

2022/23 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 2 - Last updated 12th January 2023

Exercise 2.1 (Non-existence of solution).

1. Let $I = [0, \frac{1}{2}]$. Find a solution X for

$$\frac{dX_t}{dt} = X_t^2, \quad t \in I, \quad X_0 = 1.$$

2. Does a solution to the above equation exist on $I = [0, 1]$? If yes, show that it satisfies the definition of an SDE solution from SAF lectures. In not, which property is violated?

Exercise 2.2 (Non-uniqueness of solution). Fix $T > 0$. Consider

$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}, \quad t \in [0, T], \quad X_0 = 0.$$

1. Show that $\bar{X}_t := 0$ for all $t \in [0, T]$ is a solution to the above ODE.
2. Show that $X_t := t^2$ for all $t \in [0, T]$ is also a solution.
3. Find at least two more solutions different from \bar{X} and X .

Exercise 2.3. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, let $X : \Omega \rightarrow \mathbb{R}$ be a r.v. and let $\mathcal{G} = \{\emptyset, \Omega\}$.

1. Show that there is a $c \in \mathbb{R}$ such that $\mathbb{E}[X|\mathcal{G}] = c$.
2. Show that in fact $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}X$.

Exercise 2.4. Consider the SDE

$$dX_s^{t,x} = b(X_s^{t,x}) ds + \sigma(X_s^{t,x}) dW_s, \quad t \leq s \leq T, \quad X_t^{t,x} = x.$$

Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then

$$\mathbb{P} \left[\sup_{t \leq s \leq T} |X_s^{t,x} - Y_s^{t,x}| > 0 \right] = 0.$$

Show that then the *flow property* holds i.e. for $0 \leq t \leq t' \leq T$ we have almost surely that

$$X_s^{t,x} = X_s^{t', X_{t'}^{t,x}}, \quad \forall s \in [t', T].$$