David Šiška School of Mathematics Sciences, University of Edinburgh

2018/19 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 2 - Friday 1st February 2019[1]

Exercise 2.1 (Non-existence of solution)**.**

1. Let $I = \left[0, \frac{1}{2}\right]$. Find a solution X for
$$\frac{dX_t}{dt} = X_t^2\,,\ \ t \in I\,,\ \ X_0 = 1\,.$$
2. Does a solution to the above equation exist on $I = [0,1]$? If yes, show that it satisfies the definition of an SDE solution from lectures. In not, which property is violated?

Exercise 2.2 (Non-uniqueness of solution)**.** Fix $T > 0$. Consider
$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}\,,\ \ t \in [0,T]\,,\ \ X_0 = 0\,.$$

1. Show that $\bar{X}_t := 0$ for all $t \in [0,T]$ is a solution to the above ODE.
2. Show that $X_t := t^2$ for all $t \in [0,T]$ is also a solution.
3. Find at least two more solutions different from $\bar{X}$ and X.

Exercise 2.3. Consider the SDE
$$X_t = \xi + \int_0^t b_s(X_s)\,ds + \int_0^t \sigma_s(X_s)\,dW_s\,, \qquad t \in [0,T]\,.$$
and assume that the conditions of proposition about existence and uniqueness from the lecture hold. Show that the solution to the SDE is unique in the sense that if X and Y are two solutions with $X_0 = \xi = Y_0$ then
$$\mathbb{P}\left[\sup_{0\le t\le T} |X_t - Y_t| > 0\right] = 0\,.$$
Hint: Write down the equation for the difference of X and Y mimicking the existence proof where we considered difference of X^{n+1} and X^n. Use the same basic estimates but then use Gronwall's lemma at the right point.

Exercise 2.4. Consider the SDE
$$dX_s^{t,x} = b(X_s^{t,x})\,ds + \sigma(X_s^{t,x})\,dW_s,\ \ t \le s \le T,\ \ X_t^{t,x} = x\,.$$
Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then
$$\mathbb{P}\left[\sup_{t\le s\le T} |X_s^{t,x} - Y_s^{t,x}| > 0\right] = 0\,.$$
Show that then the *flow property* holds i.e. for $0 \le t \le t' \le T$ we have almost surely that
$$X_s^{t,x} = X_s^{t',X_{t'}^{t,x}}\,, \qquad \forall s \in [t',T].$$

[1]Last updated 14th February 2019