Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results proved in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. We consider the standard Black–Scholes model for optimal investment: a risk-free asset B and a risky asset S given by

$$B_t := \exp(rt) \text{ and } S_t := S_0 \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right) t + \sigma W_t\right) .$$

Here W is a Wiener process and r, μ and σ are real constants with $\sigma > 0$. Fix $T > 0$. Let X_s denote the investment portfolio value at time $s \geq t$ and $X_t = x > 0$. There will be no cash injections and no consumption. Let $\nu = (\nu_t)_{t \in [0,T]}$ be the fraction of portfolio value invested in the risky asset. We will assume that $\mathbb{E} \int_0^T \nu_s^2 \, ds < \infty$ and that ν is adapted to the filtration generated by W. For such ν we write $\nu \in \mathcal{A}$. Let $g(x) := x^\gamma$, $\gamma \in (0,1)$ and

$$\bar{v}(t,x) := \sup_{\nu \in \mathcal{A}} \mathbb{E}\left[g(X_T^{\nu,t,x})\right] . \tag{1}$$

a) Find a candidate for the optimal control and hence show that the solution to the corresponding Bellman PDE is

$$v(t,x) = \exp\big((T-t)\beta\big) x^\gamma ,$$

where β is a constant given in terms of σ, μ, r and γ. Give an explicit expression for β. **[7 marks]**

b) Use verification theorem to check that $\bar{v} = v$ and the candidate optimal control is the true optimal control. **[8 marks]**

2. A producer with production rate $X = X_t$ at time t may allocate a portion $\alpha = \alpha_t$ of their production rate to reinvestment (thus increasing production rate) and $1 - \alpha_t$ to actual production of a storable good. Thus

$$dX_t = \gamma \alpha_t X_t \, dt \,, \quad t \in [0,T] \,, \quad X_0 = x > 0 \,,$$

where $\gamma > 0$ is a constant. The admissible controls are measurable maps $t \mapsto \alpha_t \in [0,1]$. The objective is to maximize the amount of goods produced over time $[0,T]$ i.e. maximize

$$J(x,\alpha) := \int_0^T (1-\alpha_t) X_t \, dt \,.$$

i) Use Pontryagin's maximum principle to show that an optimal control is

$$\alpha_t = \begin{cases} 0 \text{ if } Y_t < \frac{1}{\gamma} \,, \\ 1 \text{ if } Y_t > \frac{1}{\gamma} \,, \end{cases}$$

where Y_t is the solution of the adjoint (backward) equation in Pontryagin optimality. **[5 marks]**

ii) Assume that $T > \frac{1}{\gamma}$. Show that since $Y_T = 0$ we have

$$Y_t = \begin{cases} (T-t) \text{ if } t \in (T - \frac{1}{\gamma}, T] \,, \\ \frac{1}{\gamma} \exp\left(\gamma\left(T - \frac{1}{\gamma}\right) - \gamma t\right) \text{ if } t \in [0, T - \frac{1}{\gamma}] \,. \end{cases}$$

[5 marks]

iii) Hence show that the optimally controlled state is given by

$$X_t = \begin{cases} x e^{\gamma t} \text{ if } t \in (T - \frac{1}{\gamma}, T] \,, \\ x e^{\gamma\left(T - \frac{1}{\gamma}\right)} \text{ if } t \in [0, T - \frac{1}{\gamma}] \,. \end{cases}$$

[5 marks]

3. We consider a problem of optimal trade execution. Fix $T > 0$, $\lambda > 0$, $\sigma > 0$, $\kappa > 0$. The mid-price of an asset is

$$dS_t = \lambda \alpha_t \, dt + \sigma dW_t \,, \;\; t \in [0,T] \,, \;\; S_0 > 0 \,.$$

Our holding in the asset is given by

$$d\xi_t = \alpha_t \, dt \;, \;\; t \in [0,T] \,, \;\; \xi_0 \in \mathbb{R} \,.$$

Our cash account is

$$dB_t = -\alpha_t \bigg(S_t + \frac{\kappa}{2} \alpha_t \bigg) \, dt \,, \;\; t \in [0,T] \,, \;\; B_0 > 0 \,.$$

Here the control is $\alpha = \alpha_t$ representing they "buying rate". The constant $\lambda > 0$ is the "permanent price impact" while $\kappa > 0$ is the "temporary price impact".

Our task is to deliver one unit of the risky asset at time $T > 0$ and there is a quadratic penalty for missing the target. We want to do this while maximising our cash balance. Let $\mathcal{A}$ comprise processes α_t adapted to the filtration generated by W and such that $\mathbb{E} \int_0^T \alpha_t^2 \, dt < \infty$. The overall objective to maximize is, over $\alpha \in \mathcal{A}$,

$$M(S_0, \xi_0, B_0, \alpha) = \mathbb{E} \left[\frac{1}{2} |\xi_T - 1|^2 + B_T + (\xi_T - 1) S_T \right] \,.$$

a) Show that

$$\max_{\alpha \in \mathcal{A}} M(S_0, \xi_0, B_0, \alpha) = B_0 - S_0 + \xi_0 S_0 + \max_{\alpha \in \mathcal{A}} J(\xi_0, \alpha) \,,$$

where

$$J(\xi_0, \alpha) = \mathbb{E} \bigg[\int_0^T \Big(-\frac{\kappa}{2} \alpha_r^2 + \lambda \alpha_t (\xi_t - 1) \Big) \, dt + \frac{1}{2} |\xi_T - 1|^2 \bigg] \,.$$

[8 marks]

b) Find an explicit expression for the optimal control. *Hint.* You can use either the Bellman PDE or Pontryagin optimality to solve this. **[12 marks]**

[End of Paper]