There are *three* questions, each worth 25 marks. Correct answers to all questions yields the maximum of 75 marks.

Throughout the examination paper we will assume the existence of a suitable probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Results in the lectures may be used without further justification unless the question is asking specifically for the proof of a particular result.

1. Given $(s, y) \in [0, 1) \times \mathbb{R}$, consider the following stochastic control problem

$$
\begin{aligned}
V(0,0) &= \min_{\nu} J(0,0;\nu) \\
&= \min_{\nu} \mathbb{E}\left[\left(X^{\nu}_{0,0}(1)\right)^2 - \int_0^1 2\, s\, \nu(s) \mathrm{d}s \right]
\end{aligned}
$$

$$
\text{such that} \begin{cases} \mathrm{d}X^{\nu}_{0,0}(r) &= \nu(r)\mathrm{d}W(r), \qquad r \in [0,1] \\ X^{\nu}_{0,0}(0) &= 0 \\ \nu(t) &\in \mathbb{R} \quad \forall t \in [0,1],\ (\mathcal{F}_t)_{t\in[0,T]}\text{-adapted} \\ & \quad \text{and } \mathbb{E}[\int_0^1 \left(\nu(s)\right)^2 \mathrm{d}s] < \infty \end{cases}
$$

(a) Write the dynamic version of the optimal control problem above, i.e. write down $V(t, y)$ for $t \in [0, 1]$ and $y \in \mathbb{R}$.

[2 marks]

(b) Show that one can write $V(t, x) = x^2 + g(t)$ for some function $g(t)$ you need to identify. Compute further $\partial_x V(t, x)$ and $\partial_{xx} V(t, x)$.

[8 marks]

(c) Write down the HJB equation for this stochastic control problem.

[5 marks]

(d) Find a solution to the HJB equation and compute $V(0, 0)$.

[10 marks]

Solution:

(a) **Dynamic version of the Value function & objective functional**

$$\begin{aligned} V(t,x) &= \min_{\nu} J(t,x;\nu) \\ &= \min_{\nu} \mathbb{E}\left[\big(X^{\nu}_{t,x}(1)\big)^2 - \int_t^1 2\,s\,\nu(s)\mathrm{d}s \right] \end{aligned}$$

$$\text{such that} \begin{cases} \mathrm{d}X^{\nu}_{t,x}(s) &= \nu(s)\mathrm{d}W(s), \qquad s\in[t,T] \\ X^{\nu}_{t,x}(t) &= x \\ \nu(t) &\in \mathbb{R} \quad \forall t\in[0,1] \text{ and } (\mathcal{F}_t)_{t\in[0,T]}\text{-adapted} \end{cases}$$

[2 Marks]

(b) **Properties of the Value function** For this we start by computing $\mathbb{E}\left[\big(X^{\nu}_{t,x}(1)\big)^2\mathrm{d}s\right]$. From the SDE we have

$$X^{\nu}_{t,x}(s) = x + \int_t^s \nu(r)\mathrm{d}W(r) = x + X^{\nu}_{t,0}(s) \qquad \textbf{[2 Marks]}$$

and

$$\begin{aligned} \mathbb{E}\left[\big(X^{\nu}_{t,x}(1)\big)^2\right] &= x^2 + 2x\mathbb{E}\left[\int_t^1 \nu(s)\mathrm{d}W(s)\right] + \mathbb{E}\left[\Big(\int_t^1 \nu(s)\mathrm{d}W(s)\Big)^2\right] \\ &= x^2 + 0 + \mathbb{E}[\Big(X^{\nu}_{t,0}(1)\Big)^2] \\ &= x^2 + 0 + \mathbb{E}[\int_t^1 \nu^2(s)\mathrm{d}s] \qquad \textbf{[2 Marks]} \end{aligned}$$

replacing this in the expression for $V(t,x)$ easily yields that $V(t,x) = x^2 + g(t)$, namely

$$\begin{aligned} V(t,x) = \min_{\nu} J(t,x;\nu) &= \min_{\nu} \mathbb{E}\left[\big(X^{\nu}_{t,x}(1)\big)^2 - \int_t^1 2\,s\,\nu(s)\mathrm{d}s\right] \\ &= x^2 + \min_{\nu}\mathbb{E}\left[\big(X^{\nu}_{t,0}(1)\big)^2 - \int_t^1 2\,s\,\nu(s)\mathrm{d}s\right] = x^2 + V(t,0). \end{aligned}$$

[2 Marks]

The derivatives $\partial_x V(t,x) = 2x$ and $\partial_{xx}V(t,x) = 2$. **[2 Marks]**

(c) **The HJB equation** depends on the generator of the diffusion given by $\mathrm{d}X(s) = \nu(s)\mathrm{d}W(s)$, in this case $\mathcal{L}^u G(t,x) = \partial_t G + \frac{1}{2}u^2\partial_{xx}G$ with $\nu\in\mathbb{R}$ (and no drift).

Thus, the HJB equation is given by

$$\min_{u\in\mathbb{R}} \{\mathcal{L}^u V - 2\,t\,u\} = 0,$$

or

$$\begin{aligned} \partial_t V + \min_{u\in[0,1]}\left\{\frac{1}{2}u^2\partial_{xx}V - 2\,t\,u\right\} &= 0 \quad \text{for } (s,x)\in[0,1)\times\mathbb{R}, \qquad \textbf{[3 Marks]} \\ V\,(1,x) &= x^2 \quad \text{for } x\in\mathbb{R}. \qquad \textbf{[2 Marks]} \end{aligned}$$

The terminal condition $V(1,x)$ follows directly from the definition of $V(t,x)$; just set $t=1$.

(d) **Find a solution of the HJB equation.**

Since we have $V(t,x) = x^2 + V(t,0)$ we have $\partial_{xx}V = 2$ which can be replaced into the HJB equation to yield:

$$\partial_t V + \min_{\nu\in[0,1]} \left\{ \frac{1}{2}u^2 2 - 2\,t\,u \right\} = 0.$$

We now solve the minimization problem so that we can solve the HJB.

$$\begin{aligned} \min_{u\in\mathbb{R}} \left\{ \frac{1}{2}u^2 2 - 2\,t\,u \right\} &= \min_{u\in\mathbb{R}} \left\{ (u-t)^2 - t^2 \right\} \\ &\Rightarrow u^\star(t) = t \qquad \textbf{[3 Marks] For the minimization} \end{aligned}$$

replacing $u^* = t$ inside the HJB we obtain that $(u^*)^2 - 2tu^* = t^2 - 2t^2 = -t^2$ and hence

$$\partial_t V + \min_{u\in[0,1]} \left\{ \frac{1}{2}u^2 \partial_{xx}V - 2\,t\,u \right\} = 0 \quad \Leftrightarrow \quad \partial_t V - t^2 = 0.$$

This can be solved by direct integration,

$$\begin{aligned} \partial_t V = t^2 \quad &\Rightarrow \quad \int_t^1 \partial_t V(s,x)\mathrm{d}s = \int_t^1 s^2 \mathrm{d}s \\ &\Leftrightarrow \quad V(1,x) - V(t,x) = \frac{1-t^3}{3} \quad \Leftrightarrow \quad V(t,x) = x^2 - \frac{1-t^3}{3}. \qquad \textbf{[6 Marks]} \end{aligned}$$

Finally, $V(0,0) = -1/3$ **[1 Marks]**

Comment:

(a) Easy; evaluates basic Stochastic Analysis knowledge;

(b) Easy; evaluates basic Stochastic Analysis knowledge;

(c) Easy; Students must identfy the Dynkin generator and write down the HJB equation. Boundary condition must also be identified

(d) Medium. Solving the minimization problem is easy when the explicit formula for $\partial_{yy}V$ is injected;

2.

(a) Denote $V(t,x)$ as the value function of the optimal control problem below starting from time t at position x, i.e.,

$$V(t,x) := \inf_{\nu} \mathbb{E}\left[\int_t^T f\big(s, X^{\nu}_{t,x}(s), \nu(s)\big)\mathrm{d}s + h\big(X^{\nu}_{t,x}(T)\big)\right]$$

$$\text{where} \quad \begin{cases} \mathrm{d}X^{\nu}_{t,x}(s) = b\big(s, X^{\nu}_{t,x}(s), \nu(s)\big)\mathrm{d}s + \sigma\big(s, X^{\nu}_{t,x}(s), \nu(s)\big)\mathrm{d}W_s, & s \in [t,T] \\ X^{\nu}_{t,x}(t) = x \\ \nu \in \mathcal{U}_{ad}[t,T] \end{cases}.$$

Make the general assumption that the SDE coefficients satisfy assumption (S1) from class and $\mathcal{U}_{ad}$ is the usual admissibility set used in class implying that the SDE is well defined and has good properties, in particular, it holds that

$$\mathbb{E}\big[\sup_{s\in[t,T]} |X^{\nu}_{t,y}(s) - X^{\nu}_{t,x}(s)|^2\big] \le C|y-x|^2.$$

Assume additionally that the function $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz. Assume the function $f : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, i.e. $f(t,x,u)$ is Lipschitz in x uniformly in t,u and uniformly bounded in u and t.

Then, show that there exists constants $K > 0$ such that

$$|V(t,\,x) - V(t,\,y)| \le K|x-y|.$$

[12 marks]

(b) A Black-Scholes market is given where there are only one stock (with drift $a \in \mathbb{R}$ and volatility $\sigma > 0$) and one bank account with constant interest rate $r > 0$.

In this market an investor, with initial wealth $x_0 > 0$ selects among *proportion strategies* ν that are *constants* and with such a strategy the *proportion of wealth invested in the stock* is a constant throughout.

The investor seeks to maximise his expected utility at time T which is a power-type utility

$$U(x) = \log x, \qquad x > 0.$$

(i) Identify the dynamics for the underlying assets.
Show that the SDE expressing the wealth process $\big(X^{\nu}(t)\big)_{t\in[0,T]}$ is of Geometric Brownian motion type and write its explicit solution.

[5 marks]

(ii) Write clearly the optimization problem and then compute the *constant optimal proportion strategy* $\nu^{\star}$ explicitly *using* the duality theory approach, in other words, determine: the deflator process $(Y_t)_{t\in[0,T]}$ and all necessary related quantities; the optimal wealth random variable $\widehat{X}_T$ using duality theory; then the optimal wealth process $(\widehat{X}_t)_{t\in[0,T]}$; finally determine ν^{*}.

[8 marks]

Solution:

(a) Fix any $t \in [0, T)$, x and y in $\mathbb{R}$.

Step #1: Preparation. For any $\varepsilon > 0$, there exists an ε-control $\nu \in \mathcal{U}_{ad}[t, T]$ with state process $X(s) := X^{\nu^\varepsilon}_{t,x}(s)$ for $s \in [t, T]$ such that (we omit the dependencies in t, x, ν^ε)

$$\begin{aligned} V(t,x) &\geq J(t,x,\nu^\varepsilon) - \varepsilon \\ &= \mathbb{E}\left[\int_t^T f\big(s, X(s), \nu(s)\big)\mathrm{d}s + h\big(X(T)\big)\Big| X_{t,x}(t) = x\right] - \varepsilon \\ &\Leftrightarrow -V(t,x) \leq -J(t,x,\nu^\varepsilon) + \varepsilon. \end{aligned}$$

Step #2: Preparation. By definition in infimum, we have $V(t,y) \leq J(t,y,\nu^\varepsilon)$. Denote the state process associated to y and ν^ε by $Y := X^{\nu^\varepsilon}_{t,y}(s)$ for $s \in [t, T]$.

Step #3: Proof. Using step #1 and #2, we have

$$\begin{aligned} V(t,y) &- V(t,x) \\ &\leq J(t,y,\nu^\varepsilon) - J(t,x,\nu^\varepsilon) + \varepsilon \\ &= \mathbb{E}\Big[\int_t^T \Big(f\big(s, Y(s), \nu^\varepsilon(s)\big) - f\big(s, X(s), \nu^\varepsilon(s)\big)\Big)\mathrm{d}s \\ &\qquad + h\big(Y(T)\big) - h\big(X(T)\big)\ \Big| X(t) = x, Y(t) = y\Big] + \varepsilon \\ &\leq \mathbb{E}\Big[K\int_t^T |Y(s) - X(s)|\mathrm{d}s + K|Y(T) - X(T)|\ \Big| X(t) = x, Y(t) = y\Big] + \varepsilon \\ &\leq K\big(1 + (T-t)\big)\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s) - X(s)|\Big] + \varepsilon \\ &\leq K\big(1 + (T-t)\big)\mathbb{E}\Big[\sup_{s\in[t,T]} |Y(s) - X(s)|^2\Big]^{\frac{1}{2}} + \varepsilon, \end{aligned}$$

where for the first inequality we used first the ε-control and the definition of infimum (with ν^ε); for the second inequality we used the properties of the expectations (can you explain why the two expectation can be merged in this way?); the third, is simply the Lipschitz property of f and h in (S2); and, the last inequality is simply taking the sup over the time interval and the Cauchy-Schwarz inequality (with multiplication with 1).

The last line can be estimated through the result on SDEs stated in the problem. Injecting this last inequality in the above estimation we find an $L > 0\ \forall x, y$ such that $V(t,y) - V(t,x) \leq L|y - x| + \varepsilon\ \forall \varepsilon > 0$. Since ε is arbitrary we conclude $V(t,y) - V(t,x) \leq L|y - x|$.

By inspecting the above proof, we can easily prove that $V(t,x) - V(t,y) \leq L|x - y|$ follows by the same arguments. Hence $x \mapsto V(\cdot, x)$ is uniformly Lipschitz.

[12 marks]

(b) **Question (b.i):**

The wealth process. The stock and riskless process, denoted S and B respectively, have the following dynamics as postulated by the Black-Scholes market

$$\mathrm{d}S(t) = S(t)\left[a\mathrm{d}t + \sigma\mathrm{d}W(t)\right] \qquad \text{and} \qquad \mathrm{d}B(t) = rB(t)\mathrm{d}t.$$

If the strategies are constant proportions of wealth, ν for the proportion of wealth invested in the Stock and $1 - \nu$ for the proportion of wealth invested in the bank account, then equation of the wealth is given by

$$\begin{aligned}\mathrm{d}X(t) &= \frac{\nu X(t)}{S(t)}\mathrm{d}S(t) + \frac{(1-\nu)X(t)}{B(t)}\mathrm{d}B(t)\\ &= X(t)\left[(a-r)\,\nu + r\right]\mathrm{d}t + \nu\sigma X(t)\mathrm{d}W(t),\ \ X(0)=x_0,\end{aligned}$$

As the controls are constant, $X(\cdot)$ can be computed explicitly as it is a Geometric Brownian motion. The solution is given by

$$X^\nu(t) = x_0 \cdot \exp\left\{\Big((a-r)\nu + r - \frac{1}{2}\nu^2\sigma^2\Big)t\right\} \cdot \exp\left\{\nu\sigma W(t)\right\}.$$

[5 marks]

Question (b.ii): The optimization For a wealth process X^ν and a control ν the optimization problem can be written as

$$\sup_{\nu\in\mathbb{R}} \mathbb{E}\Big[U\big(X^\nu(T)\big)\Big], \qquad U(x) = \log x.$$

From the work in class on convex duals we have:

$$U(x) = \log x, \qquad U'(x) = \frac{1}{x}, \qquad I(y) = (U')^{-1}(y) = \frac{1}{y}$$

Following notation in class we define the Deflator process $(Y_t)_{t\in[0,T]}$ as

$$Y_t := D_t Z_t = \exp\big\{-rt\big\}\mathcal{E}\big(-\lambda W_T\big) = \exp\big\{-rt\big\}\exp\big\{-\lambda W_T - \frac{1}{2}\lambda^2 T\big\}$$

with λ the usual Market price of risk for this model, i.e. $\lambda = (a-r)/\sigma$.

Using the theory from class, the optimal wealth random variable is $\widehat{X}_T = I(yY_T)$ with $I(\cdot)$ as defined above for U and the number y is determined by the Lagrangian condition $\mathbb{E}[Y_T\widehat{X}_T] = \mathbb{E}[Y_T I(yY_T)] = x_0 \Leftrightarrow \mathcal{X}(y) = x_0$.

Hence,

$$\widehat{X}_T = I(yY_T) = \frac{1}{yY_T} \quad \text{and} \quad \mathcal{X}(y) = x_0 \Leftrightarrow \mathbb{E}[Y_T\frac{1}{yY_T}] = x_0 \Leftrightarrow \frac{1}{y} = x_0.$$

The optimal wealth at time $t=T$ is given by

$$\widehat{X}_T = I(yY_T) = \frac{1}{yY_T} = \frac{x_0}{Y_T}.$$

and the optimal wealth process $(\widehat{X}_t)_{t\in[0,T]}$ is given by

$$\widehat{X}_t = \frac{1}{Y_t}\mathbb{E}[Y_T\widehat{X}_t|\mathcal{F}_t] = \frac{x_0}{Y_t}, \qquad t\in[0,T].$$

By construction, the wealth dynamics is characterized by the SDE

$$\widehat{X}_t Y_t = x_0 + \int_0^t \widehat{X}_s Y_s(\nu_s^*\sigma - \lambda)\mathrm{d}W_s$$

with the budget constraint $\mathbb{E}[Y_T X_T] = x_0$ (since the market is complete).

The optimal strategy is given by $\nu = \lambda/\sigma$ as to cancel out the stochastic integral.

[8 marks]

Comment:

This question is fairly standard and straightforward; Tests ability to manipulate the objects discussed in class.

(a) *This question recalls the ideas behind* sup / inf *and* ε*-controls.*

(b) *This question is easy and is a straightforward test of the basic learning outcomes of the course. In particular, the last part of the course on Duality theory.*

3.
Let $\mathbf{B} := (B,\ B^{\perp})$ be a two-dimensional Brownian motion (BM) on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, with $\mathbb{F}$ denoting the filtration generated by $\mathbf{B}$.

Take a financial market with a bank account with interest rate of *zero*, a stock price S and its volatility Y follow

$$\mathrm{d}S_t = S_t(\mu \mathrm{d}t + Y_t \mathrm{d}B_t), \quad \mathrm{d}Y_t = a(Y_t)\mathrm{d}t + b(Y_t)\mathrm{d}W_t,\ \mu \in \mathbb{R},$$

where W is a BM given by

$$W_t := \rho B_t + \sqrt{1-\rho^2}B_t^{\perp}, \quad \rho \in [-1, 1], \quad 0 \leq t \leq T.$$

The maps a, b are deterministic such that Y is positive, and the stock's market price of risk $\lambda := \mu/Y$ is square-integrable. An agent with initial wealth $x > 0$ and logarithmic utility function $U(\cdot) = \log(\cdot)$ maximises expected utility of wealth at time T, with wealth process X generated from trading S. Admissible strategies $\pi \in \mathcal{A}$ yield non-negative X. Denote by V the convex conjugate of U. Define Z as the density process with respect to $\mathbb{P}$ of any martingale measure $\mathbb{Q} \in \mathbf{M}$, where $\mathbf{M}$ denotes the set of equivalent martingale measures.

(a) Write down a stochastic exponential formula for Z involving λ and a second adapted process $\lambda^{\perp}$.

[1 marks]

(b) Derive the dynamics of ZX, deduce that $\mathbb{E}[Z_T X_T] \leq x$, for any $\mathbb{Q} \in \mathbf{M}$, and show that $u(x) \leq v(y) + xy$, where $u(x) := \sup_{\pi\in\mathcal{A}} \mathbb{E}[U(X_T)]$ is the primal value function and $v(y) := \inf_{\mathbb{Q}\in\mathbf{M}} \mathbb{E}[V(yZ_T)]$ is the dual value function, for $y > 0$.

[5 marks]

(c) Explain the relation between the optimal terminal wealth $\hat{X}_T$ and $y\hat{Z}_T$, where $y > 0$, and $\hat{Z}_T$ is the Radon-Nikodym derivative of the dual minimiser $\hat{\mathbb{Q}} \in \mathbf{M}$, and explain how y is fixed. Hence derive a formula for $\hat{X}_T$, given that $U(\cdot) = \log(\cdot)$.

[5 marks]

(d) Derive a formula for the optimal wealth process $\hat{X} = (\hat{X}_t)_{0\leq t\leq T}$, the optimal portfolio proportion process $\hat{\theta}$, and characterise the dual minimiser $\hat{\mathbb{Q}} \in \mathbf{M}$.

[5 marks]

(e) Show that the primal value function is given by $u(x) = \log x + H$, where H is a constant which you should express in terms of λ.

[1 marks]

(f) Suppose $Y_t = \sigma \exp(\alpha W_t), t \in [0,\ T]$, for positive constants σ, α. Derive the form of $u(x)$ in this case, show that in the limit $\alpha \to 0$ we have

$$u(x) = \log x + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 T,$$

and interpret the result.

[8 marks]

Solution:

(a) *Solution*:

$$Z_t = \mathcal{E}(-\lambda \cdot B - \lambda^{\perp} \cdot B^{\perp})_t, \ 0 \leq t \leq T.$$

[1 marks]

(b) *Solution*: The Itô product rule gives

$$\mathrm{d}(Z_t X_t) = Z_t \mathrm{d}X_t + X_t \mathrm{d}Z_t + \mathrm{d}[Z, \ X]_t.$$

Use

$$\mathrm{d}X_t = Y_t \pi_t (\lambda_t \mathrm{d}t + \mathrm{d}B_t) \ , \ \mathrm{d}Z_t = -Z_t(\lambda_t \mathrm{d}B_t + \lambda_t^{\perp} \mathrm{d}B_t^{\perp}) \ ,$$

where π is the wealth in the stock, and we get

$$\mathrm{d}(Z_t X_t) = Z_t[(Y_t \pi_t - X_t \lambda_t)\mathrm{d}B_t - X_t \lambda_t^{\perp} \mathrm{d}B_t^{\perp}].$$

So ZX is a local martingale bounded from below, hence a super-martingale, and we have

$$\mathbb{E}[Z_T X_T] \leq x.$$

For any strategy $\pi \in \mathcal{A}$ and any $Z \in \mathbf{M}$ we have, for $y > 0$,

$$\begin{aligned} \mathbb{E}[U(X_T)] &\leq \mathbb{E}[U(X_T)] + y(x - \mathbb{E}[Z_T X_T]) \\ &= \mathbb{E}[U(X_T) - yZ_T X_T] + xy \\ &\leq \mathbb{E}[V(yZ_T)] + xy \qquad \text{(since } U(x) - xy \geq V(y) \text{ for } y > 0). \end{aligned}$$

Taking the supremum over π on the LHS and the infimum over $\mathbb{Q}$ on the RHS yields

$$u(x) \leq v(y) + xy.$$

[5 marks]

(c) *Solution*: We get equality in (b) if
(i) $X_T = \hat{X}_t$ and $Z_T = \hat{\hat{Z}}_t$ such that $U'(\hat{X}_T) = y\hat{Z}_T$ and
(ii) if $y > 0$ is fixed via $\mathbb{E}[\hat{Z}_T \hat{X}_T] = x$ (so the super-martingale constraint becomes a martingale constraint).
Using $U'(x) = 1/x$ we get $\hat{X}_T = 1/(y\hat{Z}_T)$, and using this in the constraint $\mathbb{E}[\hat{Z}_T \hat{X}_T] = x$ gives

$$\hat{X}_T = \frac{x}{\hat{Z}_T}.$$

[5 marks]

(d) *Solution*: $\hat{Z}\hat{X}$ is a martingale, so for $t \leq T, \hat{Z}_t \hat{X}_t = \mathbb{E}[\hat{Z}_T \hat{X}_T | \mathcal{F}_t] = x$, so

$$\hat{X}_t = \frac{x}{\hat{Z}_t}, \ 0 \leq t \leq T.$$

Comparing $\hat{Z}_t \hat{X}_t = x$ with the solution of the SDE in (b) for $\hat{Z}\hat{X}$:

$$\hat{Z}_t \hat{X}_t = x + \int_0^t \hat{Z}_s[(Y_s \hat{\pi}_s - \hat{X}_t \lambda_t)\mathrm{d}B_t - \hat{X}_t \hat{\lambda}_t^{\perp} \mathrm{d}B_t^{\perp}],$$

involving an optimal process $\hat{\lambda}^{\perp}$, gives $\sigma\hat{\pi} - \hat{X}\lambda = 0$ and $\hat{\lambda}^{\perp} = 0$, so that

$$\hat{\theta}_t := \frac{\hat{\pi}_t}{\hat{X}_t} = \frac{\lambda_t}{Y_t}, \ \hat{\lambda}_t^{\perp} = 0, \ 0 \leq t \leq T.$$

Hence the dual optimiser $\hat{\mathbb{Q}}$ is the minimal martingale measure, with $\hat{Z} = \mathcal{E}(-\lambda \cdot B)$.

[5 marks]

(e) *Solution*: Using $u(x) = \mathbb{E}[\log \hat{X}_T]$ we get

$$u(x) = \log x - \mathbb{E}[\log \hat{Z}_T] = \log x + \mathbb{E}[\frac{1}{2}\int_0^T \lambda_t^2 \mathrm{d}t] =: \log x + H.$$

[1 marks]

(f) *Solution*: We have $\lambda = \mu/Y = (\mu/\sigma)\exp(-\alpha W)$, so

$$\begin{aligned} H &= \frac{1}{2}(\frac{\mu}{\sigma})^2 \mathbb{E}[\int_0^T \exp(-2\alpha W_t)\mathrm{d}t] \\ &= \frac{1}{2}(\frac{\mu}{\sigma})^2 \int_0^T \mathbb{E}[\exp(-2\alpha W_t)]\mathrm{d}t \\ &= \frac{1}{2}(\frac{\mu}{\sigma})^2 \int_0^T \exp(2\alpha^2 t)\mathrm{d}t \\ &= \frac{1}{2}(\frac{\mu}{\sigma})^2 \frac{1}{2\alpha^2}(\exp(2\alpha^2 T) - 1)\ . \end{aligned}$$

So the value function is given by

$$u(x) = \log x + \frac{1}{2}(\frac{\mu}{\sigma})^2 \frac{1}{2\alpha^2}(\exp(2\alpha^2 T) - 1)\ .$$

To get the limit as $\alpha \to 0$ we observe directly that the expectation in the first line of the computation of H converges to T, or take a Taylor expansion of the last formula, giving

$$\frac{1}{2\alpha^2}(\exp(2\alpha^2 T) - 1) = \frac{1}{2\alpha^2}(1 + 2\alpha^2 T + O(\alpha^4) - 1) = T + O(\alpha^2)\ ,$$

so in the limit $\alpha \to 0$ we get

$$u(x) = \log x + \frac{1}{2}(\frac{\mu}{\sigma})^2 T.$$

This is the form for the case where the volatility is not stochastic (when $\alpha \to 0$ the volatility becomes the constant σ) and the market price of risk is is the constant μ/σ.

[8 marks]

Comment:

(a) Standard, and a hint that they have to realise we are in an incomplete market driven by two Brownian motions.

(b) Standard; 2 marks for the dynamics, 1 mark for for the supermartingale constraint, and 2 for the duality inequality.

(c) This should be standard, but they must take into account dual optimisation, as we are in an incomplete market; 2 marks each for the reasonings in (i) and (ii), and 1 mark for the formula for $\hat{X}_T$.

(d) Slightly less familiar, as they have done relatively less work with incomplete markets; 1 mark for $\hat{X}$*, 2 each for the optimal portfolio and dual minimiser.*

(e) Easy, just need to use that $(\lambda \cdot B)$ *is a martingale.*

(f) More tricky, and unseen: 4 marks for the computation, 2 for the asymptotics, 2 for the interpretation.