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2023/24 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 1 - Last updated 22nd January 2025

Exercise 1.1 (ODEs). Assume that $t \mapsto r(t)$ is integrable i.e. $\int_0^t |r(s)|\,ds < \infty$ holds.

1. Solve
$$dB(t) = B(t)r(t)dt, \quad B(0) = 1. \tag{1}$$
2. Is the function $t \mapsto B(t)$ continuous? Why?
3. Calculate $d(1/B(t))$.

Exercise 1.2 (Geometric Brownian motion). Let W be a real-valued Wiener martingale generating a filtration $\mathbb{F}$. Assume that $\mu \in \mathcal{A}$ and $\sigma \in \mathcal{S}$. Note that

$$\mathcal{A} := \{X = (X_s)_{s\in[0,T]} : X \text{ is adapted and } \int_0^T X_s\,ds < \infty \text{ a.s}\}$$

and

$$\mathcal{S} := \left\{X = (X)_{s\in[0,T]} : X \text{ is adapted and } \mathbb{P}\left[\int_0^T |X_s|^2\,ds < \infty\right] = 1\right\}.$$

1. Solve
$$dS(t) = S(t)\left[\mu(t)\,dt + \sigma(t)\,dW(t)\right], \quad S(0) = s. \tag{2}$$
Hint: Solve this first in the case that μ and σ are real constants. Apply Itô's formula to the process S and the function $x \mapsto \ln x$ (this is strictly speaking not allowed but it gives you the solution). Verify you have a correct solution by using Itô formula with the function $x \mapsto e^x$. Now generalize this to the case that μ and σ are the processes as above.
2. Is the function $t \mapsto S(t)$ continuous? Why?
3. Calculate $d(1/S(t))$, assuming $s \neq 0$.
4. With B given by (1) calculate $d(S(t)/B(t))$.

Lemma 1.3 (Gronwall's lemma / inequality). *Let $\lambda = \lambda(t) \geq 0$, $a = a(t)$, $b = b(t)$ and $y = y(t)$ be locally integrable, real valued functions defined on I (with $I = [0,T]$ or $I = [0,\infty)$) such that λy is also locally integrable and for almost all $t \in [0,T]$*

$$y(t) + a(t) \leq b(t) + \int_0^t \lambda(s)y(s)\,ds.$$

Then

$$y(t) + a(t) \leq b(t) + \int_0^t \lambda(s)e^{\int_s^t \lambda(r)dr}(b(s) - a(s))\,ds \quad \text{for almost all } t \in I.$$

Furthermore, if b is monotone increasing and a is non-negative, then

$$y(t) + a(t) \leq b(t)e^{\int_0^t \lambda(r)\,dr}, \quad \text{for almost all } t \in I.$$

Exercise 1.4 (On Gronwall's lemma)**.** Prove Gronwall's Lemma by following these steps:

i) Let
$$z(t) = \left(e^{-\int_0^t \lambda(r)dr}\right) \int_0^t \lambda(s)y(s)\, ds.$$
and show that
$$z'(t) \le \lambda(t)e^{-\int_0^t \lambda(r)dr}\left(b(t) - a(t)\right).$$

ii) Integrate from 0 to t to obtain the first conclusion Lemma 1.3.

iii) Obtain the second conclusion of Lemma 1.3.

Exercise 1.5. Consider the SDE
$$dX_s^{t,x} = b(X_s^{t,x})\, ds + \sigma(X_s^{t,x})\, dW_s, \;\; t \le s \le T, \;\; X_t^{t,x} = x\,.$$
Assume it has a pathwise unique solution i.e. if $Y_s^{t,x}$ is another process that satisfies the SDE then
$$\mathbb{P}\left[\sup_{t\le s\le T} |X_s^{t,x} - Y_s^{t,x}| > 0\right] = 0\,.$$
Show that then the *flow property* holds i.e. for $0 \le t \le t' \le T$ we have almost surely that
$$X_s^{t,x} = X_s^{t',X_{t'}^{t,x}}, \qquad \forall s \in [t', T].$$