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2017/18 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 4 - Friday 2rd March 2018[1] - TopHat: ????

Note that these exercises are not in the lecture notes.

Exercise 4.1 (Duality approach, complete market, no consumption with power utility)**.** Consider the problem of maximising expected utility of terminal wealth, with no consumption, for the power utility function

$$U(x) = \frac{x^p}{p},\ p < 1,\ p \neq 0,\ x \in \mathbb{R}^+,$$

in the standard d-dimensional complete market model with stock price vector S following

$$\mathrm{d}S_t = \mathbf{diag}(S_t)[\mu_t \mathrm{d}t + \sigma_t \mathrm{d}W_t],$$

with unique market price of risk process $\lambda_t = \sigma_t^{-1}(\mu_t - r)$ and deflator Y. The value function is

$$u(x) := \sup_{\theta \in \mathcal{U}} \mathbb{E}[U(X_T)],$$

where $\mathcal{U}$ denotes admissible strategies which represent fraction of wealth invested in each of the assets.

You may use the results derived in lectures for maximisation of expected utility of terminal wealth using duality.

We will need to define the process H by

$$H_t := \mathbb{E}[Y_T^q | \mathcal{F}_t], 0 \leq t \leq T, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

a) Show that the optimal terminal wealth, value function and optimal wealth process are given by

$$\hat{X}_T = \frac{x}{H_0} Y_T^{-(1-q)},\ u(x) = \frac{x^p}{p} H_0^{1-p},\ \hat{X}_t = \frac{x}{H_0}\frac{H_t}{Y_t},\ 0 \leq t \leq T.$$

b) Explain why the process H must have dynamics of the form

$$\mathrm{d}H_t = H_t \gamma_t^* \mathrm{d}W_t,$$

for some adapted d-dimensional vector γ, and deduce that the optimal portfolio proportion process $\hat{\theta}$ can be written in the form

$$\hat{\theta}_t = (\sigma_t^*)^{-1}(\lambda_t + \gamma_t)\ ,\ 0 \leq t \leq T.$$

c) Decompose Y^q according to $Y^q = \Lambda M$, for some adapted processes Λ, M, where $M := \mathcal{E}(-q\lambda \cdot W)$ is an exponential martingale, and deduce a formula for the process Λ.

[1] Last updated 21st March 2018

d) Hence show that H may be written as $H = ML$, where L is the process defined by

$$L_t := \mathbb{E}_M[\Lambda_T|F_t], \ 0 \leq t \leq T,$$

and where $\mathbb{E}_M$ denotes expectation with respect to the measure $\mathbb{P}_M$ defined by

$$\frac{\mathrm{d}\mathbb{P}_M}{\mathrm{d}\mathbb{P}} = M_T.$$

e) Explain why the $\mathbb{P}$-dynamics of L must be of the form

$$\mathrm{d}L_t = L_t \nu_t^* (q\lambda_t \mathrm{d}t + \mathrm{d}W_t)\ ,$$

for some d-dimensional adapted process ν.

f) By considering the dynamics of $H = ML$, deduce that

$$\gamma_t = \nu_t - q\lambda_t, \ 0 \leq t \leq T,$$

and hence that the optimal proportion of wealth in the stock is given by

$$\hat{\theta}_t = (\sigma_t^*)^{-1}[(1-q)\lambda_t + \nu_t], \ 0 \leq t \leq T.$$

g) If r and λ are deterministic, deduce that the formula for the optimal proportion of wealth in stock reduces to

$$\hat{\theta}_t = (\sigma_t^*)^{-1}(1-q)\lambda_t = \frac{1}{1-p}(\sigma_t^*)^{-1}\lambda_t, \quad 0 \leq t \leq T,$$

and derive formulae for the value function and optimal wealth process in this case (i.e. no ν term).