

2022/23 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 1 - Last updated 12th January 2023

Exercise 1.1 (ODEs). Assume that $t \mapsto r(t)$ is integrable i.e. $\int_0^t |r(s)| ds < \infty$ holds.

1. Solve

$$dB(t) = B(t)r(t)dt, \quad B(0) = 1. \quad (1)$$

2. Is the function $t \mapsto B(t)$ continuous? Why?

3. Calculate $d(1/B(t))$.

Exercise 1.2 (Geometric Brownian motion). Let W be a real-valued Wiener martingale generating a filtration \mathbb{F} . Assume that $\mu \in \mathcal{A}$ and $\sigma \in \mathcal{S}$. Note that

$$\mathcal{A} := \{X = (X_s)_{s \in [0, T]} : X \text{ is adapted and } \int_0^T X_s ds < \infty \text{ a.s.}\}$$

and

$$\mathcal{S} := \left\{ X = (X_s)_{s \in [0, T]} : X \text{ is adapted and } \mathbb{P} \left[\int_0^T |X_s|^2 ds < \infty \right] = 1 \right\}.$$

1. Solve

$$dS(t) = S(t) [\mu(t) dt + \sigma(t) dW(t)], \quad S(0) = s. \quad (2)$$

Hint: Solve this first in the case that μ and σ are real constants. Apply Itô's formula to the process S and the function $x \mapsto \ln x$ (this is strictly speaking not allowed but it gives you the solution). Verify you have a correct solution by using Itô formula with the function $x \mapsto e^x$. Now generalize this to the case that μ and σ are the processes as above.

2. Is the function $t \mapsto S(t)$ continuous? Why?

3. Calculate $d(1/S(t))$, assuming $s \neq 0$.

4. With B given by (1) calculate $d(S(t)/B(t))$.

Lemma 1.3 (Gronwall's lemma / inequality). Let $\lambda = \lambda(t) \geq 0$, $a = a(t)$, $b = b(t)$ and $y = y(t)$ be locally integrable, real valued functions defined on I (with $I = [0, T]$ or $I = [0, \infty)$) such that λy is also locally integrable and for almost all $t \in [0, T]$

$$y(t) + a(t) \leq b(t) + \int_0^t \lambda(s)y(s) ds.$$

Then

$$y(t) + a(t) \leq b(t) + \int_0^t \lambda(s) e^{\int_s^t \lambda(r) dr} (b(s) - a(s)) ds \quad \text{for almost all } t \in I.$$

Furthermore, if b is monotone increasing and a is non-negative, then

$$y(t) + a(t) \leq b(t) e^{\int_0^t \lambda(r) dr}, \quad \text{for almost all } t \in I.$$

Exercise 1.4 (On Gronwall's lemma). Prove Gronwall's Lemma by following these steps:

i) Let

$$z(t) = \left(e^{-\int_0^t \lambda(r) dr} \right) \int_0^t \lambda(s) y(s) ds.$$

and show that

$$z'(t) \leq \lambda(t) e^{-\int_0^t \lambda(r) dr} (b(t) - a(t)).$$

ii) Integrate from 0 to t to obtain the first conclusion Lemma 1.3.

iii) Obtain the second conclusion of Lemma 1.3.

Exercise 1.5. Consider the SDE

$$dX_s^{t,x} = b(X_s^{t,x}) ds + \sigma(X_s^{t,x}) dW_s, \quad t \leq s \leq T, \quad X_t^{t,x} = x.$$

Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then

$$\mathbb{P} \left[\sup_{t \leq s \leq T} |X_s^{t,x} - Y_s^{t,x}| > 0 \right] = 0.$$

Show that then the *flow property* holds i.e. for $0 \leq t \leq t' \leq T$ we have almost surely that

$$X_s^{t,x} = X_s^{t', X_{t'}^{t,x}}, \quad \forall s \in [t', T].$$