

## 2020/21 Semester 2

## Stochastic Control and Dynamic Asset Allocation

## Problem Sheet 2 - Last updated 8th January 2021

**Exercise 2.1.** There is a biased coin with  $p \in (0, 1)$ ,  $p \neq \frac{1}{2}$ , probability of getting heads and  $q = 1 - p$  probability of getting tails.

We will start with an initial wealth  $x = i$ ,  $i \in \mathbb{N}$  with  $i < m$ , with some  $m = 2$ .

At each turn we choose an action  $a \in \{-1, 1\}$ . By choosing  $a = 1$  we bet that the coin comes up heads and our wealth is increased by 1 if we are correct, decreased by 1 otherwise. By choosing  $a = -1$  we bet on tails and our wealth is updated accordingly.

That is, given that  $X_{n-1} = x$  and our action  $a \in \{-1, 1\}$  we have

$$\mathbb{P}(X_n = x + a \mid X_{n-1} = x, a) = p, \quad \mathbb{P}(X_n = x - a \mid X_{n-1} = x, a) = q.$$

The game terminates when either  $x = 0$  or  $x = m = 2$ . Let  $N = \min\{n \in \mathbb{N} : X_n = 0 \text{ or } X_n = m\}$ . Our aim is to maximize

$$J^\alpha(x) = \mathbb{E}[X_N^\alpha \mid X_0 = x]$$

over functions  $\alpha = \alpha(x)$  telling what action to choose in each given state.

1. Write down what the state space  $S$  and the stopping set  $\partial S$  are and write down the transition probability matrix for  $P^a$  for  $a = 1$  and for  $a = -1$ .
2. Write down the Bellman equation for the problem.
3. Assume that  $p > 1/2$ . Guess the optimal strategy. With your guess the Bellman equation is linear. Solve it for  $V$ .

**Exercise 2.2** (Non-existence of solution).

1. Let  $I = [0, \frac{1}{2}]$ . Find a solution  $X$  for

$$\frac{dX_t}{dt} = X_t^2, \quad t \in I, \quad X_0 = 1.$$

2. Does a solution to the above equation exist on  $I = [0, 1]$ ? If yes, show that it satisfies the definition of an SDE solution from SAF lectures. If not, which property is violated?

**Exercise 2.3** (Non-uniqueness of solution). Fix  $T > 0$ . Consider

$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}, \quad t \in [0, T], \quad X_0 = 0.$$

1. Show that  $\bar{X}_t := 0$  for all  $t \in [0, T]$  is a solution to the above ODE.
2. Show that  $X_t := t^2$  for all  $t \in [0, T]$  is also a solution.
3. Find at least two more solutions different from  $\bar{X}$  and  $X$ .

**Exercise 2.4.** Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space, let  $X : \Omega \rightarrow \mathbb{R}$  be a r.v. and let  $\mathcal{G} = \{\emptyset, \Omega\}$ .

1. Show that there is a  $c \in \mathbb{R}$  such that  $\mathbb{E}[X|\mathcal{G}] = c$ .
2. Show that in fact  $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$ .

**Exercise 2.5.** Consider the SDE

$$dX_s^{t,x} = b(X_s^{t,x}) ds + \sigma(X_s^{t,x}) dW_s, \quad t \leq s \leq T, \quad X_t^{t,x} = x.$$

Assume it has a pathwise unique solution i.e. if  $Y_s^{t,x}$  is another process that satisfies the SDE then

$$\mathbb{P} \left[ \sup_{t \leq s \leq T} |X_s^{t,x} - Y_s^{t,x}| > 0 \right] = 0.$$

Show that then the *flow property* holds i.e. for  $0 \leq t \leq t' \leq T$  we have almost surely that

$$X_s^{t,x} = X_s^{t', X_{t'}^{t,x}}, \quad \forall s \in [t', T].$$