Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results covered in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Let $\mu, r, \sigma, \delta, \gamma \in \mathbb{R}$ be constants s.t. $\sigma \neq 0$, $\delta > 0$, $\gamma \in (0,1)$. Consider the process

$$dX_t = X_t\big(\nu_t(\mu - r) + r - \kappa_t\big)\,dt + \nu_t \sigma X_t\,dW_t\,, t \geq 0\,,\; X_0 = x > 0$$

and let us write $X_t = X_t^{x,\nu,\kappa}$ to emphasize the dependence on the starting point x and the controls $\nu = (\nu_t)_{t\geq 0}$ and $\kappa = (\kappa_t)_{t\geq 0}$. We say that ν, κ are admissible if they are adapted and bounded. Let

$$v(x) = \sup_{\nu,\kappa} \mathbb{E}\left[\int_0^\infty e^{-\delta t}\big(\kappa_t X_t^{x,\nu,\kappa}\big)^\gamma\,dt\right],$$

where the supremum is taken over all admissible ν, κ.

Using the "guess" $v(x) = \lambda x^\gamma$ for some $\lambda > 0$ solve the HJB equation giving explicit form for the constant λ depending only on μ, r, σ, δ and γ. You don't need to use the verification theorem here. **[30 marks]**

Hint. An infinite time stochastic control problem can be written as

$$v(x) = \sup_\alpha \mathbb{E}\int_0^\infty e^{-\delta t} f^{\alpha_t}(X_t^{x,\alpha})\,dt\,,$$

where the supremum is taken over admissible controls and subject to

$$dX_t^{x,\alpha} = b^{\alpha_t}(X_t^{x,\alpha})\,dt + \sigma^{\alpha_t}(X_t^{x,\alpha})\,dW_t\,,\;\; t \in [0,\infty)\,,\;\; X_0^{x,\alpha} = x.$$

The HJB equation for this infinite-time-horizon problem is

$$\sup_{a\in A}\left[\tfrac{1}{2}(\sigma^a)^2 v'' + b^a v' - \delta v + f^a\right] = 0 \;\text{ on }\; [0,\infty)\times\mathbb{R}\,.$$

2.

(a) Assume that $g \in C^1(\mathbb{R})$. Show that $v(t,x) = g(x + (T-t))$ is a solution to

$$\begin{aligned} \partial_t v + \partial_x v &= 0 \;\text{ on }\; [0,T]\times\mathbb{R}\,,\\ v(T,x) &= g(x)\;\; \forall x \in \mathbb{R}\,. \end{aligned}$$

[5 marks]

(b) Let $\mathcal{A} = \big\{\alpha : [0,T] \to \{-1,0,1\} : \alpha \text{ is measurable}\big\}$ and let

$$v(t,x) = \inf_{\alpha\in\mathcal{A}} |X_T^{t,x,\alpha}|^2 \;\text{ where }\; X_T^{t,x,\alpha} = x + \int_t^T \alpha(s)\,ds\,.$$

Noting that the control has to take values in $\{-1,0,1\}$ guess an optimal Markovian control and hence solve the Bellman equation for v. Use this to verify that your guess is indeed an optimal control. **[25 marks]**

3. With the action space $A = \mathbb{R}$ consider the problem

$$v(t,x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[-\frac{1}{2}\int_t^T |\alpha_s|^2 \, ds + g(X_T^{t,x}) \right],$$
$$dX_s^{t,x} = \alpha_s \, ds + dW_s \,, \quad s \in [t,T]\,, \quad X_t^{t,x} = x \in \mathbb{R}\,.$$

Here $g : \mathbb{R} \to \mathbb{R}$ is smooth and bounded. We say $\alpha \in \mathcal{A}$ if it is adapted and $\mathbb{E}\int_0^T |\alpha_t|^2 dt < \infty$.

(a) Write down the HJB equation for v and hence show that

$$\partial_t v + \tfrac{1}{2}\partial_{xx} v + \tfrac{1}{2}|\partial_x v|^2 = 0 \ \text{ on } \ [0,T) \times \mathbb{R}\,,$$

with $v(T,x) = g(x)$ for $x \in \mathbb{R}$. **[10 marks]**

(b) Let $u(t,x) = e^{v(t,x)}$. Show that

$$\partial_t u + \tfrac{1}{2}\partial_{xx} u = 0 \ \text{ on } \ [0,T) \times \mathbb{R}\,,$$

with terminal condition $u(T,x) = e^{g(x)}$ for all $x \in \mathbb{R}$. **[15 marks]**

(c) Show that

$$v(t,x) = \log \int_{\mathbb{R}^d} e^{g(y)} p(T-t, y-x)\, dy\,,$$
$$p(s,z) = (2\pi s)^{-1/2} e^{-\frac{|z|^2}{2s}}\,.$$

[15 marks]

[End of Paper]