2020/21 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 4, for Week 7 - Last updated 22nd February 2021

Exercise 4.1 (Merton problem with an exponential utility, no consumption)**.** We return to the portfolio optimization problem, see Section 4.3. of the lecture notes. Unlike in Example 4.10 we consider the utility function $g(x) := -e^{-\gamma x}$, $\gamma > 0$ a constant. We will also take $r = 0$ for simplicity and assume there is no consumption ($C = 0$). With X_t denoting the wealth at time time t we have the value function given by

$$v(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[g\left(X_T^{\pi,t,x,}\right)\right] .$$

i) Write down the expression for the wealth process in terms of π, the amount of wealth invested in the risky asset and with $r = 0$, $C = 0$.

ii) Write down the HJB equation associated to the optimal control problem. Solve the HJB equation by inspecting the terminal condition and thus suggesting a possible form for the solution. Write down the optimal control explicitly.

iii) Use verification theorem to show that the solution and control obtained in previous step are indeed the value function and optimal control.

Exercise 4.2 (Unattainable optimizer)**.** Here is a simple example in which no optimal control exists, in a finite horizon setting, $T \in (0,\infty)$. Suppose that the state equation is

$$dX_s = \alpha_s\, ds + dW_s \quad s \in [t,T]\,, \quad X_t = x \in \mathbb{R}.$$

A control α is admissible ($\alpha \in \mathcal{A}$) if: α takes values in $\mathbb{R}$, is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, and $\mathbb{E}\int_0^T \alpha_s^2\, ds < \infty\,.$

Let $J(t,x,\alpha) := \mathbb{E}[|X_T^{t,x,\alpha}|^2]$. The value function is $v(t,x) := \inf_{\alpha\in\mathcal{A}} J(t,x,\alpha)$. Clearly $v(t,x) \geq 0$.

i) Show that for any $t \in [0,T]$, $x \in \mathbb{R}$, $\alpha \in \mathcal{A}$ we have $\mathbb{E}[|X_T^{t,x,\alpha}|^2] < \infty$.

ii) Show that if $\alpha_t := -cX_t$ for some constant $c \in (0,\infty)$ then $\alpha \in \mathcal{A}$ and

$$J^\alpha(t,x) = J^{cX}(t,x) = \frac{1}{2c} - \frac{1-2cx^2}{2c}e^{-2c(T-t)}.$$

Hint: with such an α, the process X is an Ornstein-Uhlenbeck process, see an earlier exercise.

iii) Conclude that $v(t,x) = 0$ for all $t \in [0,T)$, $x \in \mathbb{R}$.

iv) Show that there is no $\alpha \in \mathcal{A}$ such that $J(t,x,\alpha) = 0$. *Hint:* Suppose that there is such a α and show that this leads to a contradiction.