownian motion. But by the Girsanov theorem, we know that W and W^M are related by

$$W_t^M = W_t + \int_0^t q\lambda_s \mathrm{d}s, \quad 0 \le t \le T,$$

and so the $\mathbb{P}$-dynamics of L are of the form

$$\mathrm{d}L_t = L_t \nu_t^* (q\lambda_t \mathrm{d}t + \mathrm{d}W_t)\ .$$

f) Applying Itô's product rule to $H = ML$ we obtain

$$\begin{aligned} \mathrm{d}H_t = \mathrm{d}(M_t L_t) &= M_t \mathrm{d}L_t + L_t \mathrm{d}M_t + \mathrm{d}\{M,\ L\}_t \\ &= M_t L_t \nu_t^* (q\lambda_t \mathrm{d}t + \mathrm{d}W_t) - qM_t L_t \lambda_t^* \mathrm{d}W_t - qM_t L_t \nu_t^* \lambda_t \mathrm{d}t \\ &= H_t (\nu_t^* - q\lambda_t^*) \mathrm{d}W_t. \end{aligned}$$

Comparing this with

$$\mathrm{d}H_t = H_t \gamma_t^* \mathrm{d}W_t,$$

gives

$$\gamma_t = \nu_t - q\lambda_t, \quad 0 \le t \le T,$$

and on using the result (1) we deduce that the optimal proportion of wealth in the stock is given by

$$\hat{\theta}_t = (\sigma_t^*)^{-1}[(1-q)\lambda_t + \nu_t], \quad 0 \le t \le T. \tag{2}$$

g) If r and λ are deterministic, then so is Λ. Thus, for all $t \ge 0$, $L_t = \mathbb{E}[\Lambda_T | \mathcal{F}_t] = \Lambda_T$ (constant), which implies $dL_t = 0$ and hence $\nu = 0$. Thus the formula for the optimal portfolio reduces to

$$\hat{\theta}_t = (\sigma_t^*)^{-1}(1-q)\lambda_t = \frac{1}{1-p}(\sigma_t^*)^{-1}\lambda_t, \quad 0 \le t \le T.$$

In this case we also have, $H_0 = \mathbb{E}[Y_T^q] = \mathbb{E}[\Lambda_T M_T] = \Lambda_T$ (since $\mathbb{E}[M_T] = 1$), so using the results in part a) the value function is given by

$$\begin{aligned} u(x) &= \frac{x^p}{p} H_0^{(1-p)} \\ &= \frac{x^p}{p} \Lambda_T^{(1-p)} \\ &= \frac{x^p}{p} D_T^{-p} \exp\left(\frac{p}{2(1-p)} \int_0^T \|\lambda_t\|^2 \mathrm{d}t \right) \\ &= \frac{x^p}{p} \exp\left[p \int_0^T (r_t + \frac{1}{2(1-p)} \|\lambda_t\|^2) \mathrm{d}t \right] \end{aligned}$$

The optimal wealth process is given by

$$\hat{X}_t = \frac{x}{H_0} \frac{H_t}{Y_t} = \frac{x}{H_0} \frac{M_t L_t}{Y_t},$$

and $H_t = M_t \Lambda_T$ when r, λ are deterministic, so we obtain

$$\hat{X}_t = x \frac{M_t}{Y_t} = \frac{x}{D_t} \frac{M_t}{Z_t} = \frac{x}{D_t} \frac{\mathcal{E}(-q\lambda. \cdot W)_t}{\mathcal{E}(-\lambda W)_t}, \quad 0 \le t \le T.$$