2024/25 Semester 2

Stochastic Control and Dynamic Asset Allocation

Problem Sheet 4, for Week 8 - Last updated 12th March 2025

Exercise 4.1 (Unattainable optimizer)**.** Here is a simple example in which no optimal control exists, in a finite horizon setting, $T \in (0, \infty)$. A control α is admissible ($\alpha \in \mathcal{A}$) if: α takes values in $\mathbb{R}$, is $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, and $\mathbb{E}\int_0^T \alpha_s^2\,ds < \infty$. The state equation is

$$dX_s = \alpha_s\,ds + dW_s \quad s \in [t, T]\,, \quad X_t = x \in \mathbb{R}.$$

Let $J(t, x, \alpha) := \mathbb{E}[|X_T^{t,x,\alpha}|^2]$. The value function is $v(t, x) := \inf_{\alpha\in\mathcal{A}} J(t, x, \alpha)$. Clearly $v(t, x) \geq 0$.

i) Show that for any $t \in [0, T]$, $x \in \mathbb{R}$, $\alpha \in \mathcal{A}$ we have $\mathbb{E}[|X_T^{t,x,\alpha}|^2] < \infty$.

ii) Show that if $\alpha_t := -cX_t$ for some constant $c \in (0, \infty)$ then $\alpha \in \mathcal{A}$ and

$$J^\alpha(t, x) = J^{cX}(t, x) = \frac{1}{2c} - \frac{1 - 2cx^2}{2c} e^{-2c(T-t)}.$$

Hint: with such an α, the process X is an Ornstein-Uhlenbeck process, see an earlier exercise.

iii) Conclude that $v(t, x) = 0$ for all $t \in [0, T)$, $x \in \mathbb{R}$.

iv) Show that there is no $\alpha \in \mathcal{A}$ such that $J(t, x, \alpha) = 0$. *Hint:* Suppose that there is such a α and show that this leads to a contradiction.