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2018/19 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 5 - Friday 22nd March 2019[1]

Exercise 5.1 (Merton's problem with exponential utility, no consumption, using Pontryagin's Maximum Principle)**.** Consider a model with a risky asset $(S_t)_{t\in[0,T]}$ and a risk-free asset $(B_t)_{t\in[0,T]}$ given by

$$
\begin{aligned}
dS_t &= \mu S_t\,dt + \sigma S_t\,dW_t \;\; t\in[0,T]\,, S_0 = S\,,\\
dB_t &= rB_t\,dt \;\; t\in[0,T]\,, S_0 = S\,, B_0 = 1\,,
\end{aligned}
$$

where $\mu, r \in \mathbb{R}$ and $\sigma > 0$ are given constants. Let $(X_t)_{t\in[0,T]}$ denote the value of a self-financing investment portfolio with $X_0 = x > 0$ and let α_t denote the fraction X_t invested in the risky asset. We note that X_t depends on the investment strategy α_t and so we write $X_t = X_t^\alpha$. We will only consider α that are real-valued, adapted and such that $\mathbb{E}\int_0^T \alpha_t^2\,dt < \infty$, denoting such strategies $\mathcal{A}$ and calling them admissable. Our aim is to find the investment strategy $\hat\alpha$ which maximizes, over $\alpha\in\mathcal{A}$,

$$
J(\alpha) = \mathbb{E}\left[-\exp(-\gamma X_T^\alpha)\right]\,,
$$

for some $\gamma > 0$.

i) Use the definition of a self-financing portfolio to derive the equation for the portfolio value:
$$
dX_t = X_t\left[\alpha_t(\mu - r) + r\right]\,dt + X_t\alpha_t\sigma\,dW_t\,.
$$

ii) Write down the Hamiltonian for the problem and the adjoint BSDE for the optimal portfolio (use $\hat\alpha$ to denote the optimal control, $(\hat Y, \hat Z)$ to denote the BSDE).

iii) Justify the use of Pontryagin's maximum principle and show that it implies that
$$
\hat Z_t = -\frac{\mu - r}{\sigma}\hat Y_t\,.
$$

iv) Noting that $\hat Y_T = \gamma e^{-\gamma \hat X_T}$ use the "ansatz" $\hat Y_t = \phi_t e^{-\psi_t \hat X_t}$ with some $\phi, \psi \in C^1([0,T])$ such that $\phi_T = \gamma$ and $\psi_T = \gamma$. Hence show that
$$
\hat X_t \hat\alpha_t = e^{-r(T-t)}\frac{\mu - r}{\gamma\sigma^2}\,.
$$

[1] Last updated 21st March 2019