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2024/25 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 3, for Week 6 - Last updated 27th February 2025

Exercise 3.1 (Merton problem with an exponential utility, no consumption)**.** We return to the portfolio optimization problem, see Section 4.3. of the lecture notes. Unlike in Example 4.10 we consider the utility function $g(x) := -e^{-\gamma x}$, $\gamma > 0$ a constant. We will also take $r = 0$ for simplicity and assume there is no consumption ($C = 0$). With X_t denoting the wealth at time time t we have the value function given by

$$v(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[g\left(X_T^{\pi,t,x,} \right) \right] .$$

i) Write down the expression for the wealth process in terms of π, the amount of wealth invested in the risky asset and with $r = 0$, $C = 0$.

ii) Write down the HJB equation associated to the optimal control problem. Solve the HJB equation by inspecting the terminal condition and thus suggesting a possible form for the solution. Write down the optimal control explicitly.

Exercise 3.2. Let $r \geq 0$, $T > 0$. Assume that $v \in C^{1,2}([0,T) \times \mathbb{R})$, that there is a constant K s.t. for all t, x we have $|b(x)| + |\sigma(x)| + |\partial_x v(t,x)| \leq K$ and that

$$\begin{aligned} \partial_t v + b\partial_x v + \frac{1}{2}\sigma^2 \partial_{xx} v - rv = 0 \;\; &\text{on} \;\; [0,T) \times \mathbb{R}\,, \\ v(T,\cdot) = g \;\; &\text{on} \;\; \mathbb{R}\,. \end{aligned}$$

Show that

$$v(t,x) = e^{-r(T-t)} \mathbb{E}^{t,x}\left[\, g\big(X_T^{t,x}\big)\,\right] \qquad \forall\, (t,x) \in [0,T) \times \mathbb{R}\,,$$

where for any $(t,x) \in [0,T] \times \mathbb{R}$ the process $(X_s^{t,x})_{s \in [t,T]}$ is the solution to

$$dX_s^{t,x} = b(X_s^{t,x})\,ds + \sigma(X_s^{t,x})\,dW_s, \;\; \forall\, s \in [t,T], \qquad X_t = x\,.$$