Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results proved in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. We consider the standard Black–Scholes model for optimal investment: a risk-free asset B and a risky asset S given by

$$B_t := \exp(rt) \text{ and } S_t := S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W_t\right) .$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. Let X_s denote the investment portfolio value at time $s \geq t$ and $X_t = x > 0$. There will be no cash injections and no consumption. Let $\nu = (\nu_t)_{t\in[0,T]}$ be the fraction of portfolio value invested in the risky asset. We will assume that $\mathbb{E}\int_t^T \nu_s^2\, ds < \infty$ and that ν is adapted to the filtration generated by W. For such ν we write $\nu \in \mathcal{A}$.

a) Derive the SDE satisfied by the portfolio value process $X_s = X_s^{\nu,t,x}$. **[3 marks]**

b) Consider the control problem

$$v(t,x) := \sup_{\nu\in\mathcal{A}} \mathbb{E}\left[\ln(X_T^{\nu,t,x})\right] . \tag{1}$$

Write down the Bellman PDE that the function v must satisfy. **[3 marks]**

c) Show that

$$v(t,x) = \ln x - (T-t)\left[\tfrac{1}{2}\sigma^2\hat{u}^2 - (\mu - r)\hat{u} - r\right] ,$$

where

$$\hat{u} := \frac{\mu - r}{\sigma^2} .$$

[7 marks]

d) Use the verification theorem to prove that the v above and the optimal control you identified are indeed the solution to the optimal control problem (1). **[4 marks]**

2. We consider a simple model for optimal liquidation of an asset via market orders on an exchange over a finite time interval $[0,T]$. Let W be a real-valued Wiener process generating the filtration $\mathbb{F}$, let the process $Q = (Q_t)_{t\in[0,T]}$ represent the inventory level and let the process $S = (S_t)_{t\in[0,T]}$ represent the asset price. The control is $\alpha = (\alpha_t)_{t\in[0,T]}$ which represents the selling rate at t (if $\alpha_t > 0$) or, buying rate at t (if $\alpha_t < 0$). In our model

$$\begin{aligned} dQ_t &= -\alpha_t\, dt \\ dS_t &= \lambda \alpha_t\, dt + \sigma\, dW_t\,, \quad t \in [0,T]\,, \quad Q_0 = q > 0\,, \quad S_0 = S > 0\,. \end{aligned}$$

Here $\sigma > 0$ is the volatility of the asset and $\lambda < 0$ captures the permanent price impact of our trading. There is temporary price impact captured by the "slippage" parameter $\kappa > 0$ and the price at which we actually sell is $S_t - \kappa\alpha_t$. Finally, there is a penalty for unsold inventory at time T given by $\theta \geq 0$.

Let the set of real-valued, $\mathbb{F}$-adapted processes $\alpha = (\alpha_s)_{s\in[0,T]}$ such that $\mathbb{E}\int_0^T |\alpha_s|^2\, ds < \infty$ be denoted by $\mathcal{A}$. We wish to maximize, over $\alpha \in \mathcal{A}$, the functional

$$M(q,S,\alpha) = \mathbb{E}^{q,S,\alpha}\left[\int_0^T (S_t\alpha_t - \kappa\alpha_t^2)\, dt + Q_T S_T - \theta Q_T^2\right].$$

a) Prove that if $\alpha \in \mathcal{A}$ then $\mathbb{E}\left[\int_0^T Q_t^2\, dt\right] < \infty$. *Hint:* Use Hölder's inequality. **[3 marks]**

b) Hence show that with $\gamma := \theta + \frac{1}{2}\lambda$

$$M(q,S,\alpha) = qS - \theta q^2 + J(q,\alpha)\,, \text{ where } J(q,\alpha) := \mathbb{E}^{q,\alpha}\left[\int_0^T \left(2\gamma\alpha_t Q_t - \kappa\alpha_t^2\right)\, dt\right].$$

Hint: Use product rule to calculate $d(Q_t S_t) = \ldots$ and $d(Q_t^2) = \ldots$. **[3 marks]**

c) Write down the Hamiltonian and the adjoint BSDE $(\hat{Y}, \hat{Z})$ associated to the optimal control $\hat{\alpha}$ for the control problem

$$\max_{\alpha\in\mathcal{A}} J(q,\alpha) \text{ subject to } Q_t = q - \int_0^t \alpha_s\, ds\,.$$

[3 marks]

d) Use the Pontryagin maximum principle to show that the optimal control is

$$\hat{\alpha}_t = \frac{1}{\kappa}\left(\frac{1}{\gamma} + \frac{1}{\kappa}(T-t)\right)^{-1} \hat{Q}_t\,.$$

Hint: Make the "ansatz" that $\hat{Y}_t = 2\xi(t)\hat{Q}_t$ for some $\xi \in C^1([0,T];\mathbb{R})$. **[8 marks]**

3. Let W be an $\mathbb{R}$-valued Wiener process generating the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ i.e $\mathcal{F}_t := \sigma(W_s : s \leq t)$. Let the set of real-valued, $\mathbb{F}$-adapted processes $\alpha = (\alpha_s)_{s\in[0,T]}$ such that $\mathbb{E}\int_0^T |\alpha_s|^2\,ds < \infty$ be denoted by $\mathcal{A}$. For $x \in \mathbb{R}$ and $\alpha \in \mathcal{A}$ let

$$X_s^{t,x,\alpha} = x + \int_t^s \alpha_r\,dW_r\,.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be such that for some constants $K \geq 0$, $m \in \mathbb{N}$, it holds for all $x \in \mathbb{R}$ that $|g(x)| \leq K(1 + |x|^m)$. For $t \in [0,T] \times \mathbb{R}$ let

$$v(t,x) := \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[g(X_T^{t,x,\alpha})\right]\,.$$

Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$.

a) Prove that

$$0 \geq \partial_t v + \frac{1}{2}a^2\partial_x^2 v \text{ on } [0,T) \times \mathbb{R}\,.$$

Hint: Use the Bellman principle (DPP) and then Itô's formula. **[10 marks]**

b) Hence prove that for any $t \in [0,T)$ the function $x \mapsto v(t,x)$ is concave. **[3 marks]**

c) Hence prove that if g is concave then $v(t,\cdot) = g$. **[3 marks]**