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2021/22 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 2 - Last updated 18th January 2022

Exercise 2.1. There is a biased coin with $p \in (0,1)$, $p \neq \frac{1}{2}$, probability of getting heads and $q = 1-p$ probability of getting tails.
We will start with an initial wealth $x = i$, $i \in \mathbb{N}$ with $i < m$, with some $m = 2$.
At each turn we choose an action $a \in \{-1, 1\}$. By choosing $a = 1$ we bet that the coin comes up heads and our wealth is increased by 1 if we are correct, decreased by 1 otherwise. By choosing $a = -1$ we bet on tails and our wealth is updated accordingly.
That is, given that $X_{n-1} = x$ and our action $a \in \{-1, 1\}$ we have

$$\mathbb{P}(X_n = x + a \,|\, X_{n-1} = x, a) = p\,, \quad \mathbb{P}(X_n = x - a \,|\, X_{n-1} = x, a) = q\,.$$

The game terminates when either $x = 0$ or $x = m = 2$. Let $N = \min\{n \in \mathbb{N} : X_n = 0 \text{ or } X_n = m\}$. Our aim is to maximize

$$J^\alpha(x) = \mathbb{E}\Big[X_N^\alpha | X_0 = x\Big]$$

over functions $\alpha = \alpha(x)$ telling what action to choose in each given state.

1. Write down what the state space S and the stopping set ∂S are and write down the transition probability matrix for P^a for $a = 1$ and for $a = -1$.
2. Write down the Bellman equation for the problem.
3. Assume that $p > 1/2$. Guess the optimal strategy. With your guess the Bellman equation is linear. Solve it for V.

Exercise 2.2 (Non-existence of solution)**.**

1. Let $I = \big[0, \frac{1}{2}\big]$. Find a solution X for

$$\frac{dX_t}{dt} = X_t^2\,, \;\; t \in I\,, \;\; X_0 = 1\,.$$

2. Does a solution to the above equation exist on $I = [0, 1]$? If yes, show that it satisfies the definition of an SDE solution from SAF lectures. In not, which property is violated?

Exercise 2.3 (Non-uniqueness of solution)**.** Fix $T > 0$. Consider

$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}\,, \quad t \in [0,T]\,, \quad X_0 = 0\,.$$

1. Show that $\bar{X}_t := 0$ for all $t \in [0,T]$ is a solution to the above ODE.

2. Show that $X_t := t^2$ for all $t \in [0,T]$ is also a solution.

3. Find at least two more solutions different from $\bar{X}$ and X.

Exercise 2.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X : \Omega \to \mathbb{R}$ be a r.v. and let $\mathcal{G} = \{\emptyset, \Omega\}$.

1. Show that there is a $c \in \mathbb{R}$ such that $\mathbb{E}[X|\mathcal{G}] = c$.

2. Show that in fact $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$.

Exercise 2.5. Consider the SDE

$$dX_s^{t,x} = b(X_s^{t,x})\,ds + \sigma(X_s^{t,x})\,dW_s, \;\; t \le s \le T, \;\; X_t^{t,x} = x\,.$$

Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then

$$\mathbb{P}\left[\sup_{t \le s \le T} |X_s^{t,x} - Y_s^{t,x}| > 0\right] = 0\,.$$

Show that then the *flow property* holds i.e. for $0 \le t \le t' \le T$ we have almost surely that

$$X_s^{t,x} = X_s^{t', X_{t'}^{t,x}}, \qquad \forall s \in [t', T].$$