David Šiška School of Mathematics Sciences, University of Edinburgh

2023/24 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 2 - Last updated 7th February 2025

Exercise 2.1 (Non-existence of solution)**.**

1. Let $I = \left[0, \frac{1}{2}\right]$. Find a solution X for
$$\frac{dX_t}{dt} = X_t^2\,,\ \ t \in I\,,\ \ X_0 = 1\,.$$
2. Does a solution to the above equation exist on $I = [0,1]$? If yes, show that it satisfies the definition of an SDE solution from SAF lectures. In not, which property is violated?

Exercise 2.2 (Non-uniqueness of solution)**.** Fix $T > 0$. Consider
$$\frac{dX_t}{dt} = 2\sqrt{|X_t|}\,,\ \ t \in [0,T]\,,\ \ X_0 = 0\,.$$

1. Show that $\bar{X}_t := 0$ for all $t \in [0,T]$ is a solution to the above ODE.
2. Show that $X_t := t^2$ for all $t \in [0,T]$ is also a solution.
3. Find at least two more solutions different from $\bar{X}$ and X.

Exercise 2.3. For any $(t,x) \in [0,T] \times \mathbb{R}$, define the stochastic process $(X_s^{t,x})_{s\in[t,T]}$ as
$$X_s^{t,x} = x + W_s - W_t\,.$$
Let $\mathbb{E}^{t,x}[\,\cdot\,] := \mathbb{E}[\,\cdot\,|X_t = x]$. Define a function $v = v(t,x)$ as
$$v(t,x) = \mathbb{E}^{t,x}\left[\,g\big(X_T\big)\,\right] \qquad \forall\,(t,x) \in [0,T) \times \mathbb{R}\,.$$
Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$ and that $\big(\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T] \times \Omega; \mathbb{R})$. Show that
$$\begin{aligned}\partial_t v + \frac{1}{2}\partial_{xx} v = 0 \ \text{ on } \ [0,T) \times \mathbb{R}\,,\\ v(T,\cdot) = g \ \text{ on } \ \mathbb{R}\,.\end{aligned}$$

Hints:

i) Apply Itô formula to the function v and process $X^{t,x}$ between t and some $\tau \geq t$ a stopping time.

ii) Use the Markov and flow properties to show that $(v(s, X_s^{t,x}))_{s\in[t,T]}$ is a martingale. Use fact that the stochastic integral is a martingale (the condition $\big(\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T] \times \mathbb{R})$ ensures stochastic integral is a martingale).

iii) Convince yourself that the PDE must hold; you can use the conclusion of Exercise 2.5 below.

Exercise 2.4. Assume that X is a solution to $dX_s = b_s(X_s)\,dt + \sigma_s(X_s)\,dW_s$, $X_t = x$, $s \in [t,T]$ Assume that b and σ are bounded such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{s\in[0,T]} \sup_{x\in\mathbb{R}^d} |b_s(\omega,x)| + |\sigma_s(\omega,x)| \leq K < \infty.$$

1. Show that for any $m \in \mathbb{N}$ there is $c > 0$ (depending on T, m and bound on b and σ) such that for all $x \in \mathbb{R}^d$ we have

$$\mathbb{E}|X^{t,x}_{s'} - X^{t,x}_s|^{2m} \leq c|s'-s|^m\,.$$

2. Use Kolmogorov's continuity, the appendix in the lecture notes, to obtain Hölder continuity of sample paths of solutions.

Exercise 2.5. Let $X^{t,x}$ be the solution to the SDE $dX_s = b_s(X_s)\,dt + \sigma_s(X_s)\,dW_s$, $X_t = x$, $s \in [t,T]$ with b and σ like in Exercise 2.4.
Assume that $h : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is continuous. Assume that for any stopping time $\tau \geq t$

$$\int_t^\tau h(s, X^{t,x}_s)\,ds \leq 0 \quad \forall (t,x) \in [0,T] \times \mathbb{R}^d\,.$$

Show that $h(t,x) \leq 0$ for all $(t,x) \in [0,T] \times \mathbb{R}^d$.